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Parameterized Littlewood–Paley operators and their commutators on Herz spaces with variable exponents

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Abstract: The aim of this paper is to deal with Littlewood–Paley operators with real parameters, including the parameterized Lusin area integrals and the parameterized Littlewood–Paley g_λ^*-functions, and their commutators on Herz spaces with two variable exponents $p(\cdot)$, $q(\cdot)$. By using the properties of the Lebesgue spaces with variable exponents, the boundedness of the parameterized Littlewood–Paley operators and their commutators generated respectively by BMO function and Lipschitz function is obtained on those Herz spaces.

Key words: Parameterized Littlewood–Paley operators, commutators, Herz spaces with variable exponent, BMO spaces, Lipschitz spaces

1. Introduction and main results

The Littlewood–Paley operators, including the Lusin area integrals, the Littlewood–Paley g-functions, and g_λ^*-functions, play very important roles in harmonic analysis and PDE (see [4,12,20,25,28]). In 2009, Xue and Ding gave weighted estimates for Littlewood–Paley operators and their commutators (see [32]). In 2013, Wei and Tao proved Litttlewood–Paley operators with rough kernels are bounded on weighted $(L^q, L^p)^\alpha(\mathbb{R}^n)$ spaces (see [30]).

In 1960, the parameterized Littlewood–Paley operators were discussed by Hörmander (see [14]) for the first time. Now, let us review the definitions of the parameterized Lusin area integral and the parameterized Littlewood–Paley g_λ^*-function.

Let S^{n-1} denote the unit sphere of $\mathbb{R}^n$ equipped with Lebesgue measure $d\sigma(x')$ and $\psi(x) = \Omega(x)|x|^{-n+\rho}\chi_{\{|x|\le 1\}}$, where $0 < \rho < n$ and Ω satisfies the following conditions throughout this paper:

a) $\Omega(\lambda x) = \Omega(x)$ for all $\lambda > 0$;

b) $\int_{\mathbb{R}^n} \Omega(x')d\sigma(x') = 0$;

c) $\Omega \in L^1(S^{n-1})$.

Then the parameterized Lusin area integral S^ρ and the parameterized Littlewood–Paley g_λ^*-function $g_\lambda^{*,\,\rho}$ are defined respectively by

$$S^\rho(f)(x) = \left(\iint_{\Gamma_a(x)} |\psi_t * f(y)|^2 \frac{dydt}{t^{n+1}}\right)^{1/2}$$

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and

$$g_\lambda^{*,\ \rho}(f)(x)=\left(\iint_{\mathbb{R}_+^{n+1}}\left(\frac{t}{t+|x-y|}\right)^{\lambda n}|\psi_t*f(y)|^2\frac{dydt}{t^{n+1}}\right)^{1/2},$$

where $\Gamma(x)=\{(t,y)\in\mathbb{R}_+^{n+1}:|y-x|<t\},\quad \lambda>1$.

For general ρ, Sakamoto and Yabuta considered the L^p boundedness of S^ρ and $g_\lambda^{*,\ \rho}$ in [23]; Wei and Tao gave the boundedness of parameterized Litttlewood–Paley operators with rough kernels on weighted weak Hardy spaces in [31]. In 2009, Chen and Ding investigated the characterization of commutators for parameterized Littlewood–Paley operators (see [5,6]).

Now let us recall the definitions of the corresponding m-order commutators of the parameterized Littlewood–Paley operators above. Let $b\in L^1_{\mathrm{loc}}(\mathbb{R}^n)$, $m\in\mathbb{N}$, the commutators $[b^m,\ S^\rho]$ and $[b^m,\ g_\lambda^{*,\ \rho}]$ be defined respectively by

$$[b^m,S^\rho](f)(x)=\left(\iint_{\Gamma_a(x)}\left|\frac{1}{t^\rho}\int_{|y-x|\le t}\frac{\Omega(y-z)}{|y-z|^{n-\rho}}[b(x)-b(z)]^m f(z)dz\right|^2\frac{dydt}{t^{n+1}}\right)^{\frac{1}{2}}$$

and

$$[b^m,\ g_\lambda^{*,\ \rho}](f)(x)=\left(\iint\limits_{\mathbb{R}_+^{n+1}}\left(\frac{t}{t+|x-y|}\right)^{\lambda n}\left|\frac{1}{t^\rho}\int\limits_{|y-x|\le t}\frac{\Omega(y-z)}{|y-z|^{n-\rho}}[b(x)-b(z)]^m f(z)dz\right|^2\frac{dydt}{t^{n+1}}\right)^{\frac{1}{2}}.$$

On the other hand, Lebesgue spaces with variable exponent $L^{p(\cdot)}(\mathbb{R}^n)$ become one class of important function spaces due to the fundamental paper [21] by Kováčik and Rákosník. In the past 20 years, the theory of these spaces has progressed rapidly, and it has been widely applied in some fields such as fluid dynamics, elasticity dynamics, calculus of variations, and differential equations with nonstandard growth conditions (see [1,2,9,22,33]). In [7], Cruz-Uribe et al. studied extrapolation theorem, which leads to the boundedness of some classical operators including the commutators on $L^{p(\cdot)}(\mathbb{R}^n)$. In 2012, Wang et al. stated that higher-order commutators of Marcinkiewicz integrals are bounded on spaces $L^{p(\cdot)}(\mathbb{R}^n)$ (see [26]). Recently, Wang and Tao established the boundedness of parameterized Littlewood–Paley operators and their commutators generated respectively by BMO functions and Lipschitz functions on spaces $L^{p(\cdot)}(\mathbb{R}^n)$ (see [29]).

It is well known that Herz spaces play an important role in harmonic analysis. After the Herz spaces with one variable exponent $p(\cdot)$ were introduced in [15], the boundedness of some operators and some characterizations of these spaces were studied widely (see [16-18,27]).

In this paper, we will study Herz spaces with two variable exponents $p(\cdot)$, $q(\cdot)$. Inspired by the results mentioned previously, it is natural to ask whether the parameterized Littlewood–Paley operators S^ρ and $g_\lambda^{*,\ \rho}$ and their commutators $[b^m,\ S^\rho]$ and $[b^m,\ g_\lambda^{*,\ \rho}]$ are bounded on Herz spaces with two variable exponents or not. The purpose of this paper is to discuss this question. Before stating our main results, we need to introduce some relevant definitions and notations. Let E be a Lebesgue measurable set in $\mathbb{R}^n$ with $|E|>0$.

Definition 1.1 [21] *Let $p(\cdot): E \to [1,\infty)$ be a measurable function. The Lebesgue space with variable exponent $L^{p(\cdot)}(E)$ is defined by*

$$L^{p(\cdot)}(E) = \left\{ f \text{ is measurable}: \int_E \left(\frac{|f(x)|}{\eta}\right)^{p(x)} dx < \infty \text{ for some constant } \eta > 0 \right\},$$

and the space $L^{p(\cdot)}_{\mathrm{loc}}(E)$ is defined by

$$L^{p(\cdot)}_{\mathrm{loc}}(E) = \left\{ f \text{ is measurable}: f \in L^{p(\cdot)}(K) \text{ for all compact subsets } K \subset E \right\}.$$

It is easy to see that the Lebesgue space $L^{p(\cdot)}(E)$ is a Banach space with the following Luxemburg–Nakano norm:

$$\|f\|_{L^{p(\cdot)}(E)} = \inf\left\{ \eta > 0: \int_E \left(\frac{|f(x)|}{\eta}\right)^{p(x)} dx \le 1 \right\}.$$

Remark 1.1 (1) Noting that if the function $p(x) = p_0$ is a constant function, then $L^{p(\cdot)}(\mathbb{R}^n)$ equals $L^{p_0}(\mathbb{R}^n)$. This implies that the Lebesgue spaces with variable exponent generalize the usual Lebesgue spaces. Moreover, they have many properties in common with the usual Lebesgue spaces.

(2) Let $p_- := \operatorname{ess\,inf}\{p(x): x \in \mathbb{R}^n\}$, $p_+ := \operatorname{ess\,sup}\{p(x): x \in \mathbb{R}^n\}$. Then denote $\mathcal{P}^0(\mathbb{R}^n)$ to be the set of measurable function all $p(\cdot)$ with $p_- > 0$ and $p_+ < \infty$, $\mathcal{P}(\mathbb{R}^n)$ the set of all measurable function $p(\cdot)$ with $p_- > 1$ and $p_+ < \infty$.

Given a function $p(\cdot) \in \mathcal{P}^0(\mathbb{R}^n)$, we could define the space $L^{p(\cdot)}(\mathbb{R}^n)$ as above.

$$L^{p(\cdot)}(\mathbb{R}^n) = \left\{ f: |f|^{p_0} \in L^{q(\cdot)}(\mathbb{R}^n) \text{ for some } p_0 \text{ with } 0 < p_0 < p_- \text{ and } q(x) = \frac{p(x)}{p_0} \right\}.$$

Define a quasinorm on this space by (see [7])

$$\|f\|_{L^{p(\cdot)}(\mathbb{R}^n)} = \||f|^{p_0}\|^{1/p_0}_{L^{q(\cdot)}(\mathbb{R}^n)}.$$

(3) The Hardy–Littlewood maximal operator M is defined by

$$M(f)(x) = \sup_{Q \ni x} \frac{1}{|Q|} \int_Q |f(y)| dy.$$

Denote $\mathcal{B}(E)$ to be the set of all functions $p(\cdot) \in \mathcal{P}(E)$ satisfying the condition that M is bounded on $L^{p(\cdot)}(E)$.

Next, we will introduce the Herz spaces with two variable exponents. Before doing that we need to introduce the following function space, which is named the mixed Lebesgue sequence space (see [3]).

Definition 1.2 *Let $p(\cdot), q(\cdot) \in \mathcal{P}^0(\mathbb{R}^n)$. The mixed Lebesgue sequence space with variable exponents $l^{q(\cdot)}(L^{p(\cdot)})$ is the collection of all sequences $\{f_j\}_{j=0}^\infty$ of measurable functions on $\mathbb{R}^n$ such that*

$$\|\{f_j\}_{j=0}^\infty\|_{l^{q(\cdot)}(L^{p(\cdot)})} = \inf\left\{ \mu > 0: \varrho_{l^{q(\cdot)}(L^{p(\cdot)})}\left(\left\{\frac{f_j}{\mu}\right\}_{j=0}^\infty\right) \le 1 \right\} < \infty,$$

where

$$\varrho_{l^{q(\cdot)}(L^{p(\cdot)})}(\{f_j\}_{j=0}^{\infty})=\sum_{j=0}^{\infty}\inf\left\{\mu_j>0:\ \int_{\mathbb{R}^n}\left(\frac{|f_j(x)|}{\mu_j^{\frac{1}{q(x)}}}\right)^{p(x)}dx\le 1\right\}.$$

Noticing $q_+<\infty$*, we see that*

$$\varrho_{l^{q(\cdot)}(L^{p(\cdot)})}(\{f_j\}_{j=0}^{\infty})=\sum_{j=0}^{\infty}\left\||f_j|^{q(\cdot)}\right\|_{L^{\frac{p(\cdot)}{q(\cdot)}}}.$$

Let $B_k=\{x\in\mathbb{R}^n:|x|\le 2^k\},\ C_k=B_k\backslash B_{k-1},\ k\in\mathbb{Z},\ \chi_k=\chi_{C_k}$.

Definition 1.3 *Let* $\alpha\in\mathbb{R}^n,\ p(\cdot),q(\cdot)\in P^0(\mathbb{R}^n)$. *The nonhomogeneous Herz space with variable exponent* $K_{p(\cdot)}^{\alpha,q(\cdot)}(\mathbb{R}^n)$ *is defined by*

$$K_{p(\cdot)}^{\alpha,q(\cdot)}(\mathbb{R}^n)=\{f\in L_{loc}^{p(\cdot)}(\mathbb{R}^n\setminus\{0\}):\|f\|_{K_{p(\cdot)}^{\alpha,q(\cdot)}(\mathbb{R}^n)}<\infty\},$$

where

$$\|f\|_{K_{p(\cdot)}^{\alpha,q(\cdot)}(\mathbb{R}^n)}=\left\|\{2^{k\alpha}|f\chi_k|\}_{k=0}^{\infty}\right\|_{l^{q(\cdot)}(L^{p(\cdot)})}$$
$$=\inf\left\{\eta>0\ :\ \sum_{k=0}^{\infty}\left\|\left(\frac{2^{k\alpha}|f\chi_k|}{\eta}\right)^{q(\cdot)}\right\|_{L^{\frac{p(\cdot)}{q(\cdot)}}}\le 1\right\}.$$

The homogeneous Herz space with variable exponent $\dot{K}_{p(\cdot)}^{\alpha,q(\cdot)}(\mathbb{R}^n)$ *is defined by*

$$\dot{K}_{p(\cdot)}^{\alpha,q(\cdot)}(\mathbb{R}^n)=\{f\in L_{loc}^{p(\cdot)}(\mathbb{R}^n\setminus\{0\}):\|f\|_{\dot{K}_{p(\cdot)}^{\alpha,q(\cdot)}(\mathbb{R}^n)}<\infty\},$$

where

$$\|f\|_{\dot{K}_{p(\cdot)}^{\alpha,q(\cdot)}(\mathbb{R}^n)}=\left\|\{2^{k\alpha}|f\chi_k|\}_{k=-\infty}^{\infty}\right\|_{l^{q(\cdot)}(L^{p(\cdot)})}$$
$$=\inf\left\{\eta>0\ :\ \sum_{k=-\infty}^{\infty}\left\|\left(\frac{2^{k\alpha}|f\chi_k|}{\eta}\right)^{q(\cdot)}\right\|_{L^{\frac{p(\cdot)}{q(\cdot)}}}\le 1\right\}.$$

Remark 1.2 (1) It is easy to see that $K_{p(\cdot)}^{0,q(\cdot)}(\mathbb{R}^n)=\dot{K}_{p(\cdot)}^{0,q(\cdot)}(\mathbb{R}^n)=L^{p(\cdot)}(\mathbb{R}^n)$ and if $q(x)$ is a constant function, then the Herz spaces defined above are just the Herz spaces introduced in [15].

