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Gorenstein homological dimensions of modules over triangular matrix rings

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Abstract: Let A and B be rings, U a (B,A)-bimodule, and $T = \left(\begin{smallmatrix} A & 0 \\ U & B \end{smallmatrix}\right)$ the triangular matrix ring. In this paper, we characterize the Gorenstein homological dimensions of modules over T, and discuss when a left T-module is a strongly Gorenstein projective or strongly Gorenstein injective module.

Key words: Triangular matrix ring, Gorenstein regular ring, Gorenstein homological dimension

1. Introduction and preliminaries

Triangular matrix rings have been studied by many authors (e.g., see [15-17] and their references). Such rings play an important role in the study of the representation theory of Artin rings and algebras. The modules (left or right) over such rings can be described in a very concrete fashion and we have nice descriptions of some important classes of modules over such rings. Krylov and Tuganbaev [19] presented general properties of matrix rings and injective, projective, flat, and hereditary modules over such rings. Let A and B be rings and U a (B,A)-bimodule. We denote by T the triangular matrix ring $\left(\begin{smallmatrix} A & 0 \\ U & B \end{smallmatrix}\right)$. Using the description of T-modules, Asadollahi and Salarian [1] studied the vanishing of the extension functor Ext over T and explicitly described the structure of (right) T-modules of finite projective (resp. injective) dimension. Enochs and Torrecillas [5] described flat covers and cotorsion envelopes of modules over T.

In the 1990s Enochs, Jenda, and Torrecillas introduced the Gorenstein projective, injective, and flat modules [7, 9] and then developed Gorenstein homological algebra [8]. Zhang [20] introduced in 2013 the compatible bimodules and explicitly described the Gorenstein projective modules over triangular matrix Artin algebra. In 2014, Enochs and other authors in [6] introduced Gorenstein regular rings and characterized when a left T-module is Gorenstein projective or Gorenstein injective over such rings.

This paper is devoted to study Gorenstein homological dimensions over triangular matrix rings and is organized as follows. In Section 2, we focus on discussing the structure of left T-modules of finite Gorenstein projective (resp. injective) dimensions. Let n be a nonnegative integer, B a left Gorenstein regular ring, U_A have finite flat dimension, and ${}_BU$ have finite projective dimension. Let $M = \binom{M_1}{M_2}_{\varphi^M}$ be a left T-module. Then using the structure of left T-modules, we show that $\mathrm{Gpd}(M) \leq n$ if and only if $\mathrm{Gpd}(M_1) \leq n$, $\mathrm{Gpd}(\frac{M_2}{\mathrm{Im}\varphi^M}) \leq n$ and if $\binom{K_1}{K_2}_{\varphi^K}$ is a n-th syzygy of M, then φ^K is injective, where $\mathrm{Gpd}(M)$ denotes the Gorenstein projective

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dimension of M (Theorem 2.5). A similar dual result holds for Gorenstein injective dimension (Theorem 2.6).

Motivated by the characterization of Gorenstein projective or Gorenstein injective left T-module, in Section 3 we characterize when a left T-module is Gorenstein flat. We prove in Theorem 3.8 that if U_A and ${}_BU$ are finitely generated and have finite projective dimension, T is a Gorenstein ring; then $F = \binom{F_1}{F_2}_{\varphi^F} \in \mathcal{GF}(T)$ if and only if $F_1 \in \mathcal{GF}(A)$, $\frac{F_2}{\mathrm{Im}\varphi^F} \in \mathcal{GF}(B)$, and φ^F is injective, where $\mathcal{GF}$ denotes the class of all Gorenstein flat modules.

There is also an analogous for a free module, namely, the strongly Gorenstein projective module [3]. As observed by Bennis and Mahdou, a module is Gorenstein projective if and only if it is a direct summand of a strongly Gorenstein projective module ([3, Theorem 2.7]). Gao and Zhang [12] gave a concrete construction of strongly Gorenstein projective modules, via the existed construction of upper triangular matrix Artin algebras. Finally, we give in Section 4 the characterization of strongly Gorenstein projective (resp. injective) modules and dimensions, which extend the results in [12]. We show in Theorem 4.2 that if U_A has finite flat dimension and ${}_BU$ has finite projective dimension, B is left Gorenstein regular; then $M = \binom{M_1}{M_2}_{\varphi^M}$ is strongly Gorenstein projective if and only if M_1 and $\frac{M_2}{\mathrm{Im}\varphi^M}$ are strongly Gorenstein projective and the φ^M is injective.

