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A note on Riesz space valued measures

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Abstract: In this paper, some properties of Riesz space valued measures that are defined on an algebra of sets are obtained. The question of under what conditions $oba(\mathcal{F},E) = a(\mathcal{F},E)$ for a given Dedekind complete Riesz space E is answered under natural conditions. The outer measure, which is generated by order bounded Riesz space valued measure, is obtained and its properties are investigated. The concept of hazy convergence that is valid for real valued signed measures is extended to Riesz space valued measures. The consequences of order convergence of $f : X \to E$, which implies hazy convergence, are given.

Key words: Charge, finitely additive measure, outer measure, Riesz space valued measure

1. Introduction

For the standard definition and terminology of Riesz space theory, see [6, 10] or [5]. For general measure theory and real valued charge theory, we refer to [1, 4]. Let $\mathcal{F}$ be an algebra of subsets of any nonempty set X and let E be a Riesz space. A set function $\mu : \mathcal{F} \to E$ is called a *Riesz space valued measure* or an *additive measure* if $\mu(A \cup B) = \mu(A) + \mu(B)$ holds for each disjoint pairs of A, B in $\mathcal{F}$. We denote the space of additive measure by $a(\mathcal{F}, E)$. This space is an ordered vector space with the canonical order, $\mu \leq \nu$ if and only if $\mu(A) \leq \nu(A)$ for all A in $\mathcal{F}$. Generally, however, it fails to be a Riesz space in the case that E is a Dedekind complete Riesz space. Recall that a Riesz space E is called *Dedekind complete* if every bounded above subset of E has a supremum. A Riesz space valued measure is called *positive* if μ is E_+ valued. We should also note that for the positive part of Riesz space E we use E_+. A Riesz space valued measure is called *order bounded* if $\sup_{B \in \mathcal{F}} \mid \mu(B) \mid$ exists in E. We denote this space by $oba(\mathcal{F}, E)$. The space of $oba(\mathcal{F}, E)$ is a Dedekind complete Riesz space whenever E is a Dedekind complete Riesz space. Therefore, there are decomposition theorems as Jordan and Lebesgue decompositions for $oba(\mathcal{F}, E)$. For decomposition theorems, see [8]. Another family of measures are order countably additive measures. An order bounded measure $\mu : \mathcal{F} \to E$ is called *order countably additive* if $\mu(\bigcup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} \mu(A_i)$ holds for each pairwise disjoint sequence of (A_i) such that $\bigcup_{i=1}^{\infty} A_i$ is in $\mathcal{F}$, where the sum appearing in the right side of the equality is in the sense of order convergence. We denote countably additive measures space by $obca(\mathcal{F}, E)$ and note that $obca(\mathcal{F}, E)$ is a projection band of $oba(\mathcal{F}, E)$. For details, see [8].

In the second section, some elementary properties of order bounded and countably additive Riesz space valued measures are given. It is interesting that none of the studies about Riesz space valued measures in

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the literature include these elementary observations. Afterwards, our study will turn to give answers to the following question: "When is $oba(\mathcal{F},E)=a(\mathcal{F},E)$?" The first answer follows by the characterization of the order bounded additive measures by using cardinality of algebra on which the measure is defined. Essentially, this conclusion can be considered as a generalization of [3], which states the same result for the real case. On the other hand, we indicate another characterization of order boundedness by putting some conditions on the Riesz space E. This approach is directly the consequence of [9]. We get through the second section by giving some equivalent conditions of order countably additivity for Riesz space valued measures.

In the last section, we consider an order bounded additive measure with a fixed strictly positive order bounded continuous functional taken from the order continuous dual of the Dedekind complete Riesz space E. We define a real valued function by using these tools and we show that this function is a real valued outer measure. We proceed with some order and decomposition properties of the outer measure, which depends on the order bounded positive measure and on the strictly positive order continuous functional. Then we continue in a similar manner as classical Lebesgue integral theory by proceeding with the definition of null set and null function and their properties. We show that the set of null functions is an order ideal in the Riesz space of all E valued functions defined on the algebra $\mathcal{F}$. At the end of the last section, properties of hazy convergence are given and the relation between order convergence and hazy convergence is shown.