(2) If $q_1(\cdot),q_2(\cdot)\in\mathcal{P}^0(\mathbb{R}^n)$ satisfying $(q_1)_+\le(q_2)_-$, then

$$\dot{K}_{p(\cdot)}^{\alpha,q_1(\cdot)}(\mathbb{R}^n)\subset\dot{K}_{p(\cdot)}^{\alpha,q_2(\cdot)}(\mathbb{R}^n),\quad K_{p(\cdot)}^{\alpha,q_1(\cdot)}(\mathbb{R}^n)\subset K_{p(\cdot)}^{\alpha,q_2(\cdot)}(\mathbb{R}^n).$$

In fact, if letting $q_1(\cdot),q_2(\cdot)\in\mathcal{P}^0(\mathbb{R}^n)$ and $(q_1)_+\le(q_2)_-$, then $\frac{q_2(\cdot)}{q_1(\cdot)}\in\mathcal{P}^0(\mathbb{R}^n)$ and $\frac{q_2(\cdot)}{q_1(\cdot)}\ge 1$. Thus, by Lemma

2.4 and Remark 3.1 in the following passage, for any $f \in \dot{K}^{\alpha,q_1(\cdot)}_{p(\cdot)}(\mathbb{R}^n)$, we have

$$\sum_{k=-\infty}^{\infty} \left\| \left(\frac{2^{k\alpha}|f\chi_k|}{\eta} \right)^{q_2(\cdot)} \right\|_{L^{\frac{p(\cdot)}{q_2(\cdot)}}} \le \sum_{k=-\infty}^{\infty} \left\| \left(\frac{2^{k\alpha}|f\chi_k|}{\eta} \right)^{q_1(\cdot)} \right\|_{L^{\frac{p(\cdot)}{q_1(\cdot)}}}^{p_k}$$
$$\le \left\{ \sum_{k=-\infty}^{\infty} \left\| \left(\frac{2^{k\alpha}|f\chi_k|}{\eta} \right)^{q_1(\cdot)} \right\|_{L^{\frac{p(\cdot)}{q_1(\cdot)}}}^{p_k} \right\}^{p_*} \le 1$$

where

$$p_k = \begin{cases} \left(\frac{q_2(\cdot)}{q_1(\cdot)}\right)_-, & \frac{2^{k\alpha}|f\chi_k|}{\eta} \le 1, \\ \left(\frac{q_2(\cdot)}{q_1(\cdot)}\right)_+, & \frac{2^{k\alpha}|f\chi_k|}{\eta} > 1. \end{cases}, \quad p_* = \begin{cases} \min\limits_{k\in\mathbb{N}} p_k, & \sum_{k=0}^{\infty} a_k \le 1, \\ \max\limits_{k\in\mathbb{N}} p_k, & \sum_{k=0}^{\infty} a_k > 1. \end{cases}$$

This implies that $\dot{K}^{\alpha,q_1(\cdot)}_{p(\cdot)}(\mathbb{R}^n) \subset \dot{K}^{\alpha,q_2(\cdot)}_{p(\cdot)}(\mathbb{R}^n)$. A similar argument yields $K^{\alpha,q_1(\cdot)}_{p(\cdot)}(\mathbb{R}^n) \subset K^{\alpha,q_2(\cdot)}_{p(\cdot)}(\mathbb{R}^n)$.

Definition 1.4 [11] *Let $\Omega \in L^q(S^{n-1})$ for $q \ge 1$. Then the integral modulus $\omega_q(\delta)$ of L^q continuity of Ω is defined by*

$$\omega_q(\delta) = \sup_{\|\rho\|<\delta} \left(\int_{S^{n-1}} |\Omega(\rho x') - \Omega(x')|^q d\sigma(x') \right)^{1/q}, \quad 1 \le q < \infty$$

and

$$\omega_\infty(\delta) = \sup_{\|\rho\|<\delta} |\Omega(\rho x') - \Omega(x')|,$$

where $0 < \delta \le 1$, ρ denotes the rotation on $\mathbb{R}^n$, and $\|\rho\| = \sup\limits_{x'\in S^{n-1}} |\rho x' - x'|$.

Definition 1.5 [24] *For $0 < \beta \le 1$, the Lipschitz space $Lip_\beta(\mathbb{R}^n)$ is defined by*

$$Lip_\beta(\mathbb{R}^n) = \left\{ f : \ \|f\|_{Lip_\beta} = \sup_{x,y\in\mathbb{R}^n; x\ne y} \frac{|f(x) - f(y)|}{|x-y|^\beta} < \infty \right\}.$$

Our main results in this paper are formulated as follows.

Theorem 1.1 *Suppose that $p_1(\cdot) \in \mathcal{B}(\mathbb{R}^n)$,$q_1(\cdot), q_2(\cdot) \in \mathcal{P}^0(\mathbb{R}^n)$ with $(q_2)_- \ge (q_1)_+$, $\rho > n/2$ and $\Omega \in L^2(S^{n-1})$ satisfying*

$$\int_0^1 \frac{\omega_2(\delta)}{\delta}(1 + |\log\delta|)^\sigma d\delta < \infty, \quad \sigma > 2. \tag{1.1}$$

If $-n\delta_{12} < \alpha < n\delta_{11}$, where δ_{11}, δ_{12} are the constants in Lemma 2.2, then the operator S^ρ is bounded from $\dot{K}^{\alpha,q_2(\cdot)}_{p_1(\cdot)}(\mathbb{R}^n)$ $\left(K^{\alpha,q_2(\cdot)}_{p_1(\cdot)}(\mathbb{R}^n)\right)$ to $\dot{K}^{\alpha,q_1(\cdot)}_{p_1(\cdot)}(\mathbb{R}^n)$ $\left(K^{\alpha,q_1(\cdot)}_{p_1(\cdot)}(\mathbb{R}^n)\right)$.

Theorem 1.2 *Suppose that $p_1(\cdot) \in \mathcal{B}(\mathbb{R}^n)$,$q_1(\cdot), q_2(\cdot) \in \mathcal{P}^0(\mathbb{R}^n)$ with $(q_2)_- \ge (q_1)_+$, $\rho > n/2$, $\lambda > 2$ and $\Omega \in L^2(S^{n-1})$ satisfying (1.1). If $-n\delta_{12} < \alpha < n\delta_{11}$, where δ_{11}, δ_{12} are the constants in Lemma 2.2, then the operator $g^{*,\ \rho}_\lambda$ is bounded from $\dot{K}^{\alpha,q_2(\cdot)}_{p_1(\cdot)}(\mathbb{R}^n)$ $\left(K^{\alpha,q_2(\cdot)}_{p_1(\cdot)}(\mathbb{R}^n)\right)$ to $\dot{K}^{\alpha,q_1(\cdot)}_{p_1(\cdot)}(\mathbb{R}^n)$ $\left(K^{\alpha,q_1(\cdot)}_{p_1(\cdot)}(\mathbb{R}^n)\right)$.*

Theorem 1.3 *Let* $b \in BMO(\mathbb{R}^n)$ *and* $m \in \mathbb{N}$. *Suppose that* $p_1(\cdot) \in \mathcal{B}(\mathbb{R}^n)$, $q_1(\cdot), q_2(\cdot) \in \mathcal{P}^0(\mathbb{R}^n)$ *with* $(q_2)_- \geq (q_1)_+$, $\rho > n/2$ *and* $\Omega \in L^2(S^{n-1})$ *satisfying (1.1). If* $-n\delta_{12} < \alpha < n\delta_{11}$, *where* δ_{11}, δ_{12} *are the constants in Lemma 2.2, then the commutator* $[b^m, S^\rho]$ *is bounded from* $\dot{K}^{\alpha,q_2(\cdot)}_{p_1(\cdot)}(\mathbb{R}^n)$ $\left(K^{\alpha,q_2(\cdot)}_{p_1(\cdot)}(\mathbb{R}^n)\right)$ *to* $\dot{K}^{\alpha,q_1(\cdot)}_{p_1(\cdot)}(\mathbb{R}^n)$ $\left(K^{\alpha,q_1(\cdot)}_{p_1(\cdot)}(\mathbb{R}^n)\right)$.

Theorem 1.4 *Let* $b \in BMO(\mathbb{R}^n)$ *and* $m \in \mathbb{N}$. *Suppose that* $p_1(\cdot) \in \mathcal{B}(\mathbb{R}^n)$, $q_1(\cdot), q_2(\cdot) \in \mathcal{P}^0(\mathbb{R}^n)$ *with* $(q_2)_- \geq (q_1)_+$, $\rho > n/2$, $\lambda > 2$ *and* $\Omega \in L^2(S^{n-1})$ *satisfying (1.1). If* $-n\delta_{12} < \alpha < n\delta_{11}$, *where* δ_{11}, δ_{12} *are the constants in Lemma 2.2, then the commutator* $[b^m, g_\lambda^{*,\ \rho}]$ *is bounded from* $\dot{K}^{\alpha,q_2(\cdot)}_{p_1(\cdot)}(\mathbb{R}^n)$ $\left(K^{\alpha,q_2(\cdot)}_{p_1(\cdot)}(\mathbb{R}^n)\right)$ *to* $\dot{K}^{\alpha,q_1(\cdot)}_{p_1(\cdot)}(\mathbb{R}^n)$ $\left(K^{\alpha,q_1(\cdot)}_{p_1(\cdot)}(\mathbb{R}^n)\right)$.

Theorem 1.5 *Let* $b \in Lip_\beta(\mathbb{R}^n)$, $m \in \mathbb{N}$, *and* $\Omega \in L^2(S^{n-1})$. *Suppose that* $\rho > n/2$, $0 < \beta < \min\{1,\ n/m\}$, $q_1(\cdot), q_2(\cdot) \in \mathcal{P}^0(\mathbb{R}^n)$ *with* $(q_2)_- \geq (q_1)_+$ *and* $p_1(\cdot), p_2(\cdot) \in \mathcal{B}(\mathbb{R}^n)$ *be such that* $(p_1)_+ < \frac{n}{m\beta}$, $1/p_1(x) - 1/p_2(x) = m\beta/n$, *and* $p_2(\cdot)(n - m\beta)/n \in \mathcal{B}(\mathbb{R}^n)$. *If* $-n\delta_{22} < \alpha < n\delta_{11}$, *where* δ_{11}, δ_{22} *are the constants in Lemma 2.2, then the commutator* $[b^m, S^\rho]$ *is bounded from* $\dot{K}^{\alpha,q_1(\cdot)}_{p_1(\cdot)}(\mathbb{R}^n)$ $\left(K^{\alpha,q_1(\cdot)}_{p_1(\cdot)}(\mathbb{R}^n)\right)$ *to* $\dot{K}^{\alpha,q_2(\cdot)}_{p_2(\cdot)}(\mathbb{R}^n)$ $\left(K^{\alpha,q_2(\cdot)}_{p_2(\cdot)}(\mathbb{R}^n)\right)$.

Theorem 1.6 *Let* $b \in Lip_\beta(\mathbb{R}^n)$, $m \in \mathbb{N}$, *and* $\Omega \in L^2(S^{n-1})$. *Suppose that* $\rho > n/2$, $\lambda > 2$, $0 < \beta < \min\{1,\ n/m\}$, $q_1(\cdot), q_2(\cdot) \in \mathcal{P}^0(\mathbb{R}^n)$ *with* $(q_2)_- \geq (q_1)_+$ *and* $p_1(\cdot), p_2(\cdot) \in \mathcal{B}(\mathbb{R}^n)$ *be such that* $(p_1)_+ < \frac{n}{m\beta}$, $1/p_1(x) - 1/p_2(x) = m\beta/n$, *and* $p_2(\cdot)(n - m\beta)/n \in \mathcal{B}(\mathbb{R}^n)$. *If* $-n\delta_{22} < \alpha < n\delta_{11}$, *where* δ_{11}, δ_{22} *are the constants in Lemma 2.2, then the commutator* $[b^m, g_\lambda^{*,\ \rho}]$ *is bounded from* $\dot{K}^{\alpha,q_1(\cdot)}_{p_1(\cdot)}(\mathbb{R}^n)$ $\left(K^{\alpha,q_1(\cdot)}_{p_1(\cdot)}(\mathbb{R}^n)\right)$ *to* $\dot{K}^{\alpha,q_2(\cdot)}_{p_2(\cdot)}(\mathbb{R}^n)$ $\left(K^{\alpha,q_2(\cdot)}_{p_2(\cdot)}(\mathbb{R}^n)\right)$.

We now make some conventions. Throughout this paper, $|E|$ denotes the Lebesgue measure of $E \subset \mathbb{R}^n$. Given a function f, we denote the mean value of f on E by $f_E =: \frac{1}{|E|}\int_E f(x)dx$. $p'(\cdot)$ means the conjugate exponent of $p(\cdot)$, namely $1/p(x) + 1/p'(x) = 1$ holds. C always means a positive constant independent of the main parameters and may change from one occurrence to another.