Throughout this paper, all rings are associative rings with identity, and all modules are unitary. As usual, $\mathrm{pd}(M)$, $\mathrm{id}(M)$, and $\mathrm{fd}(M)$ denote the projective, injective, and flat dimensions of a left R-module M, respectively.

Let R be a ring. A left R-module M is called *Gorenstein projective* if there exists an exact sequence

$$\cdots \longrightarrow P^{-2} \longrightarrow P^{-1} \longrightarrow P^0 \longrightarrow P^1 \longrightarrow \cdots$$

of projective left R-modules with $M \cong \mathrm{Ker}(P^0 \to P^1)$ such that $\mathrm{Hom}_R(-, Q)$ leaves the sequence exact for any projective left R-module Q. The Gorenstein injective modules are defined dually. A left R-module M is said to be *Gorenstein flat* if there exists an exact sequence

$$\cdots \longrightarrow F^{-2} \longrightarrow F^{-1} \longrightarrow F^0 \longrightarrow F^1 \longrightarrow \cdots$$

of flat left R-modules with $M \cong \mathrm{Ker}(F^0 \to F^1)$ such that $I \otimes_R -$ leaves the sequence exact for any injective right R-module I. We denote by $\mathcal{GP}(R)$, $\mathcal{GI}(R)$, and $\mathcal{GF}(R)$ the class of all Gorenstein projective, injective and flat R-modules, respectively. For any R-module M, $\mathrm{Gpd}(M)$, $\mathrm{Gid}(M)$, and $\mathrm{Gfd}(M)$ denote the Gorenstein projective, injective, and flat dimension of M, respectively, and $\mathrm{glGpd}(R)$ and $\mathrm{glGid}(R)$ denote the global Gorenstein projective and injective dimensions of R, respectively.

A complex $\mathbf{C}$ of modules is a sequence

$$\cdots \longrightarrow C^{n-1} \xrightarrow{d^{n-1}} C^n \xrightarrow{d^n} C^{n+1} \longrightarrow \cdots$$

of R-modules and R-homomorphisms such that $d^n d^{n-1} = 0$ for all $n \in \mathbb{Z}$. A complex $\mathbf{C}$ is exact if for each n, $\mathrm{Ker}d^n = \mathrm{Im}d^{n-1}$.

If $\mathcal{C}$ is an abelian category and $\mathbf{f} : R\text{-Mod} \to \mathcal{C}$ is an additive covariant functor, $\mathbf{f}(\mathbf{C})$ will be the complex

$$\cdots \longrightarrow \mathbf{f}(C^{n-1}) \xrightarrow{\mathbf{f}(d^{n-1})} \mathbf{f}(C^n) \xrightarrow{\mathbf{f}(d^n)} \mathbf{f}(C^{n+1}) \longrightarrow \cdots .$$

We say that $\mathbf{C}$ is $\mathbf{f}$-exact if $\mathbf{f}(\mathbf{C})$ is exact.

Let $\mathcal{C}$ be a Grothendieck category. $\mathcal{C}$ is said to be Gorenstein if it satisfies:

(1) The classes of all objects with finite projective dimension and with finite injective dimension coincide.

(2) The finitistic projective and injective dimensions of $\mathcal{C}$ are finite.

(3) $\mathcal{C}$ has a generator with finite projective dimension.

Definition 1.1 *([6, Definition 2.1]) A ring R is said to be left Gorenstein regular if the category R-Mod is Gorenstein.*

Each Gorenstein ring (that is, a two-sided noetherian ring with finite left and right self-injective dimensions) is left and right Gorenstein regular (see [8, Theorem 9.1.11]), and the converse is true precisely when the ring is two-sided noetherian. A equivalent formulation for Gorenstein regular rings is given in [6, Proposition 2.2], which is more convenient to use. That is, a ring R is left (resp. right) Gorenstein regular if and only if each projective left (resp. right) R-module has finite injective dimension and each injective left (resp. right) R-module has finite projective dimension.