2. Order bounded and order countably additive measures

We start this section by indicating elementary properties of additive and order bounded additive Riesz space valued measures. The proofs of some routine facts will be omitted.

Theorem 2.1 *Let E be a Dedekind complete Riesz space and $\mathcal{F}$ be an algebra of subsets of a nonempty set X and $\mu \in a(\mathcal{F},\ E)$. Then the following statements are true.*
(i) If $A_1, A_2, ..., A_n$ are finitely many of pairwise disjoint sets in $\mathcal{F}$, then

$$\mu(\bigcup_{i=1}^{n} A_i) = \sum_{i=1}^{n} \mu(A_i).$$

(ii) If $A, B \in \mathcal{F}$ and $A \subset B$, then $\mu(B-A) = \mu(B) - \mu(A)$.

(iii) If $F_0, F_1, F_2, ..., F_n$ are finitely many of sets in $\mathcal{F}$ such that $F_0 \subset \bigcup\limits_{i=1}^{n} F_i$ and μ is positive, then

$$\mu(F_0) \leq \sum_{i=1}^{n} \mu(F_i).$$

In particular,

$$\mu(\bigcup_{i=1}^{n} F_i) \leq \sum_{i=1}^{n} \mu(F_i).$$

(iv) If (E_n) is a sequence of pairwise disjoint sets in $\mathcal{F}$ such that $\bigcup_{n\geq 1} E_n \subset A$ for $A \in \mathcal{F}$, and μ is positive, then

$$\sum_{n=1}^{\infty} \mu(E_n) \leq \mu(A).$$

The following theorem, which is one of the core results of the manuscript, establishes the equivalence of order boundedness and additivity for Riesz space valued measures for which the corresponding algebra is finite. The real case is obtained in [3]; we include the proof of the general case below for the sake of completeness, since it appears along similar lines of the one given for the real case.

Theorem 2.2 *Let $\mathcal{F}$ be an algebra on X and E be a Dedekind complete Riesz space; then the following are equivalent.*
(i) $\mathcal{F}$ is finite.
(ii) Each additive measure $\mu : \mathcal{F} \to E$ is order bounded.

Proof $(i) \Rightarrow (ii)$ Let $\mathcal{F}$ be a finite algebra, say $\mathcal{F} = \{A_1, A_2, ..., A_n\}$. Since E is a Riesz space, $\sup\{|\mu(A_1)|, |\mu(A_2)|, ..., |\mu(A_n)|\}$ exists in E. Thus, μ is order bounded.
$(ii) \Rightarrow (i)$ Assume that $\mathcal{F}$ is not a finite algebra and each additive measure $\mu : \mathcal{F} \to E$ is order bounded. We will contradict by finding a Riesz space valued measure that is not order bounded. We can easily show that there exists an infinite sequence (C_n) of mutually disjoint nonvoid sets in $\mathcal{F}$. Now let us select points $c_n \in C_n$, and for each n define the function from $\mathcal{F}$ by

$$p_n(A) = \begin{cases} 1, & c_n \in A \\ 0, & c_n \notin A. \end{cases}$$

Note that for each $n \in \mathbb{N}$, p_n is a real valued measure. Let $v : \mathcal{F} \to \mathbb{R}$ be defined by

$$v(A) = \sum_{n=1}^{\infty} \frac{\pi^n}{5^n} p_n(A).$$

Then v is finitely additive. The sequence $\mathcal{U} = (\frac{\pi^n}{5^n})$ is linearly independent over the rational field, and we may imbed $\mathcal{U}$ in a Hamel base $\mathcal{B}$ for the real numbers. Let e be a strictly positive element in E and define the Riesz spaces valued function $\alpha : \mathcal{B} \to \mathbb{R}$ by

$$\alpha(x) = \begin{cases} ne, & x \in \mathcal{U} \\ 0, & x \in \mathcal{B} \setminus \mathcal{U}. \end{cases}$$