2. Preliminary lemmas

In this section, we need some conclusions that will be used in the proofs of our main results.

Lemma 2.1 [21] *(Generalized Hölder's Inequality) Let* $p(\cdot)$, $p_1(\cdot)$, $p_2(\cdot) \in \mathcal{P}(\mathbb{R}^n)$.

(1) For any $f \in L^{p(\cdot)}(\mathbb{R}^n)$, $g \in L^{p'(\cdot)}(\mathbb{R}^n)$,

$$\int_{\mathbb{R}^n} |f(x)g(x)|dx \leq C_p \|f\|_{L^{p(\cdot)}(\mathbb{R}^n)} \|g\|_{L^{p'(\cdot)}(\mathbb{R}^n)},$$

where $C_p = 1 + 1/p_- - 1/p_+$.

(2) For any $f \in L^{p_1(\cdot)}(\mathbb{R}^n)$, $g \in L^{p_2(\cdot)}(\mathbb{R}^n)$, *when* $1/p(x) = 1/p_1(x) + 1/p_2(x)$, *we have*

$$\|f(x)g(x)\|_{L^{p(\cdot)}(\mathbb{R}^n)} \leq C_{p_1,p_2}\|f\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}\|g\|_{L^{p_2(\cdot)}(\mathbb{R}^n)},$$

where $C_{p_1,p_2} = (1 + 1/p_{1_-} - 1/p_{1_+})^{1/p_-}$.

Lemma 2.2 [10, 19] *If* $p_i(\cdot) \in \mathcal{B}(\mathbb{R}^n)$ $(i = 1, 2)$, *then there exist constants* $\delta_{i1}, \delta_{i2}, C > 0$, *such that for all balls* $B \subset \mathbb{R}^n$ *and all measurable subsets* $S \subset B$,

$$\frac{\|\chi_B\|_{L^{p_i(\cdot)}(\mathbb{R}^n)}}{\|\chi_S\|_{L^{p_i(\cdot)}(\mathbb{R}^n)}} \leq C\frac{|B|}{|S|}, \quad \frac{\|\chi_S\|_{L^{p_i'(\cdot)}(\mathbb{R}^n)}}{\|\chi_B\|_{L^{p_i'(\cdot)}(\mathbb{R}^n)}} \leq C\left(\frac{|S|}{|B|}\right)^{\delta_{i1}}, \quad \frac{\|\chi_S\|_{L^{p_i(\cdot)}(\mathbb{R}^n)}}{\|\chi_B\|_{L^{p_i(\cdot)}(\mathbb{R}^n)}} \leq C\left(\frac{|S|}{|B|}\right)^{\delta_{i2}}.$$

Lemma 2.3 [17] *If* $p(\cdot) \in \mathcal{B}(\mathbb{R}^n)$, *then there exists constant* $C > 0$, *such that for all balls* $B \subset \mathbb{R}^n$,

$$\frac{1}{|B|}\|\chi_B\|_{L^{p(\cdot)}(\mathbb{R}^n)}\|\chi_B\|_{L^{p'(\cdot)}(\mathbb{R}^n)} \leq C.$$

Lemma 2.4 *Let* $p(\cdot), q(\cdot) \in \mathcal{P}^0(\mathbb{R}^n)$. *If* $f \in L^{p(\cdot)q(\cdot)}(\mathbb{R}^n)$, *then*

$$\min\left(\|f\|^{q_+}_{L^{p(\cdot)q(\cdot)}},\ \|f\|^{q_-}_{L^{p(\cdot)q(\cdot)}}\right) \leq \left\||f|^{q(\cdot)}\right\|_{L^{p(\cdot)}} \leq \max\left(\|f\|^{q_+}_{L^{p(\cdot)q(\cdot)}},\ \|f\|^{q_-}_{L^{p(\cdot)q(\cdot)}}\right).$$

Proof Since $p(\cdot), q(\cdot) \in \mathcal{P}^0(\mathbb{R}^n)$, it is easy to see that $p(\cdot)q(\cdot) \in \mathcal{P}^0(\mathbb{R}^n)$. If $\|f\|_{L^{p(\cdot)q(\cdot)}} \leq 1$, then

$$\begin{aligned}\int_{\mathbb{R}^n}\left(\frac{|f(x)|^{q(x)}}{\|f\|^{q_+}_{L^{p(\cdot)q(\cdot)}}}\right)^{p(x)} dx &= \int_{\mathbb{R}^n}\left(\frac{|f(x)|}{\|f\|^{\frac{q_+}{q(x)}}_{L^{p(\cdot)q(\cdot)}}}\right)^{p(x)q(x)} dx \\ &\geq \int_{\mathbb{R}^n}\left(\frac{|f(x)|}{\|f\|_{L^{p(\cdot)q(\cdot)}}}\right)^{p(x)q(x)} dx = 1.\end{aligned}$$

On the other hand,

$$\begin{aligned}\int_{\mathbb{R}^n}\left(\frac{|f(x)|^{q(x)}}{\|f\|^{q_-}_{L^{p(\cdot)q(\cdot)}}}\right)^{p(x)} dx &= \int_{\mathbb{R}^n}\left(\frac{|f(x)|}{\|f\|^{\frac{q_-}{q(x)}}_{L^{p(\cdot)q(\cdot)}}}\right)^{p(x)q(x)} dx \\ &\leq \int_{\mathbb{R}^n}\left(\frac{|f(x)|}{\|f\|_{L^{p(\cdot)q(\cdot)}}}\right)^{p(x)q(x)} dx = 1.\end{aligned}$$

From these, it follows that

$$\|f\|^{q_+}_{L^{p(\cdot)q(\cdot)}} \leq \left\||f|^{q(\cdot)}\right\|_{L^{p(\cdot)}} \leq \|f\|^{q_-}_{L^{p(\cdot)q(\cdot)}}.$$

In a similar way, we can prove that if $\|f\|_{L^{p(\cdot)q(\cdot)}} > 1$, then

$$\|f\|^{q_-}_{L^{p(\cdot)q(\cdot)}} \leq \left\||f|^{q(\cdot)}\right\|_{L^{p(\cdot)}} \leq \|f\|^{q_+}_{L^{p(\cdot)q(\cdot)}}.$$

The proof of Lemma 2.4 is finished. □

Lemma 2.5 [17] *Let $b \in BMO(\mathbb{R}^n)$; m is a positive integer, and there exist constants $C > 0$, such that for any $k, j \in \mathbb{Z}$ with $k > j$,*

(1) $C^{-1}\|b\|_*^m \le \sup_B \frac{1}{\|\chi_B\|_{L^{p(\cdot)}(\mathbb{R}^n)}} \|(b - b_B)^m \chi_B\|_{L^{p(\cdot)}(\mathbb{R}^n)} \le C\|b\|_*^m$;

(2) $\|(b - b_{B_j})^m \chi_{B_k}\|_{L^{p(\cdot)}(\mathbb{R}^n)} \le C(k-j)^m \|b\|_*^m \|\chi_{B_k}\|_{L^{p(\cdot)}(\mathbb{R}^n)}$.

Lemma 2.6 [10] *If $p(\cdot)_1 \in \mathcal{B}(\mathbb{R}^n)$, then there exist constants $0 < \delta < 1$, $C > 0$, such that for all $Y \in \mathcal{Y}$, all nonnegative numbers t_Q and all $f \in L^1_{loc}(\mathbb{R}^n)$ with $f_Q \ne 0$ $(Q \in Y)$,*

$$\left\| \sum_{Q \in Y} t_Q \left| \frac{f}{f_Q} \right|^\delta \chi_Q \right\|_{L^{p(\cdot)}(\mathbb{R}^n)} \le C \left\| \sum_{Q \in Y} t_Q \chi_Q \right\|_{L^{p(\cdot)}(\mathbb{R}^n)},$$

where $\mathcal{Y}$ denotes all families of disjoint and open cube in $\mathbb{R}^n$.

Lemma 2.7 [8] *Let $f \in Lip_\beta(\mathbb{R}^n)$, $0 < \beta < 1$, $1 \le p < \infty$, $B_1 \subset B_2$. We have*

(1) $\|f\|_{Lip_\beta(\mathbb{R}^n)} \approx \sup_B \frac{1}{|B|^{\beta/n}} \left(\frac{1}{|B|} \int_B |f(x) - f_B|^p dx \right)^{1/p}$;

(2) $|f_{B_1} - f_{B_2}| \leqslant C\|f\|_{Lip_\beta(\mathbb{R}^n)} |B_2|^{\beta/n}$.

Lemma 2.8 *Let $b \in Lip_\beta(\mathbb{R}^n)$; m is a positive integer, and there exist constants $C > 0$, such that for any $k, j \in \mathbb{Z}$ with $k > j$,*

(1) $C^{-1}\|b\|^m_{Lip_\beta(\mathbb{R}^n)} \le |B|^{-m\beta/n} \|\chi_B\|^{-1}_{L^{p(\cdot)}(\mathbb{R}^n)} \|(b - b_B)^m \chi_B\|_{L^{p(\cdot)}(\mathbb{R}^n)} \le C\|b\|^m_{Lip_\beta(\mathbb{R}^n)}$;

(2) $\|(b - b_{B_j})^m \chi_{B_k}\|_{L^{p(\cdot)}(\mathbb{R}^n)} \le C|B_k|^{m\beta/n} \|b\|^m_{Lip_\beta(\mathbb{R}^n)} \|\chi_{B_k}\|_{L^{p(\cdot)}(\mathbb{R}^n)}$.

Proof For (1), applying Lemmas 2.1 and 2.3, we have, for any ball B,

$$\begin{aligned} &|B|^{-m\beta/n-1} \int_B |b(x) - b_B|^m dx \\ \le &C|B|^{-m\beta/n-1} \|(b - b_B)^m \chi_B\|_{L^{p(\cdot)}(\mathbb{R}^n)} \|\chi_B\|_{L^{p'(\cdot)}(\mathbb{R}^n)} \\ \le &C|B|^{-m\beta/n} \|\chi_B\|^{-1}_{L^{p(\cdot)}(\mathbb{R}^n)} \|(b - b_B)^m \chi_B\|_{L^{p(\cdot)}(\mathbb{R}^n)}. \end{aligned}$$

On the other hand, we can take a cube Q_B : $B \subset Q_B \subset \sqrt{n}B$.

By Lemma 2.6, there exists a constant $0 < \delta < 1$ independent of B such that for all $f \in L^1_{loc}(\mathbb{R}^n)$,

$$\||f|^\delta \chi_{Q_B}\|_{L^{p(\cdot)}(\mathbb{R}^n)} \le C|f_{Q_B}|^\delta \|\chi_{Q_B}\|_{L^{p(\cdot)}(\mathbb{R}^n)}.$$

Now we put $f(x) = (b(x) - b_B)^{m/\delta} \chi_B$. Noting $m/\delta > 1$, we have

$$|f_{Q_B}|^\delta = [|Q_B|^{-1} \int_B |b(x) - b_B|^{m/\delta} dx]^\delta \le C|B|^{m\beta/n} \|b\|^m_{Lip_\beta(\mathbb{R}^n)}.$$

It follows from Lemma 2.2 that

$$\|\chi_{Q_B}\|_{L^{p(\cdot)}(\mathbb{R}^n)} \le \frac{\|\chi_{\sqrt{n}B}\|_{L^{p(\cdot)}(\mathbb{R}^n)}}{\|\chi_B\|_{L^{p(\cdot)}(\mathbb{R}^n)}} \|\chi_B\|_{L^{p(\cdot)}(\mathbb{R}^n)} \le C \frac{|\sqrt{n}B|}{|B|} \|\chi_B\|_{L^{p(\cdot)}(\mathbb{R}^n)} \le C\|\chi_B\|_{L^{p(\cdot)}(\mathbb{R}^n)}.$$

Thus,

$$\|(b-b_B)^m\chi_B\|_{L^{p(\cdot)}(\mathbb{R}^n)}\le C|B|^{m\beta/n}\|b\|^m_{Lip_\beta(\mathbb{R}^n)}\|\chi_B\|_{L^{p(\cdot)}(\mathbb{R}^n)}.$$

Therefore, (1) holds.