Let A and B be rings and U a (B, A)-bimodule. We denote by T the triangular matrix ring $\left(\begin{smallmatrix} A & 0 \\ U & B \end{smallmatrix}\right)$. According to [14, Theorem 1.5] T-Mod is equivalent to the category whose objects are triples $M = \binom{M_1}{M_2}_{\varphi^M}$, where $M_1 \in A$-Mod, $M_2 \in B$-Mod and $\varphi^M : U \otimes M_1 \to M_2$ is a B-homomorphism, and whose morphisms between two objects $\binom{M_1}{M_2}_{\varphi^M}$ and $N = \binom{N_1}{N_2}_{\varphi^N}$ are pairs $\binom{f_1}{f_2}$ such that $f_1 \in \mathrm{Hom}_A(M_1, N_1)$, $f_2 \in \mathrm{Hom}_B(M_2, N_2)$, satisfying that the diagram

$$\begin{tikzcd} U \otimes M_1 \arrow[r, "1_U \otimes f_1"] \arrow[d, "\varphi^M"'] & U \otimes N_1 \arrow[d, "\varphi^N"] \\ M_2 \arrow[r, "f_2"] & N_2 \end{tikzcd}$$

is commutative. In the rest of the paper we identify T-Mod with this category and, whenever there is no possible confusion, we omit the homomorphism φ^M. Consequently, through the paper, a left T-module is a pair $\binom{M_1}{M_2}$. Given such a module M, we denote by $\widetilde{\varphi^M}$ the morphism from M_1 to $\mathrm{Hom}_B(U, M_2)$ given by $\widetilde{\varphi^M}(m)(u) = \varphi^M(u \otimes m)$ for each $m \in M_1$, $u \in U$.

There are some functors between the category T-module and the product A-Mod $\times$ B-Mod:

- $\mathbf{p} : A\text{-Mod} \times B\text{-Mod} \to T\text{-Mod}$ is defined as follows: for each object (M_1, M_2) of A-Mod $\times$ B-Mod, let $\mathbf{p}(M_1, M_2) = \binom{M_1}{(U \otimes M_1) \oplus M_2}$, with the obvious map. And, for any morphism (f_1, f_2) in A-Mod $\times$ B-Mod, let $\mathbf{p}(f_1, f_2) = \binom{f_1}{(U \otimes f_1) \oplus f_2}$.

- $\mathbf{h} : A\text{-Mod} \times B\text{-Mod} \to T\text{-Mod}$ is defined as follows: for each object (M_1, M_2) of A-Mod $\times$ B-Mod, let $\mathbf{h}(M_1, M_2) = \binom{M_1 \oplus \mathrm{Hom}_B(U, M_2)}{M_2}$ with the obvious map. And, for any morphism (f_1, f_2) in A-Mod$\times$$B$-Mod, let $\mathbf{h}(f_1, f_2) = \binom{f_1 \oplus \mathrm{Hom}_B(U, f_2)}{f_2}$.

- $\mathbf{q} : T\text{-Mod} \to A\text{-Mod} \times B\text{-Mod}$ is defined, for each left T-module $\binom{M_1}{M_2}$, as $\mathbf{q}\binom{M_1}{M_2} = (M_1, M_2)$, and for any morphism $\binom{f_1}{f_2}$ in T-Mod as $\mathbf{q}\binom{f_1}{f_2} = (f_1, f_2)$.

It is easy to see that $\mathbf{p}$ is a left adjoint of $\mathbf{q}$, $\mathbf{h}$ is a right adjoint of $\mathbf{q}$, and that $\mathbf{q}$ is exact. In particular, $\mathbf{p}$ preserves projective objects and $\mathbf{h}$ preserves injective objects.