Since $\mathcal{B}$ is a dominated subspace of the Riesz space $\mathbb{R}$, from Kantorovich's theorem (see [2]) α has a positive linear extension $\bar{\alpha}$ to all real numbers. Now if one defines the positive measure μ by

$$\mu(C) = \bar{\alpha}(v(C)), \quad (C \in \mathcal{F}),$$

then a finitely additive Riesz space valued measure that is not order bounded is obtained. Therefore, $\mathcal{F}$ is finite. □

Another equivalence for order boundedness can be given by putting the so-called PR condition on the Riesz space E. Let us now define the PR property as given in [3] and [7].

Definition 2.3 *Let E be a Riesz space and (h_k) be an increasing sequence of positive elements in E. If for each $e \in E$ there exists $t > 0$ and k such that $|e| \leq th_k$ then E has the PR property.*

We need one more boundedness definition, which is familiar, to measure theory reader.

Definition 2.4 *Let μ be a measure from $\mathcal{F}$ to Dedekind complete Riesz space E where $\mathcal{F}$ is an algebra of subsets of nonempty set X. Then μ is called s-bounded if $o-\lim \mu(A_n) = 0$ holds for each pairwise disjoint sequence (A_n).*

The following theorem is a direct consequence of [9].

Theorem 2.5 *Let $\mathcal{F}$ be an algebra of subset of X and E be a Riesz space that has the PR property. Then the following are equivalent:*
(1) $\mu : \mathcal{F} \to E$ is order bounded.
(2) $\mu : \mathcal{F} \to E$ is s-bounded.

We end this section by giving equivalent conditions for order countably additivity.

Theorem 2.6 *Let $\mu \in a(\mathcal{F}, E)$. Then the following are equivalent.*
(1) μ is order countably additive.
(2) For each increasing sequence (A_n) in $\mathcal{F}$ such that $A = \bigcup_{n\geq 1} A_n \in \mathcal{F}$, one has

$$o-\lim \mu(A_n) = \mu(A).$$

(3) For each decreasing sequence (A_n) in $\mathcal{F}$ such that $A = \bigcap_{n\geq 1} A_n \in \mathcal{F}$, one has

$$o-\lim \mu(A_n) = \mu(A).$$

(4) For each decreasing sequence (A_n) in $\mathcal{F}$ such that $\bigcap_{n\geq 1} A_n = \emptyset$, one has

$$o-\lim \mu(A_n) = 0.$$

3. Outer measure and hazy convergence

Throughout this section $\mathcal{F}$ will be an algebra of a subset of a nonempty set X and $\tilde{e}$ will be a fixed strictly positive functional taken from the order continuous dual of the Dedekind complete Riesz space E. Also, let μ be a positive and order bounded E valued measure defined on $\mathcal{F}$.

Theorem 3.1 *For a Dedekind complete Riesz space valued positive measure μ, the function $\mu^* : \wp(X) \to [0, \infty)$ given by*

$$\mu^*(A) = \inf\{\tilde{e}(\mu(B)) : A \subset B, B \in \mathcal{F}\}$$

is a real valued outer measure for each strictly positive order continuous functional $\tilde{e}$.

Proof Let us first observe that $\mu^*(\emptyset) = 0$. We have

$$0 \leq \mu^*(\emptyset) = \inf\{\tilde{e}(\mu(B)); \emptyset \subset B, B \in \mathcal{F}\} \leq \tilde{e}(\mu(\emptyset)) = 0.$$

For the monotonicity of μ^*, assume that $A \subset C \subset X$, then

$$\begin{aligned}\mu^*(A) &= Inf\{\tilde{e}(\mu(B)); A \subset B, B \in \mathcal{F}\}\\ &\leq Inf\{\tilde{e}(\mu(B)); A \subset C \subset B, B \in \mathcal{F}\}\\ &= \mu^*(C).\end{aligned}$$