Next, we consider (2). By Lemma 2.7 and the result (1) in this lemma, for all $k,j\in\mathbb{Z}$ with $k>j$, we have

$$\begin{aligned}&\|(b-b_{B_j})^m\chi_{B_k}\|_{L^{p(\cdot)}(\mathbb{R}^n)}\\ \le& C\|(b-b_{B_k})^m\chi_{B_k}\|_{L^{p(\cdot)}(\mathbb{R}^n)}+C\|(b_{B_k}-b_{B_j})^m\chi_{B_k}\|_{L^{p(\cdot)}(\mathbb{R}^n)}\\ \le& C|B_k|^{m\beta/n}\|b\|^m_{Lip_\beta(\mathbb{R}^n)}\|\chi_{B_k}\|_{L^{p(\cdot)}(\mathbb{R}^n)}.\end{aligned}$$

Therefore, (2) also holds. □

Lemma 2.9 [29] *Suppose that $p(\cdot)\in\mathcal{B}(\mathbb{R}^n)$, $\rho>n/2$, $\lambda>2$, and $\Omega\in L^2(S^{n-1})$ satisfying (1.1). Then there exists a constant $C>0$ independent of f such that*

$$\|S^\rho(f)\|_{L^{p(\cdot)}(\mathbb{R}^n)}\le C\|f\|_{L^{p(\cdot)}(\mathbb{R}^n)},\quad \|g_\lambda^{*,\ \rho}(f)\|_{L^{p(\cdot)}(\mathbb{R}^n)}\le C\|f\|_{L^{p(\cdot)}(\mathbb{R}^n)}.$$

Lemma 2.10 [29] *Let $b\in BMO(\mathbb{R}^n)$ and $m\in\mathbb{N}$. Suppose that $p(\cdot)\in\mathcal{B}(\mathbb{R}^n)$, $\rho>n/2$, $\lambda>2$, and $\Omega\in L^2(S^{n-1})$ satisfying (1.1). Then there exists a constant $C>0$ independent of f such that*

$$\|[b^m,S^\rho](f)\|_{L^{p(\cdot)}(\mathbb{R}^n)}\le C\|b\|_*^m\|f\|_{L^{p(\cdot)}(\mathbb{R}^n)},$$

$$\|[b^m,g_\lambda^{*,\ \rho}](f)\|_{L^{p(\cdot)}(\mathbb{R}^n)}\le C\|b\|_*^m\|f\|_{L^{p(\cdot)}(\mathbb{R}^n)}.$$

Lemma 2.11 [29] *Let $b\in Lip_\beta(\mathbb{R}^n)$, $m\in\mathbb{N}$, and $\Omega\in L^2(S^{n-1})$. Suppose that $\rho>n/2$, $\lambda>2$, $0<\beta<\min\{1,\ n/m\}$, and $p(\cdot)\in\mathcal{P}(\mathbb{R}^n)$ be such that $p_+<\frac{n}{m\beta}$. Define $q(\cdot)$ by $1/p(x)-1/q(x)=m\beta/n$. If $q(\cdot)(n-m\beta)/n\in\mathcal{B}(\mathbb{R}^n)$, then there exists a constant $C>0$ independent of f such that*

$$\|[b^m,S^\rho](f)\|_{L^{q(\cdot)}(\mathbb{R}^n)}\le C\|b\|^m_{Lip_\beta(\mathbb{R}^n)}\|f\|_{L^{p(\cdot)}(\mathbb{R}^n)},$$

$$\|[b^m,g_\lambda^{*,\ \rho}](f)\|_{L^{q(\cdot)}(\mathbb{R}^n)}\le C\|b\|^m_{Lip_\beta(\mathbb{R}^n)}\|f\|_{L^{p(\cdot)}(\mathbb{R}^n)}.$$

3. Proof of main theorems

Before starting the proofs of our results, we state a simple inequality as a remark that will be used in the following proofs first.

Remark 3.1 Let $k\in\mathbb{N}$, $a_k\ge0$, $1\le p_k<\infty$. Then

$$\sum_{k=0}^\infty a_k^{p_k}\le\left(\sum_{k=0}^\infty a_k\right)^{p_*},$$

where

$$p_*=\begin{cases}\min\limits_{k\in\mathbb{N}}p_k, & \sum_{k=0}^\infty a_k\le1,\\ \max\limits_{k\in\mathbb{N}}p_k, & \sum_{k=0}^\infty a_k>1.\end{cases}$$

It is easy to check that

$$S^{\rho}(f)(x) \le 2^{n\lambda} g_{\lambda}^{*,\ \rho}(f)(x), \quad [b^m, S^{\rho}](f)(x) \le 2^{n\lambda}[b^m, g_{\lambda}^{*,\ \rho}](f)(x), \quad \text{for} \quad m \in \mathbb{N}.$$

Therefore, it is enough to consider the operators $g_{\lambda}^{*,\ \rho}$ and $[b^m, g_{\lambda}^{*,\ \rho}]$ in the proofs of our results. That is to say, it suffices to prove Theorem 1.2, Theorem 1.4, and Theorem 1.6.

Proof of Theorem 1.2 We will just give the proof in the case of $\dot{K}_{p(\cdot)}^{\alpha,q(\cdot)}(\mathbb{R}^n)$. The same proof is also valid for the nonhomogeneous case.

Let $f \in \dot{K}_{p_1(\cdot)}^{\alpha,q_1(\cdot)}(\mathbb{R}^n)$. We write

$$f(x) = \sum_{j=-\infty}^{\infty} f(x)\chi_j(x) \triangleq \sum_{j=-\infty}^{\infty} f_j(x).$$

By the definition of the norm in $\dot{K}_{p(\cdot)}^{\alpha,q(\cdot)}(\mathbb{R}^n)$,

$$\|g_{\lambda}^{*,\ \rho}(f)\|_{\dot{K}_{p_1(\cdot)}^{\alpha,q_2(\cdot)}(\mathbb{R}^n)} = \inf\left\{\eta > 0 \ : \ \sum_{k=-\infty}^{\infty} \left\|\left(\frac{2^{k\alpha}|g_{\lambda}^{*,\ \rho}(f)\chi_k|}{\eta}\right)^{q_2(\cdot)}\right\|_{L^{\frac{p_1(\cdot)}{q_2(\cdot)}}} \le 1\right\}.$$

Since

$$\begin{aligned}
\left\|\left(\frac{2^{k\alpha}|g_{\lambda}^{*,\ \rho}(f)\chi_k|}{\eta}\right)^{q_2(\cdot)}\right\|_{L^{\frac{p_1(\cdot)}{q_2(\cdot)}}} &\le \left\|\left(\frac{2^{k\alpha}|\sum_{j=-\infty}^{\infty} g_{\lambda}^{*,\ \rho}(f_j)\chi_k|}{\eta_{11}+\eta_{12}+\eta_{13}}\right)^{q_2(\cdot)}\right\|_{L^{\frac{p_1(\cdot)}{q_2(\cdot)}}} \\
&\le C\left\|\left(\frac{2^{k\alpha}|\sum_{j=-\infty}^{k-3} g_{\lambda}^{*,\ \rho}(f_j)\chi_k|}{\eta_{11}}\right)^{q_2(\cdot)}\right\|_{L^{\frac{p_1(\cdot)}{q_2(\cdot)}}} \\
&\quad + C\left\|\left(\frac{2^{k\alpha}|\sum_{j=k-2}^{k+2} g_{\lambda}^{*,\ \rho}(f_j)\chi_k|}{\eta_{12}}\right)^{q_2(\cdot)}\right\|_{L^{\frac{p_1(\cdot)}{q_2(\cdot)}}} \\
&\quad + C\left\|\left(\frac{2^{k\alpha}|\sum_{j=k+2}^{\infty} g_{\lambda}^{*,\ \rho}(f_j)\chi_k|}{\eta_{13}}\right)^{q_2(\cdot)}\right\|_{L^{\frac{p_1(\cdot)}{q_2(\cdot)}}},
\end{aligned}$$

where

$$\eta_{11} = \left\|\left\{2^{k\alpha}|\sum_{j=-\infty}^{k-3} g_{\lambda}^{*,\ \rho}(f_j)\chi_k|\right\}_{k=-\infty}^{\infty}\right\|_{l^{q_2(\cdot)}(L^{p_1(\cdot)})},$$

$$\eta_{12} = \left\|\left\{2^{k\alpha}|\sum_{j=k-2}^{k+2} g_{\lambda}^{*,\ \rho}(f_j)\chi_k|\right\}_{k=-\infty}^{\infty}\right\|_{l^{q_2(\cdot)}(L^{p_1(\cdot)})},$$

$$\eta_{13} = \left\|\left\{2^{k\alpha}|\sum_{j=k+3}^{\infty} g_{\lambda}^{*,\ \rho}(f_j)\chi_k|\right\}_{k=-\infty}^{\infty}\right\|_{l^{q_2(\cdot)}(L^{p_1(\cdot)})}$$

and

$$\eta = \eta_{11} + \eta_{12} + \eta_{13}.$$

Thus,

$$\sum_{k=-\infty}^{\infty} \left\| \left(\frac{2^{k\alpha} |g_\lambda^{*,\ \rho}(f)\chi_k|}{\eta} \right)^{q_2(\cdot)} \right\|_{L^{\frac{p_1(\cdot)}{q_2(\cdot)}}} \leq C.$$

That is,

$$\|g_\lambda^{*,\ \rho}(f)\|_{\dot{K}^{\alpha, q_2(\cdot)}_{p_1(\cdot)}(\mathbb{R}^n)} \leq C\eta = C(\eta_{11} + \eta_{12} + \eta_{13}).$$

This implies that we only need to prove $\eta_{11},\ \eta_{12}\ ,\ \eta_{13} \leq C\|f\|_{\dot{K}^{\alpha, q_1(\cdot)}_{p_1(\cdot)}(\mathbb{R}^n)}$ Denote $\eta_0 = \|f\|_{\dot{K}^{\alpha, q_1(\cdot)}_{p_1(\cdot)}(\mathbb{R}^n)}$.

We consider η_{12} first. Applying Lemma 2.4, we obtain

$$\begin{aligned} &\sum_{k=-\infty}^{\infty} \left\| \left(\frac{2^{k\alpha} |\sum_{j=k-2}^{k+2} g_\lambda^{*,\ \rho}(f_j)\chi_k|}{\eta_0} \right)^{q_2(\cdot)} \right\|_{L^{\frac{p_1(\cdot)}{q_2(\cdot)}}} \\ \leq &\sum_{k=-\infty}^{\infty} \left\| \frac{2^{k\alpha} |\sum_{j=k-2}^{k+2} g_\lambda^{*,\ \rho}(f_j)\chi_k|}{\eta_0} \right\|_{L^{p_1(\cdot)}}^{(q_2^1)_k} \\ \leq &C \sum_{k=-\infty}^{\infty} \left(\sum_{j=k-2}^{k+2} \left\| \frac{2^{k\alpha} |g_\lambda^{*,\ \rho}(f_j)\chi_k|}{\eta_0} \right\|_{L^{p_1(\cdot)}} \right)^{(q_2^1)_k}, \end{aligned}$$

where

$$(q_2^1)_k = \begin{cases} (q_2)_-, & \left\| \left(\frac{2^{k\alpha} |\sum_{j=k-2}^{k+2} g_\lambda^{*,\ \rho}(f_j)\chi_k|}{\eta_0} \right)^{q_2(\cdot)} \right\|_{L^{\frac{p_1(\cdot)}{q_2(\cdot)}}} \leq 1, \\ (q_2)_+, & \left\| \left(\frac{2^{k\alpha} |\sum_{j=k-2}^{k+2} g_\lambda^{*,\ \rho}(f_j)\chi_k|}{\eta_0} \right)^{q_2(\cdot)} \right\|_{L^{\frac{p_1(\cdot)}{q_2(\cdot)}}} > 1. \end{cases}$$

From Lemma 2.8, it follows

$$\begin{aligned} &\sum_{k=-\infty}^{\infty} \left\| \left(\frac{2^{k\alpha} |\sum_{j=k-2}^{k+2} g_\lambda^{*,\ \rho}(f_j)\chi_k|}{\eta_0} \right)^{q_2(\cdot)} \right\|_{L^{\frac{p_1(\cdot)}{q_2(\cdot)}}} \\ \leq &C \sum_{k=-\infty}^{\infty} \left(\sum_{j=k-2}^{k+2} \left\| \frac{2^{k\alpha} |f_j|}{\eta_0} \right\|_{L^{p_1(\cdot)}} \right)^{(q_2^1)_k} \\ \leq &C \sum_{k=-\infty}^{\infty} \left\| \frac{2^{k\alpha} |f\chi_k|}{\eta_0} \right\|_{L^{p_1(\cdot)}}^{(q_2^1)_k}. \end{aligned}$$

Since $f\in \dot{K}^{\alpha,q_1(\cdot)}_{p_1(\cdot)}(\mathbb{R}^n)$, it is easy to see that $\left\|\frac{2^{k\alpha}|f\chi_k|}{\eta_0}\right\|_{L^{p_1(\cdot)}}\leq 1$ and $\sum\limits_{k=-\infty}^{\infty}\left\|\left(\frac{2^{k\alpha}|f\chi_k|}{\eta_0}\right)^{q_1(\cdot)}\right\|_{L^{\frac{p_1(\cdot)}{q_1(\cdot)}}}\leq 1$. Hence, by Lemma 2.4 and Remark 3.1, we see that

$$\begin{aligned}
&\sum_{k=-\infty}^{\infty}\left\|\left(\frac{2^{k\alpha}|\sum_{j=k-2}^{k+2}g_{\lambda}^{*,\ \rho}(f_j)\chi_k|}{\eta_0}\right)^{q_2(\cdot)}\right\|_{L^{\frac{p_1(\cdot)}{q_2(\cdot)}}}\\
\leq &C\sum_{k=-\infty}^{\infty}\left\|\left(\frac{2^{k\alpha}|f\chi_k|}{\eta_0}\right)^{q_1(\cdot)}\right\|_{L^{\frac{p_1(\cdot)}{q_1(\cdot)}}}^{\frac{(q_2^1)_k}{(q_1)_+}}\\
\leq &C\left\{\sum_{k=-\infty}^{\infty}\left\|\left(\frac{2^{k\alpha}|f\chi_k|}{\eta_0}\right)^{q_1(\cdot)}\right\|_{L^{\frac{p_1(\cdot)}{q_1(\cdot)}}}\right\}^{q_*}\leq C.
\end{aligned}$$

Here $(p_1)_+\leq(p_2)_-\leq(q_2^1)_k$ and $q_*=\min\limits_{k\in\mathbb{N}}\frac{(q_2^1)_k}{(q_1)_+}$.

Consequently, we have $\eta_{12}\leq C\eta_0\leq C\|f\|_{\dot{K}^{\alpha,q_1(\cdot)}_{p_1(\cdot)}(\mathbb{R}^n)}$.