Note that a sequence of T-modules

$$0 \to \begin{pmatrix} M_1' \\ M_2' \end{pmatrix} \to \begin{pmatrix} M_1 \\ M_2 \end{pmatrix} \to \begin{pmatrix} M_1'' \\ M_2'' \end{pmatrix} \to 0$$

is exact if and only if both sequence $0 \to M_1' \to M_1 \to M_1'' \to 0$ of A-modules and $0 \to M_2' \to M_2 \to M_2'' \to 0$ of B-modules are exact.

2. Gorenstein projective (resp. injective) dimensions

In this section, we describe explicitly the structure of left T-modules of finite Gorenstein projective (resp. injective) dimensions. We start with the following lemma, which is useful in the following arguments.

Lemma 2.1 *([6, Theorem 3.5]) Let U_A have finite flat dimension, ${}_BU$ have finite projective dimension, and B be left Gorenstein regular. Let $M = \binom{M_1}{M_2}_{\varphi^M}$ be a left T-module. Then the following are equivalent:*

(1) *M is Gorenstein projective;*

(2) *M_1 and $\frac{M_2}{\mathrm{Im}\varphi^M}$ are Gorenstein projective and the φ^M is injective.*

Proposition 2.2 *Let B be a left Gorenstein regular ring, U a (B,A)-bimodule, $M_1 \in \mathcal{GP}(A)$. If U is projective as left B-module and has finite flat dimension as right A-module, then $U \otimes_A M_1 \in \mathcal{GP}(B)$.*

Proof Assume that U is projective as left B-module and has finite flat dimension as right A-module, $M_1 \in \mathcal{GP}(A)$, let

$$\mathbf{P}: \qquad \cdots \longrightarrow P^{-1} \longrightarrow P^0 \xrightarrow{\partial^0} P^1 \longrightarrow \cdots$$

be an exact sequence consisting of projective left A-modules that are $\mathrm{Hom}_B(-,Q)$-exact for each projective left A-module and such that $M_1 \cong \mathrm{Ker}\partial^0_{\mathbf{P}}$. Using the hypothesis and [6, Lemma 2.3], we get that $U \otimes_A \mathbf{P}$ is an exact sequence of projective B-module such that $U \otimes M_1 \cong \mathrm{Ker}(1_U \otimes \partial^0_{\mathbf{P}})$. Since B is left Gorenstein regular, we know that $U \otimes_A M_1 \in \mathcal{GP}(B)$ by [6, Proposition 2.6]. □

In the following we consider that if left T-module $M = \binom{M_1}{M_2}_{\varphi^M}$ is Gorenstein projective, whether M_2 is a Gorenstein projective B-module.

Proposition 2.3 *Let T be left Gorenstein regular, U be projective as left B-module and have finite flat dimension as right A-module. If $G = \binom{G_1}{G_2}_{\varphi^G} \in \mathcal{GP}(T)$, then $G_1 \in \mathcal{GP}(A)$, $G_2 \in \mathcal{GP}(B)$ and the morphism φ^G is injective.*

Proof Suppose $G = \binom{G_1}{G_2}_{\varphi^G} \in \mathcal{GP}(T)$. By Lemma 2.1, there exists a short exact sequence

$$0 \longrightarrow U \otimes G_1 \xrightarrow{\varphi^G} G_2 \longrightarrow \frac{G_2}{\mathrm{Im}\varphi^G} \longrightarrow 0,$$

where G_1 and $\frac{G_2}{\mathrm{Im}\varphi^G}$ are Gorenstein projective. Since U is projective as left B-module and has finite flat dimension as right A-module, it follows from Proposition 2.2 that $U \otimes_A G_1 \in \mathcal{GP}(B)$. Thus $G_2 \in \mathcal{GP}(B)$ by [18, Theorem 2.5]. □

Asadollahi and Salarian [1] establish a relationship between the projective (resp. injective) dimension of modules over T and over A and B. Let n be a nonnegative integer. They proved that $\mathrm{pd}\binom{M_1}{M_2}_{\varphi^M} \leq n$ if and only if $\mathrm{pd}(M_1) \leq n$, $\mathrm{pd}(\frac{M_2}{\mathrm{Im}\varphi^M}) \leq n$ and the map related to the n-th syzygy of $\binom{M_1}{M_2}_{\varphi^M}$ is injective. We have similar arguments about Gorenstein projective dimension and Gorenstein injective dimension.