This shows that μ^* is monotone. In particular, if $A \in \mathcal{F}$, then $\mu^*(A) = \tilde{e}(\mu(A))$. For this, by Theorem 2.1 (iii), $\tilde{e}(\mu(A))$ is a lower bound of the set

$$\{e^{\sim}(\mu(B)); A \subset C \subset B, B \in \mathcal{F}\}.$$

On the other hand,

$$\begin{aligned}\mu^*(A) &= Inf\{\tilde{e}(\mu(B)); A \subset B, B \in \mathcal{F}\}\\ &\leq \tilde{e}(\mu(A)),\end{aligned}$$

so we get $\mu^*(A) = \tilde{e}(\mu(A))$ for $A \in \mathcal{F}$. For subadditivity, let $A, B \subset X$ and $\varepsilon > 0$ be arbitrary. From the definition of μ^* there exist $A \subset A_1 \in \mathcal{F}, B \subset B_1 \in \mathcal{F}$ such that $\tilde{e}(\mu(A_1)) \leq \tilde{e}(\mu(A)) + \frac{\varepsilon}{2}$ and $\tilde{e}(\mu(B_1)) \leq \tilde{e}(\mu(B)) + \frac{\varepsilon}{2}$. Consequently,

$$\begin{aligned}\mu^*(A \cup B) &\leq \mu^*(A_1 \cup B_1) = \tilde{e}(\mu(A_1 \cup B_1))\\ &\leq \mu^*(A) + \mu^*(B) + \varepsilon.\end{aligned}$$

Since ε is arbitrary , $\mu^*(A \cup B) \leq \mu^*(A) + \mu^*(B)$. □

Note that from the lattice property of the space $oba(\mathcal{F}, E)$, there is no positivity condition on μ in order to generate the outer measure. If we follow the procedure above for $|\mu|$, then we can generate the outer measure. Decompositions in $oba(\mathcal{F}, E)$ are valid if we generate the outer measure from these decompositions, as is shown next.

Proposition 3.2 *Let μ_1 and μ_2 be positive measures in $oba(\mathcal{F}, E)$. Then we have*

$$(\mu_1 + \mu_2)^* = \mu_1^* + \mu_2^*.$$

Particularly, for each $\mu \in oba(\mathcal{F}, E)$, we have

$$|\mu|^* = \mu_+^* + \mu_-^*.$$

Proof Let μ_1, μ_2 be positive measures on $\mathcal{F}$ and $A \subset X$ so that

$$\begin{aligned}(\mu_1 + \mu_2)^*(A) &= \inf\{\tilde{e}((\mu_1 + \mu_2)(B)); A \subset B, B \in \mathcal{F}\}\\ &= \inf\{\tilde{e}((\mu_1)) + \tilde{e}((\mu_2)(B)); A \subset B, B \in \mathcal{F}\}\\ &\geq \inf\{\tilde{e}((\mu_1)); A \subset B, B \in \mathcal{F}\} + \inf\{\tilde{e}((\mu_2)(B)); A \subset B, B \in \mathcal{F}\}\\ &= \mu_1^*(A) + \mu_2^*(A).\end{aligned}$$

Consequently, $(\mu_1+\mu_2)^* \geq \mu_1^* + \mu_2^*$. On the other hand, let $\varepsilon > 0$ be arbitrary. There exist $A \subset B_1 \in \mathcal{F}, A \subset B_2 \in \mathcal{F}$ such that $\tilde{e}(\mu_1(B_1)) \leq \mu_1^*(A) + \frac{\varepsilon}{2}$ and $\tilde{e}(\mu_2(B_2)) \leq \mu_2^*(A) + \frac{\varepsilon}{2}$. Consequently,

$$\begin{aligned}(\mu_1+\mu_2)^*(A) &\leq \tilde{e}(\mu_1+\mu_2)(B_1\cap B_2)\\ &= \tilde{e}(\mu_1(B_1\cap B_2)) + \tilde{e}(\mu_2)(B_1\cap B_2)\\ &\leq \mu_1^*(A) + \mu_2^*(A) + \varepsilon.\end{aligned}$$

Since ε is arbitrary, $(\mu_1+\mu_2)^* \leq \mu_1^* + \mu_2^*$. Thus, $(\mu_1+\mu_2)^* = \mu_1^* + \mu_2^*$. Since $oba(\mathcal{F}, E)$ is a Riesz space, from decomposition of $|\mu| = \mu_+ + \mu_-$, one can immediately get $|\mu|^* = \mu_+^* + \mu_-^*$. □

Let us continue with the definition and properties of null sets and null functions.