Now let us turn to η_{11}. Using Minkowski's inequality, we have

$$\begin{aligned}
g_{\lambda}^{*,\ \rho}(f_j)(x)=&\left(\int_0^{\infty}\int_{\mathbb{R}^n}\left(\frac{t}{t+|x-y|}\right)^{\lambda n}\left|\frac{1}{t^{\rho}}\int_{|y-z|\leq t}\frac{\Omega(y-z)}{|y-z|^{n-\rho}}f_j(z)dz\right|^2\frac{dydt}{t^{n+1}}\right)^{1/2}\\
\leq &C\int_{\mathbb{R}^n}|f_j(z)|\left(\int_0^{\infty}\int_{|y-z|\leq t}\left(\frac{t}{t+|x-y|}\right)^{\lambda n}\frac{|\Omega(y-z)|^2}{|y-z|^{2n-2\rho}}\frac{dydt}{t^{2\rho+n+1}}\right)^{1/2}dz\\
\leq &C\int_{\mathbb{R}^n}|f_j(z)|\left(\int_0^{|x|}\int_{|y-z|\leq t}\left(\frac{t}{t+|x-y|}\right)^{\lambda n}\frac{|\Omega(y-z)|^2}{|y-z|^{2n-2\rho}}\frac{dydt}{t^{2\rho+n+1}}\right)^{1/2}dz\\
&+C\int_{\mathbb{R}^n}|f_j(z)|\left(\int_{|x|}^{\infty}\int_{|y-z|\leq t}\left(\frac{t}{t+|x-y|}\right)^{\lambda n}\frac{|\Omega(y-z)|^2}{|y-z|^{2n-2\rho}}\frac{dydt}{t^{2\rho+n+1}}\right)^{1/2}dz.
\end{aligned}$$

Let $x\in C_k,\ z\in C_j,\ j\leq k-3$; then we have $t+|y-z|\geq|x-y|+|y-z|\geq|x|-|z|\geq\frac{3}{4}|x|$. Hence, we see that

$$\begin{aligned}
&\int_0^{|x|}\int_{|y-z|\leq t}\left(\frac{t}{t+|x-y|}\right)^{\lambda n}\frac{|\Omega(y-z)|^2}{|y-z|^{2n-2\rho}}\frac{dydt}{t^{2\rho+n+1}}\\
\leq &C\int_0^{|x|}|x|^{-\lambda n}\int_{|y-z|\leq t}\frac{|\Omega(y-z)|^2}{|y-z|^{2n-2\rho}}\frac{dydt}{t^{2\rho+n-\lambda n+1}}\\
\leq &C\int_0^{|x|}|x|^{-\lambda n}\int_0^t r^{2\rho-n-1}dr\int_{S^{n-1}}|\Omega(y')|^2d\sigma(y')\frac{dt}{t^{2\rho+n-\lambda n+1}}\\
\leq &C\|\Omega\|^2_{L^2(S^{n-1})}\int_0^{|x|}|x|^{-\lambda n}t^{\lambda n-2n-1}dt\\
\leq &C|x|^{-2n}.
\end{aligned}$$

and

$$\begin{aligned}
&\int_{|x|}^{\infty}\int_{|y-z|\le t}\left(\frac{t}{t+|x-y|}\right)^{\lambda n}\frac{|\Omega(y-z)|^2}{|y-z|^{2n-2\rho}}\frac{dydt}{t^{2\rho+n+1}}\\
\le&\int_{|x|}^{\infty}\int_{|y-z|\le t}\frac{|\Omega(y-z)|^2}{|y-z|^{2n-2\rho}}\frac{dydt}{t^{2\rho+n+1}}\\
\le&\int_{|x|}^{\infty}\int_{0}^{t}r^{2\rho-n-1}dr\int_{S^{n-1}}|\Omega(y')|^2d\sigma(y')\frac{dt}{t^{2\rho+n+1}}\\
\le&C\|\Omega\|^2_{L^2(S^{n-1})}\int_{|x|}^{\infty}t^{-2n-1}dt\\
\le&C|x|^{-2n}.
\end{aligned}$$

Combined with the above estimates, we obtain

$$g_\lambda^{*,\ \rho}(f_j)(x)\le C|x|^{-n}\|f_j\|_{L^1(\mathbb{R}^n)}.$$

Thus, from Lemmas 2.1–2.4 and $\left\|\left(\frac{|2^{j\alpha}f\chi_j|}{\eta_0}\right)^{q_1(\cdot)}\right\|_{L^{p_1(\cdot)q_1(\cdot)}}\le 1$, it follows that

$$\begin{aligned}
&\sum_{k=-\infty}^{\infty}\left\|\left(\frac{2^{k\alpha}|\sum_{j=-\infty}^{k-3}g_\lambda^{*,\ \rho}(f_j)\chi_k|}{\eta_0}\right)^{q_2(\cdot)}\right\|_{L^{\frac{p_1(\cdot)}{q_2(\cdot)}}}\\
\le&C\sum_{k=-\infty}^{\infty}\left\|\left(\frac{2^{k\alpha}|\sum_{j=-\infty}^{k-3}|x|^{-n}\|f_j\|_{L^1(\mathbb{R}^n)}\chi_k|}{\eta_0}\right)^{q_2(\cdot)}\right\|_{L^{\frac{p_1(\cdot)}{q_2(\cdot)}}}\\
\le&C\sum_{k=-\infty}^{\infty}\left\|\frac{2^{k\alpha}|\sum_{j=-\infty}^{k-3}|x|^{-n}\|f_j\|_{L^1(\mathbb{R}^n)}\chi_k|}{\eta_0}\right\|_{L^{p_1(\cdot)}}^{(q_2^2)_k}\\
\le&C\sum_{k=-\infty}^{\infty}\left(2^{k(\alpha-n)}\sum_{j=-\infty}^{k-3}\left\|\frac{f_j}{\eta_0}\right\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}\|\chi_j\|_{L^{p_1'(\cdot)}}\|\chi_k\|_{L^{p_1(\cdot)}}\right)^{(q_2^2)_k}\\
\le&C\sum_{k=-\infty}^{\infty}\left(2^{k(\alpha-n)}\sum_{j=-\infty}^{k-3}\left\|\frac{f\chi_j}{\eta_0}\right\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}\frac{\|\chi_{B_j}\|_{L^{p_1'(\cdot)}(\mathbb{R}^n)}}{\|\chi_{B_k}\|_{L^{p_1'(\cdot)}(\mathbb{R}^n)}}|B_k|\right)^{(q_2^2)_k}\\
\le&C\sum_{k=-\infty}^{\infty}\left\{\sum_{j=-\infty}^{k-3}2^{(k-j)(\alpha-n\delta_{11})}\left\|\left(\frac{|2^{j\alpha}f\chi_j|}{\eta_0}\right)^{q_1(\cdot)}\right\|_{L^{p_1(\cdot)q_1(\cdot)}(\mathbb{R}^n)}^{\frac{1}{(q_1)_+}}\right\}^{(q_2^2)_k}.
\end{aligned}$$

where

$$(q_2^2)_k=\begin{cases}(q_2)_-, & \left\|\left(\frac{2^{k\alpha}|\sum_{j=-\infty}^{k-3}g_\lambda^{*,\ \rho}(f_j)\chi_k|}{\eta_0}\right)^{q_2(\cdot)}\right\|_{L^{\frac{p_1(\cdot)}{q_2(\cdot)}}}\le 1,\\ (q_2)_+, & \left\|\left(\frac{2^{k\alpha}|\sum_{j=-\infty}^{k-3}g_\lambda^{*,\ \rho}(f_j)\chi_k|}{\eta_0}\right)^{q_2(\cdot)}\right\|_{L^{\frac{p_1(\cdot)}{q_2(\cdot)}}}>1.\end{cases}$$

If $(q_1)_+ < 1$, then by Remark 3.1 and the fact $(p_1)_+ \le (p_2)_- \le (q_2^1)_k$, we have

$$\begin{aligned}
&\sum_{k=-\infty}^{\infty}\left\|\left(\frac{2^{k\alpha}|\sum_{j=-\infty}^{k-3} g_\lambda^{*,\ \rho}(f_j)\chi_k|}{\eta_0}\right)^{q_2(\cdot)}\right\|_{L^{\frac{p_1(\cdot)}{q_2(\cdot)}}} \\
\le & C\sum_{k=-\infty}^{\infty}\left\{\sum_{j=-\infty}^{k-3}2^{(k-j)(\alpha-n\delta_{11})}\left\|\left(\frac{|2^{j\alpha}f\chi_j|}{\eta_0}\right)^{q_1(\cdot)}\right\|_{L^{p_1(\cdot)q_1(\cdot)}(\mathbb{R}^n)}\right\}^{\frac{(q_2^2)_k}{(q_1)_+}} \\
\le & C\left\{\sum_{j=-\infty}^{\infty}\left\|\left(\frac{|2^{j\alpha}f\chi_j|}{\eta_0}\right)^{q_1(\cdot)}\right\|_{L^{p_1(\cdot)q_1(\cdot)}(\mathbb{R}^n)}\sum_{k=j+3}^{\infty}2^{(k-j)(\alpha-n\delta_{11})}\right\}^{q_*}\le C
\end{aligned}$$

where $q_* = \min\limits_{k\in\mathbb{N}} \frac{(q_2^2)_k}{(q_1)_+}$.

If $(q_1)_+ \ge 1$, then $(q_2^2)_k \ge (q_2)_- \ge (q_1)_+ \ge 1$. Thus, for $\alpha < n\delta_{11}$, it follows from Hölder's inequality and Remark 3.1 that

$$\begin{aligned}
&\sum_{k=-\infty}^{\infty}\left\|\left(\frac{2^{k\alpha}|\sum_{j=-\infty}^{k-3} g_\lambda^{*,\ \rho}(f_j)\chi_k|}{\eta_0}\right)^{q_2(\cdot)}\right\|_{L^{\frac{p_1(\cdot)}{q_2(\cdot)}}} \\
\le & C\sum_{k=-\infty}^{\infty}\left\{\sum_{j=-\infty}^{k-3}2^{(k-j)(\alpha-n\delta_{11})(q_1)_+/2}\left\|\left(\frac{|2^{j\alpha}f\chi_j|}{\eta_0}\right)^{q_1(\cdot)}\right\|_{L^{p_1(\cdot)q_1(\cdot)}(\mathbb{R}^n)}\right\}^{\frac{(q_2^2)_k}{(q_1)_+}} \\
&\times\left(\sum_{j=-\infty}^{k-3}2^{(k-j)(\alpha-n\delta_{11})((q_1)_+)'/2}\right)^{\frac{(q_2^2)_k}{((q_1)_+)'}} \\
\le & C\left\{\sum_{j=-\infty}^{\infty}\left\|\left(\frac{|2^{j\alpha}f\chi_j|}{\eta_0}\right)^{q_1(\cdot)}\right\|_{L^{p_1(\cdot)q_1(\cdot)}(\mathbb{R}^n)}\sum_{k=j+3}^{\infty}2^{(k-j)(\alpha-n\delta_{11})(q_1)_+/2}\right\}^{q_*}\le C
\end{aligned}$$

where $q_* = \min\limits_{k\in\mathbb{N}} \frac{(q_2^2)_k}{(q_1)_+}$.

This implies that

$$\eta_{11} \le C\eta_0 \le C\|f\|_{\dot{K}^{\alpha,q_1(\cdot)}_{p_1(\cdot)}(\mathbb{R}^n)}.$$

Finally, we estimate η_{13}. Applying Minkowski's inequality, we have

$$\begin{aligned}
g_\lambda^{*,\ \rho}(f_j)(x) =& \left(\int_0^\infty\int_{\mathbb{R}^n}\left(\frac{t}{t+|x-y|}\right)^{\lambda n}\left|\frac{1}{t^\rho}\int_{|y-z|\le t}\frac{\Omega(y-z)}{|y-z|^{n-\rho}}f_j(z)dz\right|^2\frac{dydt}{t^{n+1}}\right)^{1/2} \\
\le & C\int_{\mathbb{R}^n}|f_j(z)|\left(\int_0^\infty\int_{|y-z|\le t}\left(\frac{t}{t+|x-y|}\right)^{\lambda n}\frac{|\Omega(y-z)|^2}{|y-z|^{2n-2\rho}}\frac{dydt}{t^{2\rho+n+1}}\right)^{1/2}dz
\end{aligned}$$

$$\leq C\int_{\mathbb{R}^n}|f_j(z)|\left(\int_0^{2^j}\int_{|y-z|\leq t}\left(\frac{t}{t+|x-y|}\right)^{\lambda n}\frac{|\Omega(y-z)|^2}{|y-z|^{2n-2\rho}}\frac{dydt}{t^{2\rho+n+1}}\right)^{1/2}dz$$
$$+C\int_{\mathbb{R}^n}|f_j(z)|\left(\int_{2^j}^{\infty}\int_{|y-z|\leq t}\left(\frac{t}{t+|x-y|}\right)^{\lambda n}\frac{|\Omega(y-z)|^2}{|y-z|^{2n-2\rho}}\frac{dydt}{t^{2\rho+n+1}}\right)^{1/2}dz.$$