The following lemma is quoted from [1].

Lemma 2.4 *Let $J = Te_1$, where $e_1 = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$. Then for each left T-module $X = \binom{X_1}{X_2}_{\varphi^X}$, there is a isomorphism $\frac{T}{J} \otimes X \cong \left(\begin{smallmatrix} 0 \\ \frac{X_2}{\mathrm{Im}\varphi^X} \end{smallmatrix}\right)$.*

Theorem 2.5 *Let n be a nonnegative integer, B a left Gorenstein regular ring, U_A have finite flat dimension, and ${}_BU$ have finite projective dimension. Let $M = \binom{M_1}{M_2}_{\varphi^M}$ be a left T-module. Then $\mathrm{Gpd}(M) \leq n$ if and only if $\mathrm{Gpd}(M_1) \leq n$, $\mathrm{Gpd}(\frac{M_2}{\mathrm{Im}\varphi^M}) \leq n$ and if $K = \binom{K_1}{K_2}_{\varphi^K}$ is a n-th syzygy of M, then φ^K is injective.*

Proof $(\Rightarrow)$ Let $\mathrm{Gpd}(M) \leq n$. Then there exists an exact sequence of T-modules

$$\mathbf{P}: \ 0 \longrightarrow \begin{pmatrix} G_1^n \\ G_2^n \end{pmatrix} \xrightarrow{\binom{\partial_1^n}{\partial_2^n}} \cdots \longrightarrow \begin{pmatrix} G_1^1 \\ G_2^1 \end{pmatrix} \xrightarrow{\binom{\partial_1^1}{\partial_2^1}} \begin{pmatrix} G_1^0 \\ G_2^0 \end{pmatrix} \xrightarrow{\binom{\partial_1^0}{\partial_2^0}} \begin{pmatrix} M_1 \\ M_2 \end{pmatrix} \longrightarrow 0,$$

where $\binom{G_1^i}{G_2^i} \in \mathcal{GP}(T)$ for $i = 0, \cdots, n$. Thus the sequence

$$\mathbf{P}_1 : 0 \longrightarrow G_1^n \longrightarrow \cdots \longrightarrow G_1^1 \longrightarrow M_1 \longrightarrow 0$$

is exact. Since $\binom{G_1^i}{G_2^i} \in \mathcal{GP}(T)$, then $G_1^i \in \mathcal{GP}(A)$ by Lemma 2.1, and so $\mathrm{Gpd}(M_1) \leq n$. By Lemma 2.4, because $\frac{T}{J}$ is projective in T-Mod, we can apply the functor $\frac{T}{J} \otimes -$ on $\mathbf{P}$ to get the following exact sequence

$$\overline{\mathbf{P}_2}: \quad 0 \longrightarrow \frac{G_2^n}{\mathrm{Im}\varphi^n} \longrightarrow \cdots \longrightarrow \frac{G_2^0}{\mathrm{Im}\varphi^0} \longrightarrow \frac{M_2}{\mathrm{Im}\varphi^M} \longrightarrow 0,$$

where $\frac{G_2^i}{\mathrm{Im}\varphi^i} \in \mathcal{GP}(B)$ for $i = 0, \cdots, n$. Thus $\mathrm{Gpd}(\frac{M_2}{\mathrm{Im}\varphi^M}) \leq n$.

If $K = \binom{K_1}{K_2}_{\varphi^K}$ is a n-th syzygy of M, since $\mathrm{Gpd}(M) \leq n$, $\binom{K_1}{K_2}$ has to be Gorenstein projective by [18, Theorem 2.20], and the morphism φ^K is injective by Lemma 2.1.