Definition 3.3 *A subset A of X is called a μ-null set if $|\mu|^*(A) = 0$.*

The following theorem can be easily obtained from Theorem 3.1.

Theorem 3.4 *Let $(X, \mathcal{F}, \mu)$ be a measure space. Then the following hold.*
(1) $\emptyset$ is a μ-null set.
(2) $B \subset A$ is a μ-null set whenever A is a μ-null set.
(3) The finite union of μ-null sets is a μ-null set.

Definition 3.5 *(i) Let $\varepsilon > 0$ be given. Then f is called a μ-null function if*

$$|\mu|^*(\{x \in X : \tilde{e}(|f(x)|) > \varepsilon\}) = 0.$$

(ii) Two functions f and g are called almost everywhere equal if $f - g$ is a μ-null function. In this case we write "f=g μ-a.e."
(iii) f is called dominated almost everywhere by g if $f \leq g + h$ for a μ-null function h.

The set of all E valued null functions is an order ideal in the Riesz space of all E valued functions, as is shown next.

Theorem 3.6 *(1) Let f, g be μ-null functions. Then $cf + dg$ and $|f|$ are μ-null functions for real numbers c, d.*
(2) Let f be a μ-null function. If $|g| \leq |f|$, then g is a μ-null function.

Proof (1) Let f and g be two null functions on X and let c, d be two real numbers and $\varepsilon > 0$ be given, so that

$$\begin{aligned}\mu^*(\{x &\in X : \tilde{e}(|cf(x)+dg(x)|) > \varepsilon\}) \leq \mu^*(\{x \in X : \tilde{e}(|cf(x)| + |dg(x)|) > \varepsilon\})\\ &\leq \mu^*(\{x \in X : \tilde{e}(|cf(x)|) > \frac{\varepsilon}{2}\} \cup \{x \in X; \tilde{e}(|dg(x)|) > \frac{\varepsilon}{2}\})\\ &\leq \mu^*(\{x \in X : \tilde{e}(|f(x)|) > \frac{\varepsilon}{2\,|c|}\}) + \mu^*(\{x \in X : \tilde{e}(|g(x)|) > \frac{\varepsilon}{2\,|d|}\})\\ &= 0.\end{aligned}$$

Thus, $cf+dg$ is a null function. Since $|f|(x)=|f(x)|,(\forall x\in X)$, $|f|$ is a null function.
(2) Let $|g|\le|f|$ μ-a.e. and f be a μ-null function. Since $|g|\le|f|$ μ-a.e., there exists a null function h on X such that $|g|\le|f|+h$. From (1), $|f|+h$ is a null function. Since

$$\{x\in X;\tilde{e}(|g(x)|)>\varepsilon\}\subset\{x\in X;\tilde{e}(||f|(x)+h(x)|)>\varepsilon\},$$

from which

$$\mu^*(\{x\in X;\tilde{e}(|g(x)|)>\varepsilon\})\le\mu^*(\{x\in X;\tilde{e}(||f|(x)+h(x)|)\})=0,$$

g is a null function. □

With the null set concept, we have the following convergence type, which can be called hazy convergence as in the real case.

Definition 3.7 *Let (f_n) be a sequence of functions defined on X to E. The sequence (f_n) is called convergent to a function f hazily if for each $\varepsilon>0$,*

$$\lim_{n\to\infty}|\mu|^*(\{x\in X:\tilde{e}(|f_n(x)-f(x)|)>\varepsilon\})=0.$$

The limit of a hazily convergent sequence is unique, as expected, which is a consequence of the next theorem.