Let $x\in C_k,\ z\in C_j,\ j\geq k+3$; then we have $t+|y-z|\geq|x-y|+|y-z|\geq|z|-|x|\geq\frac{3}{4}|z|$. Hence, by a similar argument in the estimate η_{11}, we obtain

$$\begin{aligned}&\int_0^{2^j}\int_{|y-z|\leq t}\left(\frac{t}{t+|x-y|}\right)^{\lambda n}\frac{|\Omega(y-z)|^2}{|y-z|^{2n-2\rho}}\frac{dydt}{t^{2\rho+n+1}}\\ \leq& C\int_0^{2^j}|z|^{-\lambda n}\int_{|y-z|\leq t}\frac{|\Omega(y-z)|^2}{|y-z|^{2n-2\rho}}\frac{dydt}{t^{2\rho+n-\lambda n+1}}\\ \leq& C\|\Omega\|^2_{L^2(S^{n-1})}\int_0^{2^j}|z|^{-\lambda n}t^{\lambda n-2n-1}dt\\ \leq& C2^{-2jn}.\end{aligned}$$

and

$$\begin{aligned}&\int_{2^j}^{\infty}\int_{|y-z|\leq t}\left(\frac{t}{t+|x-y|}\right)^{\lambda n}\frac{|\Omega(y-z)|^2}{|y-z|^{2n-2\rho}}\frac{dydt}{t^{2\rho+n+1}}\\ \leq&\int_{2^j}^{\infty}\int_{|y-z|\leq t}\frac{|\Omega(y-z)|^2}{|y-z|^{2n-2\rho}}\frac{dydt}{t^{2\rho+n+1}}\\ \leq& C2^{-2jn}.\end{aligned}$$

Combined with the above estimates, we obtain

$$g_\lambda^{*,\ \rho}(f_j)(x)\leq C2^{-jn}\|f_j\|_{L^1(\mathbb{R}^n)}.$$

Thus, from Lemmas 2.1–2.4 and $\left\|\left(\frac{|2^{j\alpha}f\chi_j|}{\eta_0}\right)^{q_1(\cdot)}\right\|_{L^{p_1(\cdot)q_1(\cdot)}}\leq 1$, it follows that

$$\begin{aligned}&\sum_{k=-\infty}^{\infty}\left\|\left(\frac{2^{k\alpha}|\sum_{j=k+3}^{\infty}g_\lambda^{*,\ \rho}(f_j)\chi_k|}{\eta_0}\right)^{q_2(\cdot)}\right\|_{L^{\frac{p_1(\cdot)}{q_2(\cdot)}}}\\ \leq& C\sum_{k=-\infty}^{\infty}\left\|\left(\frac{2^{k\alpha}|\sum_{j=k+3}^{\infty}2^{-jn}\|f_j\|_{L^1(\mathbb{R}^n)}\chi_k|}{\eta_0}\right)^{q_2(\cdot)}\right\|_{L^{\frac{p_1(\cdot)}{q_2(\cdot)}}}\\ \leq& C\sum_{k=-\infty}^{\infty}\left(2^{k\alpha}\sum_{j=k+3}^{\infty}2^{-jn}\left\|\frac{f_j}{\eta_0}\right\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}\|\chi_j\|_{L^{p_1'(\cdot)}}\|\chi_k\|_{L^{p_1(\cdot)}}\right)^{(q_2^3)_k}\end{aligned}$$

$$\leq C\sum_{k=-\infty}^{\infty}\left(2^{k\alpha}\sum_{j=k+3}^{\infty}2^{-jn}\left\|\frac{f\chi_j}{\eta_0}\right\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}\frac{\|\chi_{B_k}\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}}{\|\chi_{B_j}\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}}|B_j|\right)^{(q_2^3)_k}$$

$$\leq C\sum_{k=-\infty}^{\infty}\left\{\sum_{j=k+3}^{\infty}2^{(k-j)(\alpha+n\delta_{12})}\left\|\left(\frac{|2^{j\alpha}f\chi_j|}{\eta_0}\right)^{q_1(\cdot)}\right\|_{L^{p_1(\cdot)q_1(\cdot)}(\mathbb{R}^n)}^{\frac{1}{(q_1)_+}}\right\}^{(q_2^3)_k},$$

where

$$(q_2^3)_k=\begin{cases}(q_2)_-, & \left\|\left(\frac{2^{k\alpha}|\sum_{j=k+3}^{\infty}g_\lambda^{*,\ \rho}(f_j)\chi_k|}{\eta_0}\right)^{q_2(\cdot)}\right\|_{L^{\frac{p_1(\cdot)}{q_2(\cdot)}}}\leq 1,\\ (q_2)_+, & \left\|\left(\frac{2^{k\alpha}|\sum_{j=k+3}^{\infty}g_\lambda^{*,\ \rho}(f_j)\chi_k|}{\eta_0}\right)^{q_2(\cdot)}\right\|_{L^{\frac{p_1(\cdot)}{q_2(\cdot)}}}>1.\end{cases}$$

Noticing that $(q_2)_-\geq(q_1)_+$ and $\alpha>-n\delta_{12}$, by a similar argument about η_{11}, we have

$$\eta_{13}\leq C\eta_0\leq C\|f\|_{\dot{K}_{p_1(\cdot)}^{\alpha,q_1(\cdot)}(\mathbb{R}^n)}.$$

This completes the proof of Theorem 1.2. □

Proof of Theorem 1.4 In this proof, we only prove the case of $\dot{K}_{p(\cdot)}^{\alpha,q(\cdot)}(\mathbb{R}^n)$ as before. The same proof is also valid for the nonhomogeneous case.

Let $b\in BMO(\mathbb{R}^n)$, $f\in\dot{K}_{p_1(\cdot)}^{\alpha,q_1(\cdot)}(\mathbb{R}^n)$. We write

$$f(x)=\sum_{j=-\infty}^{\infty}f(x)\chi_j(x)\triangleq\sum_{j=-\infty}^{\infty}f_j(x).$$

By the definition of the norm in $\dot{K}_{p(\cdot)}^{\alpha,q(\cdot)}(\mathbb{R}^n)$,

$$\|[b^m,\ g_\lambda^{*,\ \rho}](f)\|_{\dot{K}_{p_1(\cdot)}^{\alpha,q_2(\cdot)}(\mathbb{R}^n)}=\inf\left\{\eta>0\ :\ \sum_{k=-\infty}^{\infty}\left\|\left(\frac{2^{k\alpha}|[b^m,\ g_\lambda^{*,\ \rho}](f)\chi_k|}{\eta}\right)^{q_2(\cdot)}\right\|_{L^{\frac{p_1(\cdot)}{q_2(\cdot)}}}\leq 1\right\}.$$

Since

$$\begin{aligned}\left\|\left(\frac{2^{k\alpha}|[b^m,\ g_\lambda^{*,\ \rho}](f)\chi_k|}{\eta}\right)^{q_2(\cdot)}\right\|_{L^{\frac{p_1(\cdot)}{q_2(\cdot)}}}\leq &C\left\|\left(\frac{2^{k\alpha}|\sum_{j=-\infty}^{k-3}[b^m,\ g_\lambda^{*,\ \rho}](f_j)\chi_k|}{\eta_{21}}\right)^{q_2(\cdot)}\right\|_{L^{\frac{p_1(\cdot)}{q_2(\cdot)}}}\\ &+C\left\|\left(\frac{2^{k\alpha}|\sum_{j=k-2}^{k+2}[b^m,\ g_\lambda^{*,\ \rho}]\chi_k|}{\eta_{22}}\right)^{q_2(\cdot)}\right\|_{L^{\frac{p_1(\cdot)}{q_2(\cdot)}}}\\ &+C\left\|\left(\frac{2^{k\alpha}|\sum_{j=k+2}^{\infty}[b^m,\ g_\lambda^{*,\ \rho}](f_j)\chi_k|}{\eta_{23}}\right)^{q_2(\cdot)}\right\|_{L^{\frac{p_1(\cdot)}{q_2(\cdot)}}},\end{aligned}$$

here,

$$\eta_{21}=\left\|\left\{2^{k\alpha}|\sum_{j=-\infty}^{k-3}[b^m,\ g_\lambda^{*,\ \rho}](f_j)\chi_k|\right\}_{k=-\infty}^{\infty}\right\|_{l^{q_2(\cdot)}(L^{p_1(\cdot)})},$$

$$\eta_{22} = \left\| \left\{ 2^{k\alpha} \left| \sum_{j=k-2}^{k+2} [b^m,\ g_\lambda^{*,\ \rho}](f_j)\chi_k \right| \right\}_{k=-\infty}^{\infty} \right\|_{l^{q_2(\cdot)}(L^{p_1(\cdot)})},$$

$$\eta_{23} = \left\| \left\{ 2^{k\alpha} \left| \sum_{j=k+3}^{\infty} [b^m,\ g_\lambda^{*,\ \rho}](f_j)\chi_k \right| \right\}_{k=-\infty}^{\infty} \right\|_{l^{q_2(\cdot)}(L^{p_1(\cdot)})}$$

and

$$\eta = \eta_{21} + \eta_{22} + \eta_{23}.$$

A similar argument of the proof Theorem 1.2 yields

$$\|[b^m,\ g_\lambda^{*,\ \rho}](f)\|_{\dot{K}^{\alpha, q_2(\cdot)}_{p_1(\cdot)}(\mathbb{R}^n)} \leq C\eta = C(\eta_{21} + \eta_{22} + \eta_{23}).$$

Hence, it is enough to prove $\eta_{21},\ \eta_{22}\ ,\ \eta_{23} \leq C\|b\|_*^m \|f\|_{\dot{K}^{\alpha, q_1(\cdot)}_{p_1(\cdot)}(\mathbb{R}^n)}$ Denote $\eta_0 = \|f\|_{\dot{K}^{\alpha, q_1(\cdot)}_{p_1(\cdot)}(\mathbb{R}^n)}$.

We estimate η_{22} first. Noting that $[b^m, g_\lambda^{*,\ \rho}]$ is bounded on $L_1^p(\cdot)(\mathbb{R}^n)$ (Lemma 2.10), as argued about η_{12} in the proof of Theorem 1.2 , we immediately get

$$\sum_{k=-\infty}^{\infty} \left\| \left(\frac{2^{k\alpha} | \sum_{j=k-2}^{k+2} [b^m,\ g_\lambda^{*,\ \rho}](f_j)\chi_k |}{\eta_0 \|b\|_*^m} \right)^{q_2(\cdot)} \right\|_{L^{\frac{p_1(\cdot)}{q_2(\cdot)}}} \leq C.$$

That is $\eta_{22} \leq C\eta_0\|b\|_*^m \leq C\|b\|_*^m \|f\|_{\dot{K}^{\alpha, q_1(\cdot)}_{p_1(\cdot)}(\mathbb{R}^n)}$.

Now let us turn to η_{21}. Let $x \in C_k,\ j \leq k-3,\quad \text{supp}\ f_j \subset C_j$. By the estimation of $g_\lambda^{*,\ \rho}(f_j)$ in the proof of Theorem 1.2, we have

$$g_\lambda^{*,\ \rho}(f_j)(x) \leq C|x|^{-n} \|f_j\|_{L^1(\mathbb{R}^n)}.$$

From this, it follows

$$[b^m,\ g_\lambda^{*,\ \rho}](f_j)(x) = |g_\lambda^{*,\ \rho}[(b(x) - b)^m f_j](x)| \leq C|x|^{-n} \|(b(\cdot) - b)^m f_j\|_{L^1(\mathbb{R}^n)}.$$

Thus, using Lemma 2.4, we obtain

$$\begin{aligned}
&\sum_{k=-\infty}^{\infty} \left\| \left(\frac{2^{k\alpha} | \sum_{j=-\infty}^{k-3} [b^m,\ g_\lambda^{*,\ \rho}](f_j)\chi_k |}{\eta_0 \|b\|_*^m} \right)^{q_2(\cdot)} \right\|_{L^{\frac{p_1(\cdot)}{q_2(\cdot)}}} \\
\leq& C \sum_{k=-\infty}^{\infty} \left\| \left(\frac{2^{k\alpha} | \sum_{j=-\infty}^{k-3} |x|^{-n} \|(b(\cdot) - b)^m f_j\|_{L^1(\mathbb{R}^n)}\chi_k |}{\eta_0 \|b\|_*^m} \right)^{q_2(\cdot)} \right\|_{L^{\frac{p_1(\cdot)}{q_2(\cdot)}}} \\
\leq& C \sum_{k=-\infty}^{\infty} \left\| \frac{2^{k\alpha} | \sum_{j=-\infty}^{k-3} |x|^{-n} \|(b(\cdot) - b)^m f_j\|_{L^1(\mathbb{R}^n)}\chi_k |}{\eta_0 \|b\|_*^m} \right\|_{L^{p_1(\cdot)}}^{(q_2^2)_k}
\end{aligned}$$

$$
\leq C\sum_{k=-\infty}^{\infty}\left(2^{k(\alpha-n)}\sum_{j=-\infty}^{k-3}\left\|\frac{|(b-b_j)^m f_j|}{\eta_0\|b\|_*^m}\right\|_{L^1(\mathbb{R}^n)}\|\chi_k\|_{L^{p_1(\cdot)}}\right)^{(q_2^2)_k}
$$

$$
+C\sum_{k=-\infty}^{\infty}\left(2^{k(\alpha-n)}\sum_{j=-\infty}^{k-3}\left\|\frac{|f_j|}{\eta_0}\right\|_{L^1(\mathbb{R}^n)}\frac{1}{\|b\|_*^m}\|(b-b_j)^m\chi_{B_k}\|_{L^{p_1(\cdot)}}\right)^{(q_2^2)_k},
$$

here

$$
(q_2^2)_k=\begin{cases}(q_2)_-, & \left\|\left(\frac{2^{k\alpha}|\sum_{j=-\infty}^{k-3}[b^m,\ g_\lambda^{*,\ \rho}](f_j)\chi_k|}{\eta_0\|b\|_*^m}\right)^{q_2(\cdot)}\right\|_{L^{\frac{p_1(\cdot)}{q_2(\cdot)}}}\leq 1,\\ (q_2)_+, & \left\|\left(\frac{2^{k\alpha}|\sum_{-j=\infty}^{k-3}[b^m,\ g_\lambda^{*,\ \rho}]\chi_k|}{\eta_0\|b\|_*^m}\right)^{q_2(\cdot)}\right\|_{L^{\frac{p_1(\cdot)}{q_2(\cdot)}}}>1.\end{cases}
$$