Theorem 3.8 *Let (f_n) be a sequence of functions that is hazily convergent to f and g. Then $f=g$ μ-a.e. Furthermore, if (f_n) is a sequence of functions that hazily converges to f and $f=g$ μ-a.e., then (f_n) is hazily convergent to g.*

Proof For the first part, observe that for any $\varepsilon>0$,

$$\begin{aligned}\{x &\in X:\tilde{e}(|f_n(x)-g(x)|)>\varepsilon\}\subset\\ \{x &\in X:\tilde{e}(|f_n(x)-f(x)|)>\frac{\varepsilon}{2}\}\cup\{x\in X:\tilde{e}(|f(x)-g(x)|)>\frac{\varepsilon}{2}\}.\end{aligned}$$

For the second part, observe that for any $\varepsilon>0$,

$$\begin{aligned}\{x &\in X:\tilde{e}(|f(x)-g(x)|)>\varepsilon\}\subset\\ \{x &\in X:\tilde{e}(|f_n(x)-f(x)|)>\frac{\varepsilon}{2}\}\cup\{x\in X:\tilde{e}(|f_n(x)-g(x)|)>\frac{\varepsilon}{2}\}.\end{aligned}$$

□

The following theorem gives us the algebraic properties of hazy convergence.

Theorem 3.9 *Let (f_n) and (g_n) be a sequence from X to E such that it is hazily convergent to f and g, respectively. Then the following hold.*
(1) If c and d are two real numbers, then sequence (cf_n+dg_n) hazily converges to $cf+dg$.
(2) The sequence $(f_n\vee g_n)$ hazily converges to $f\vee g$.

Proof (1) is obvious by the following inclusion:

$$\begin{aligned}\{x &\in X;\tilde{e}(|cf_n(x)+dg_n(x)-cf(x)-dg(x)|)>\varepsilon\}\subset\\ \{x &\in X;\tilde{e}(|f_n(x)-f(x)|)>\frac{\varepsilon}{2|c|}\}\cup\{x\in X;\tilde{e}(|g_n(x)-g(x)|)>\frac{\varepsilon}{2|d|}\}.\end{aligned}$$

(2) By the Birkhoff inequality we know that

$$|f_n \vee g_n - f \vee g| \leq |f_n - f| + |g_n - g|.$$

Now (2) is obvious by the following inclusion:

$$\begin{aligned} \{x &\in X; \tilde{e}(|f_n \vee g_n(x) - f \vee g(x)|) > \varepsilon\} \subset \\ \{x &\in X; \tilde{e}(|f_n(x) - f(x)|) > \frac{\varepsilon}{2}\} \cup \{x \in X; \tilde{e}(|g_n(x) - g(x)|) > \frac{\varepsilon}{2}\}. \end{aligned}$$

□

Finally, we will give the relation between order convergence and hazy convergence.

Theorem 3.10 *If (f_n) is a sequence of functions that converges in order to $f : X \to E$ in E^X, then (f_n) hazily converges to f.*

Proof Since the sequence (f_n) is order convergent in E^X, for each $x \in X$ there exists a sequence $p_n(x) \downarrow \theta$ such that $|f_n(x) - f(x)| \leq p_n$ for all n. Since $\tilde{e}$ is a strictly positive order continuous functional, then for each $x \in X$,

$$\tilde{e}(|f_n(x) - f(x)|) \leq \tilde{e}(p_n(x)).$$

It then follows that

$$\lim_{n\to\infty} \tilde{e}(|f_n(x) - f(x)|) \leq \lim_{n\to\infty} \tilde{e}(p_n(x)) = 0.$$

This means

$$\lim_{n\to\infty} \mu^*(\{x \in X; \tilde{e}(|f_n(x) - f(x)|) > \varepsilon\}) = 0.$$

Consequently, the sequence of (f_n) converges to f hazily. □

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