Applying the generalized Hölder's inequality and Lemma 2.5, we know that

$$
\sum_{k=-\infty}^{\infty}\left\|\left(\frac{2^{k\alpha}|\sum_{j=-\infty}^{k-3}[b^m,\ g_\lambda^{*,\ \rho}](f_j)\chi_k|}{\eta_0\|b\|_*^m}\right)^{q_2(\cdot)}\right\|_{L^{\frac{p_1(\cdot)}{q_2(\cdot)}}}
$$

$$
\leq C\sum_{k=-\infty}^{\infty}\left(2^{k(\alpha-n)}\sum_{j=-\infty}^{k-3}\left\|\frac{|f_j|}{\eta_0}\right\|_{L^{p_1}(\mathbb{R}^n)}\frac{1}{\|b\|_*^m}\|(b-b_j)^m\chi_{B_j}\|_{L^{p_1'(\cdot)}}\|\chi_k\|_{L^{p_1(\cdot)}}\right)^{(q_2^2)_k}
$$

$$
+C\sum_{k=-\infty}^{\infty}\left(2^{k(\alpha-n)}\sum_{j=-\infty}^{k-3}\left\|\frac{|f_j|}{\eta_0}\right\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}\|\chi_j\|_{L^{p_1'(\cdot)}}(k-j)^m\|\chi_{B_k}\|_{L^{p_1(\cdot)}}\right)^{(q_2^2)_k}
$$

$$
\leq C\sum_{k=-\infty}^{\infty}\left(2^{k(\alpha-n)}\sum_{j=-\infty}^{k-3}(k-j)^m|B_k|\frac{\|\chi_{B_j}\|_{L^{p_1'(\cdot)}}}{\|\chi_{B_k}\|_{L^{p_1'(\cdot)}}}\left\|\frac{|f_j|}{\eta_0}\right\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}\right)^{(q_2^2)_k}.
$$

Furthermore, by the same argument as η_{11} in the proof of Theorem 1.2, we have

$$
\sum_{k=-\infty}^{\infty}\left\|\left(\frac{2^{k\alpha}|\sum_{j=-\infty}^{k-3}[b^m,\ g_\lambda^{*,\ \rho}](f_j)\chi_k|}{\eta_0\|b\|_*^m}\right)^{q_2(\cdot)}\right\|_{L^{\frac{p_1(\cdot)}{q_2(\cdot)}}}
$$

$$
\leq C\sum_{k=-\infty}^{\infty}\left(\sum_{j=-\infty}^{k-3}(k-j)^m 2^{(k-j)(\alpha-n\delta_{11})}\left\|\left(\frac{|f\chi_j|}{\eta_0}\right)^{q_1(\cdot)}\right\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}^{\frac{1}{(q_1)_+}}\right)^{(q_2^2)_k}
$$

$$
\leq C\begin{cases}C\left\{\sum_{j=-\infty}^{\infty}\left\|\left(\frac{|2^{j\alpha}f\chi_j|}{\eta_0}\right)^{q_1(\cdot)}\right\|_{L^{p_1(\cdot)q_1(\cdot)}(\mathbb{R}^n)}\sum_{k=j+3}^{\infty}(k-j)^m 2^{(k-j)(\alpha-n\delta_{11})}\right\}^{q_*}, & (p_1)_+\leq 1,\\ C\left\{\sum_{j=-\infty}^{\infty}\left\|\left(\frac{|2^{j\alpha}f\chi_j|}{\eta_0}\right)^{q_1(\cdot)}\right\|_{L^{p_1(\cdot)q_1(\cdot)}(\mathbb{R}^n)}\sum_{k=j+3}^{\infty}(k-j)^m 2^{(k-j)(\alpha-n\delta_{11})(q_1)_+/2}\right\}^{q_*}, & (p_1)_+>1.\end{cases}
$$

$$
\leq C.
$$

where $q_* = \min\limits_{k\in\mathbb{N}} \frac{(q_2^2)_k}{(q_1)_+}$.

This implies that

$$\eta_{21} \le C\eta_0 \|b\|_*^m \le C\|b\|_*^m \|f\|_{\dot{K}_{p_1(\cdot)}^{\alpha, q_1(\cdot)}(\mathbb{R}^n)}.$$

Finally, we estimate η_{23}. Let $x \in C_k,\ j \ge k+3,\ \ \mathrm{supp}\ f_j \subset C_j$. By the estimation of $g_\lambda^{*,\ \rho}(f_j)$ in the proof of Theorem 1.2, we have

$$g_\lambda^{*,\ \rho}(f_j)(x) \le C2^{-jn}\|f_j\|_{L^1(\mathbb{R}^n)}.$$

From this, it follows

$$[b^m,\ g_\lambda^{*,\ \rho}](f_j)(x) = |g_\lambda^{*,\ \rho}[(b(x)-b)^m f_j](x)| \le C2^{-jn}\|(b(\cdot)-b)^m f_j\|_{L^1(\mathbb{R}^n)}.$$

Thus, when $\alpha > -n\delta_{12}$, as the similar way to estimate η_{21} before, we see that

$$\begin{aligned}
&\sum_{k=-\infty}^{\infty} \left\| \left(\frac{2^{k\alpha} |\sum_{j=k+3}^{\infty} [b^m,\ g_\lambda^{*,\ \rho}](f_j)\chi_k|}{\eta_0\|b\|_*^m} \right)^{q_2(\cdot)} \right\|_{L^{\frac{p_1(\cdot)}{q_2(\cdot)}}} \\
\le& C\sum_{k=-\infty}^{\infty} \left\| \frac{2^{k\alpha} |\sum_{j=k+3}^{\infty} 2^{-jn}\|(b(\cdot)-b)^m f_j\|_{L^1(\mathbb{R}^n)}\chi_k|}{\eta_0\|b\|_*^m} \right\|_{L^{p_1(\cdot)}}^{(q_2^3)_k} \\
\le& C\sum_{k=-\infty}^{\infty} \left(2^{k\alpha} \sum_{j=k+3}^{\infty} 2^{jn} \left\| \frac{|(b-b_k)^m f_j|}{\eta_0\|b\|_*^m} \right\|_{L^1(\mathbb{R}^n)} \|\chi_k\|_{L^{p_1(\cdot)}} \right)^{(q_2^3)_k} \\
&+ C\sum_{k=-\infty}^{\infty} \left(2^{k\alpha} \sum_{j=k+3}^{\infty} 2^{jn} \left\| \frac{|f_j|}{\eta_0} \right\|_{L^1(\mathbb{R}^n)} \frac{\|(b-b_k)^m \chi_{B_k}\|_{L^{p_1(\cdot)}}}{\|b\|_*^m} \right)^{(q_2^3)_k} \\
\le& C\sum_{k=-\infty}^{\infty} \left(2^{k\alpha} \sum_{j=k+3}^{\infty} 2^{jn}(k-j)^m |B_j| \frac{\|\chi_{B_k}\|_{L^{p_1(\cdot)}}}{\|\chi_{B_j}\|_{L^{p_1(\cdot)}}} \left\| \frac{|f_j|}{\eta_0} \right\|_{L^{p_1(\cdot)}(\mathbb{R}^n)} \right)^{(q_2^3)_k} \\
\le& C\sum_{k=-\infty}^{\infty} \left(\sum_{j=k+3}^{\infty} (j-k)^m 2^{(k-j)(\alpha+n\delta_{12})} \left\| \left(\frac{|f\chi_j|}{\eta_0} \right)^{q_1(\cdot)} \right\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}^{\frac{1}{(q_1)_+}} \right)^{(q_2^3)_k} \le C,
\end{aligned}$$

here

$$(q_2^3)_k = \begin{cases} (q_2)_-, & \left\| \left(\frac{2^{k\alpha} |\sum_{j=k+3}^{\infty} [b^m,\ g_\lambda^{*,\ \rho}](f_j)\chi_k|}{\eta_0\|b\|_*^m} \right)^{q_2(\cdot)} \right\|_{L^{\frac{p_1(\cdot)}{q_2(\cdot)}}} \le 1, \\ (q_2)_+, & \left\| \left(\frac{2^{k\alpha} |\sum_{j=k+3}^{\infty} [b^m,\ g_\lambda^{*,\ \rho}]\chi_k|}{\eta_0\|b\|_*^m} \right)^{q_2(\cdot)} \right\|_{L^{\frac{p_1(\cdot)}{q_2(\cdot)}}} > 1. \end{cases}$$

Hence,

$$\eta_{23} \le C\eta_0 \|b\|_*^m \le C\|b\|_*^m \|f\|_{\dot{K}_{p_1(\cdot)}^{\alpha, q_1(\cdot)}(\mathbb{R}^n)}.$$

This finishes the proof of Theorem 1.4. □

Proof of Theorem 1.6 In this proof, we only prove the case of $\dot{K}_{p(\cdot)}^{\alpha,q(\cdot)}(\mathbb{R}^n)$ as before. The same proof is also valid for the nonhomogeneous case.

Let $b \in Lip_\beta(\mathbb{R}^n),\ 0<\beta<1,\ f \in \dot{K}_{p_1(\cdot)}^{\alpha,q_1(\cdot)}(\mathbb{R}^n)$. We write

$$f(x) = \sum_{j=-\infty}^{\infty} f(x)\chi_j(x) \triangleq \sum_{j=-\infty}^{\infty} f_j(x).$$

By the definition of the norm in $\dot{K}_{p(\cdot)}^{\alpha,q(\cdot)}(\mathbb{R}^n)$,

$$\|[b^m,\ g_\lambda^{*,\ \rho}](f)\|_{\dot{K}_{p_2(\cdot)}^{\alpha,q_2(\cdot)}(\mathbb{R}^n)} = \inf\left\{\eta>0\ :\ \sum_{k=-\infty}^{\infty}\left\|\left(\frac{2^{k\alpha}|[b^m,\ g_\lambda^{*,\ \rho}](f)\chi_k|}{\eta}\right)^{q_2(\cdot)}\right\|_{L^{\frac{p_2(\cdot)}{q_2(\cdot)}}} \le 1\right\}.$$

Since

$$\begin{aligned}\left\|\left(\frac{2^{k\alpha}|[b^m,\ g_\lambda^{*,\ \rho}](f)\chi_k|}{\eta}\right)^{q_2(\cdot)}\right\|_{L^{\frac{p_2(\cdot)}{q_2(\cdot)}}} \le& C\left\|\left(\frac{2^{k\alpha}|\sum_{j=-\infty}^{k-3}[b^m,\ g_\lambda^{*,\ \rho}](f_j)\chi_k|}{\eta_{31}}\right)^{q_2(\cdot)}\right\|_{L^{\frac{p_2(\cdot)}{q_2(\cdot)}}}\\ &+C\left\|\left(\frac{2^{k\alpha}|\sum_{j=k-2}^{k+2}[b^m,\ g_\lambda^{*,\ \rho}]\chi_k|}{\eta_{32}}\right)^{q_2(\cdot)}\right\|_{L^{\frac{p_2(\cdot)}{q_2(\cdot)}}}\\ &+C\left\|\left(\frac{2^{k\alpha}|\sum_{j=k+2}^{\infty}[b^m,\ g_\lambda^{*,\ \rho}](f_j)\chi_k|}{\eta_{33}}\right)^{q_2(\cdot)}\right\|_{L^{\frac{p_2(\cdot)}{q_2(\cdot)}}},\end{aligned}$$

here,

$$\eta_{31} = \left\|\left\{2^{k\alpha}|\sum_{j=-\infty}^{k-3}[b^m,\ g_\lambda^{*,\ \rho}](f_j)\chi_k|\right\}_{k=-\infty}^{\infty}\right\|_{l^{q_2(\cdot)}(L^{p_2(\cdot)})},$$

$$\eta_{32} = \left\|\left\{2^{k\alpha}|\sum_{j=k-2}^{k+2}[b^m,\ g_\lambda^{*,\ \rho}](f_j)\chi_k|\right\}_{k=-\infty}^{\infty}\right\|_{l^{q_2(\cdot)}(L^{p_2(\cdot)})},$$

$$\eta_{33} = \left\|\left\{2^{k\alpha}|\sum_{j=k+3}^{\infty}[b^m,\ g_\lambda^{*,\ \rho}](f_j)\chi_k|\right\}_{k=-\infty}^{\infty}\right\|_{l^{q_2(\cdot)}(L^{p_2(\cdot)})}$$

and

$$\eta = \eta_{31}+\eta_{32}+\eta_{33}.$$

A similar argument of the proof Theorem 1.2 yields

$$\|[b^m,\ g_\lambda^{*,\ \rho}](f)\|_{\dot{K}_{p_2(\cdot)}^{\alpha,q_2(\cdot)}(\mathbb{R}^n)} \le C\eta = C(\eta_{31}+\eta_{32}+\eta_{33}).$$

We are now going to estimate $\eta_{31},\ \eta_{32}$, and η_{33}. Denote $\eta_0 = \|f\|_{\dot{K}_{p_1(\cdot)}^{\alpha,q_1(\cdot)}(\mathbb{R}^n)}$.

We estimate η_{32} first. Noting that $[b^m, g_\lambda^{*,\ \rho}]$ is bounded from $L_1^p(\cdot)(\mathbb{R}^n)$ to $L_2^p(\cdot)(\mathbb{R}^n)$ (Lemma 2.11), as argued about η_{12} in the proof of Theorem 1.2 , we immediately get

$$\sum_{k=-\infty}^{\infty}\left\|\left(\frac{2^{k\alpha}|\sum_{j=k-2}^{k+2}[b^m,\ g_\lambda^{*,\ \rho}](f_j)\chi_k|}{\eta_0\|b\|^m_{Lip_\beta(\mathbb{R}^n)}}\right)^{q_2(\cdot)}\right\|_{L^{\frac{p_2(\cdot)}{q_2(\cdot)}}} \le C.$$

That is $\eta_{32} \le C\eta_0\|b\|^m_{Lip_\beta(\mathbb{R}^n)} \le C\|b\|^m_{Lip_\beta(\mathbb{R}^n)}\|f\|_{\dot{K}^{\alpha,q_1(\cdot)}_{p_1(\cdot)}(\mathbb{R}^n)}$.

Now let us turn to η_{31}. Let $x\in C_k,\ j\le k-3,$ supp $f_j\subset C_j$. By the estimation of η_{21} in the proof of Theorem 1.4 and the generalized Hölder's inequality, we have

$$\begin{aligned}
&\sum_{k=-\infty}^{\infty}\left\|\left(\frac{2^{k\alpha}|\sum_{j=-\infty}^{k-3}[b^m,\ g_\lambda^{*,\ \rho}](f_j)\chi_k|}{\eta_0\|b\|^m_{Lip_\beta(\mathbb{R}^n)}}\right)^{q_2(\cdot)}\right\|_{L^{\frac{p_2(\cdot)}{q_2(\cdot)}}}\\
\le& C\sum_{k=-\infty}^{\infty}\left(2^{k(\alpha-n)}\sum_{j=-\infty}^{k-3}\left\|\frac{|(b-b_j)^m f_j|}{\eta_0\|b\|^m_{Lip_\beta(\mathbb{R}^n)}}\right\|_{L^1(\mathbb{R}^n)}\|\chi_k\|_{L^{p_2(\cdot)}}\right)^{(q_2^2)_k}\\
&+C\sum_{k=-\infty}^{\infty}\left(2^{k(\alpha-n)}\sum_{j=-\infty}^{k-3}\left\|\frac{|f_j|}{\eta_0}\right\|_{L^1(\mathbb{R}^n)}\frac{\|(b-b_j)^m\chi_{B_k}\|_{L^{p_2(\cdot)}}}{\|b\|^m_{Lip_\beta(\mathbb{R}^n)}}\right)^{(q_2^2)_k}\\
\le& C\sum_{k=-\infty}^{\infty}\left(2^{k(\alpha-n)}\sum_{j=-\infty}^{k-3}\left\|\frac{|f_j|}{\eta_0}\right\|_{L^{p_1}(\mathbb{R}^n)}\frac{1}{\|b\|^m_{Lip_\beta(\mathbb{R}^n)}}\|(b-b_j)^m\chi_{B_j}\|_{L^{p_1'(\cdot)}}\|\chi_k\|_{L^{p_2(\cdot)}}\right)^{(q_2^2)_k}\\
&+C\sum_{k=-\infty}^{\infty}\left(2^{k(\alpha-n)}\sum_{j=-\infty}^{k-3}\left\|\frac{|f_j|}{\eta_0}\right\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}\|\chi_{B_j}\|_{L^{p_1'(\cdot)}}\frac{\|(b-b_j)^m\chi_{B_k}\|_{L^{p_2(\cdot)}}}{\|b\|^m_{Lip_\beta(\mathbb{R}^n)}}\right)^{(q_2^2)_k},
\end{aligned}$$

here

$$(q_2^2)_k=\begin{cases}(q_2)_-, & \left\|\left(\frac{2^{k\alpha}|\sum_{j=-\infty}^{k-3}[b^m,\ g_\lambda^{*,\ \rho}](f_j)\chi_k|}{\eta_0\|b\|^m_{Lip_\beta(\mathbb{R}^n)}}\right)^{q_2(\cdot)}\right\|_{L^{\frac{p_1(\cdot)}{q_2(\cdot)}}}\le 1,\\ (q_2)_+, & \left\|\left(\frac{2^{k\alpha}|\sum_{-j=\infty}^{k-3}[b^m,\ g_\lambda^{*,\ \rho}]\chi_k|}{\eta_0\|b\|^m_{Lip_\beta(\mathbb{R}^n)}}\right)^{q_2(\cdot)}\right\|_{L^{\frac{p_1(\cdot)}{q_2(\cdot)}}}>1.\end{cases}$$

Noting the fact that if $1/p_1(x)-1/p_2(x)=m\beta/n$, then (see [13])

$$C_1|B|^{m\beta/n}\|\chi_B\|_{L^{p_2(\cdot)}(\mathbb{R}^n)}\le\|\chi_B\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}\le C_2|B|^{m\beta/n}\|\chi_B\|_{L^{p_2(\cdot)}(\mathbb{R}^n)}.$$

Therefore, together this and applying Lemmas 2.2–2.4 as well as Lemma 2.8, we know that

$$\begin{aligned}
&\sum_{k=-\infty}^{\infty}\left\|\left(\frac{2^{k\alpha}|\sum_{j=-\infty}^{k-3}[b^m,\ g_\lambda^{*,\ \rho}](f_j)\chi_k|}{\eta_0\|b\|^m_{Lip_\beta(\mathbb{R}^n)}}\right)^{q_2(\cdot)}\right\|_{L^{\frac{p_2(\cdot)}{q_2(\cdot)}}}\\
\leq& C\sum_{k=-\infty}^{\infty}\left(2^{k(\alpha-n)}\sum_{j=-\infty}^{k-3}\left\|\frac{|f_j|}{\eta_0}\right\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}\frac{|B_j|^{m\beta/n}}{|B_k|^{m\beta/n}}\|\chi_{B_j}\|_{L^{p_1'(\cdot)}}\|\chi_{B_k}\|_{L^{p_1(\cdot)}}\right)^{(q_2^2)_k}\\
&+C\sum_{k=-\infty}^{\infty}\left(2^{k(\alpha-n)}\sum_{j=-\infty}^{k-3}\left\|\frac{|f_j|}{\eta_0}\right\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}\|\chi_{B_j}\|_{L^{p_1'(\cdot)}}\|\chi_{B_k}\|_{L^{p_1(\cdot)}}\right)^{(q_2^2)_k}\\
\leq& C\sum_{k=-\infty}^{\infty}\left(2^{k(\alpha-n)}\sum_{j=-\infty}^{k-3}|B_k|\frac{\|\chi_{B_j}\|_{L^{p_1'(\cdot)}}}{\|\chi_{B_k}\|_{L^{p_1'(\cdot)}}}\left\|\frac{|f_j|}{\eta_0}\right\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}\right)^{(q_2^2)_k}\\
\leq& C\sum_{k=-\infty}^{\infty}\left(\sum_{j=-\infty}^{k-3}2^{(k-j)(\alpha-n\delta_{11})}\left\|\left(\frac{|f\chi_j|}{\eta_0}\right)^{q_1(\cdot)}\right\|^{\frac{1}{(q_1)_+}}_{L^{p_1(\cdot)}(\mathbb{R}^n)}\right)^{(q_2^2)_k}.
\end{aligned}$$

Furthermore, by the same argument as η_{11} in the proof of Theorem 1.2, we immediately get

$$\sum_{k=-\infty}^{\infty}\left\|\left(\frac{2^{k\alpha}|\sum_{j=-\infty}^{k-3}[b^m,\ g_\lambda^{*,\ \rho}](f_j)\chi_k|}{\eta_0\|b\|^m_{Lip_\beta(\mathbb{R}^n)}}\right)^{q_2(\cdot)}\right\|_{L^{\frac{p_1(\cdot)}{q_2(\cdot)}}}\leq C.$$

This implies that

$$\eta_{31}\leq C\eta_0\|b\|^m_{Lip_\beta(\mathbb{R}^n)}\leq C\|b\|^m_{Lip_\beta(\mathbb{R}^n)}\|f\|_{\dot{K}^{\alpha,q_1(\cdot)}_{p_1(\cdot)}(\mathbb{R}^n)}.$$

Finally, we estimate η_{33}. Let $x\in C_k,\ j\geq k+3,\ \ \operatorname{supp} f_j\subset C_j$. By the estimation of η_{23} in the proof of Theorem 1.4 and the generalized Hölder's inequality, we have

$$\begin{aligned}
&\sum_{k=-\infty}^{\infty}\left\|\left(\frac{2^{k\alpha}|\sum_{j=k+3}^{\infty}[b^m,\ g_\lambda^{*,\ \rho}](f_j)\chi_k|}{\eta_0\|b\|^m_{Lip_\beta(\mathbb{R}^n)}}\right)^{q_2(\cdot)}\right\|_{L^{\frac{p_2(\cdot)}{q_2(\cdot)}}}\\
\leq& C\sum_{k=-\infty}^{\infty}\left(2^{k\alpha}\sum_{j=k+3}^{\infty}2^{-jn}\left\|\frac{|(b-b_k)^m f_j|}{\eta_0\|b\|^m_{Lip_\beta(\mathbb{R}^n)}}\right\|_{L^1(\mathbb{R}^n)}\|\chi_k\|_{L^{p_2(\cdot)}}\right)^{(q_2^3)_k}\\
&+C\sum_{k=-\infty}^{\infty}\left(2^{k(\alpha-n)}\sum_{j=k+3}^{\infty}2^{-jn}\left\|\frac{|f_j|}{\eta_0}\right\|_{L^1(\mathbb{R}^n)}\frac{\|(b-b_k)^m\chi_{B_k}\|_{L^{p_2(\cdot)}}}{\|b\|^m_{Lip_\beta(\mathbb{R}^n)}}\right)^{(q_2^3)_k},
\end{aligned}$$

here

$$(q_2^3)_k=\begin{cases}(q_2)_-, & \left\|\left(\frac{2^{k\alpha}|\sum_{j=k+3}^{\infty}[b^m,\ g_\lambda^{*,\ \rho}](f_j)\chi_k|}{\eta_0}\right)^{q_2(\cdot)}\right\|_{L^{\frac{p_1(\cdot)}{q_2(\cdot)}}}\leq 1,\\ (q_2)_+, & \left\|\left(\frac{2^{k\alpha}|\sum_{j=k+3}^{\infty}[b^m,\ g_\lambda^{*,\ \rho}]\chi_k|}{\eta_0}\right)^{q_2(\cdot)}\right\|_{L^{\frac{p_1(\cdot)}{q_2(\cdot)}}}>1.\end{cases}$$

Observe that $1/p_1(x) - 1/p_2(x) = m\beta/n$ implies $1/p_2'(x) - 1/p_1'(x) = m\beta/n$. Hence, when $\alpha > -n\delta_{22}$, as argued about η_{31} before, we obtain

$$\sum_{k=-\infty}^{\infty} \left\| \left(\frac{2^{k\alpha} |\sum_{j=k+3}^{\infty} [b^m,\ g_\lambda^{*,\ \rho}](f_j)\chi_k|}{\eta_0 \|b\|_{Lip_\beta(\mathbb{R}^n)}^m} \right)^{q_2(\cdot)} \right\|_{L^{\frac{p_2(\cdot)}{q_2(\cdot)}}}$$

$$\leq C \sum_{k=-\infty}^{\infty} \left(2^{k(\alpha-n)} \sum_{j=k+3}^{\infty} 2^{-jn} |B_j| \frac{\|\chi_{B_k}\|_{L^{p_2(\cdot)}}}{\|\chi_{B_j}\|_{L^{p_2(\cdot)}}} \left\| \frac{|f_j|}{\eta_0} \right\|_{L^{p_1(\cdot)}(\mathbb{R}^n)} \right)^{(q_2^3)_k}$$

$$\leq C \sum_{k=-\infty}^{\infty} \left(\sum_{j=k+3}^{\infty} 2^{(k-j)(\alpha+n\delta_{22})} \left\| \left(\frac{|f_{\chi_j}|}{\eta_0} \right)^{q_1(\cdot)} \right\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}^{\frac{1}{(q_1)_+}} \right)^{(q_2^3)_k} \leq C.$$

and

$$\eta_{33} \leq C\eta_0 \|b\|_{Lip_\beta(\mathbb{R}^n)}^m \leq C \|b\|_{Lip_\beta(\mathbb{R}^n)}^m \|f\|_{\dot{K}_{p_1(\cdot)}^{\alpha, q_1(\cdot)}(\mathbb{R}^n)}.$$

Summing up the estimations of η_{31}, η_{32}, and η_{33}, it follows

$$\|[b^m,\ g_\lambda^{*,\ \rho}](f)\|_{\dot{K}_{p_2(\cdot)}^{\alpha, q_2(\cdot)}(\mathbb{R}^n)} \leq C \|b\|_{Lip_\beta(\mathbb{R}^n)}^m \|f\|_{\dot{K}_{p_1(\cdot)}^{\alpha, q_1(\cdot)}(\mathbb{R}^n)}.$$

This accomplishes the proof of Theorem 1.6. □

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