$(\Leftarrow)$ Let $\mathrm{Gpd}(M_1) \leq n$, $\mathrm{Gpd}(\frac{M_2}{\mathrm{Im}\varphi^M}) \leq n$, and φ^K be injective for any n-th syzygy $\binom{K_1}{K_2}_{\varphi^K}$ of M. Then there exists an exact sequence

$$\mathbf{P}': \quad 0 \longrightarrow \begin{pmatrix} K_1^n \\ K_2^n \end{pmatrix} \longrightarrow \begin{pmatrix} P_1^{n-1} \\ P_2^{n-1} \end{pmatrix} \longrightarrow \cdots \longrightarrow \begin{pmatrix} P_1^1 \\ P_2^1 \end{pmatrix} \longrightarrow \begin{pmatrix} P_1^0 \\ P_2^0 \end{pmatrix} \longrightarrow \begin{pmatrix} M_1 \\ M_2 \end{pmatrix} \longrightarrow 0.$$

Since every projective T-module is Gorenstein projective, it suffices to verify $\binom{K_1^n}{K_2^n} \in \mathcal{GP}(T)$.

The sequence above induces the following exact sequence

$$0 \longrightarrow K_1^n \longrightarrow P_1^{n-1} \longrightarrow \cdots \longrightarrow P_1^0 \longrightarrow M_1 \longrightarrow 0.$$

As $\mathrm{Gpd}(M_1) \leq n$, we know that $K_1^n \in \mathcal{GP}(A)$. Apply the functor $\frac{T}{J} \otimes -$ on $\mathbf{P'}$, we get the following exact sequence

$$0 \longrightarrow \frac{K_2^n}{\mathrm{Im}\varphi^K} \longrightarrow \cdots \longrightarrow \frac{P_2^0}{\mathrm{Im}\varphi^0} \longrightarrow \frac{M_2}{\mathrm{Im}\varphi^M} \longrightarrow 0.$$

We have $\frac{K_2^n}{\mathrm{Im}\varphi^K} \in \mathcal{GP}(B)$ since $\mathrm{Gpd}(\frac{M_2}{\mathrm{Im}\varphi^M}) \leq n$. We get that $\binom{K_1^n}{K_2^n} \in \mathcal{GP}(T)$ by Lemma 2.1. Thus $\mathrm{Gpd}(M) \leq n$. $\square$

Using the dual argument of Lemma 2.1 and Theorem 2.5, one can prove the following theorem.

Theorem 2.6 *Let n be a nonnegative integer, A a left Gorenstein regular ring, U_A have finite flat dimension, and ${}_BU$ have finite projective dimension. Let $M = \binom{M_1}{M_2}_{\varphi^M}$ be a left T-module. Then* $\mathrm{Gid}(M) \leq n$ *if and only if* $\mathrm{Gid}(M_2) \leq n$, $\mathrm{Gid}(\mathrm{Ker}\widetilde{\varphi^M}) \leq n$ *and if $\binom{L_1}{L_2}_{\varphi^L}$ is a n-th cosyzygy of M, then $\widetilde{\varphi^L}$ is surjective.*

For finiteness of Gorenstein projective (resp. injective) dimensions, we have the following result.

Theorem 2.7 *Let U_A have finite flat dimension and ${}_BU$ have finite projective dimension. Let $M = \binom{M_1}{M_2}_{\varphi^M}$ be a left T-module.*

(1) Suppose that B is left Gorenstein regular, then $\mathrm{Gpd}(M) < \infty$ *if and only if* $\mathrm{Gpd}(M_1) < \infty$, $\mathrm{Gpd}(M_2) < \infty$.

(2) Suppose that A is left Gorenstein regular, then $\mathrm{Gid}(M) < \infty$ *if and only if* $\mathrm{Gid}(M_1) < \infty$, $\mathrm{Gid}(M_2) < \infty$.

Proof (1)($\Rightarrow$) Suppose $\mathrm{Gpd}(M) < \infty$. By Theorem 2.5, we get $\mathrm{Gpd}(M_1) \leq n$. As a consequence of [10, Theorem 2.27], B is left Gorenstein regular if and only if $\mathrm{glGpd}(B) < \infty$ and $\mathrm{glGid}(B) < \infty$, Thus $\mathrm{Gpd}(M_2) < \infty$ by the hypothesis.

($\Leftarrow$) Suppose that $\mathrm{Gpd}(M_1) < \infty$ and $\mathrm{Gpd}(M_2) < \infty$. Fix a Gorenstein projective resolution of M_1,

$$0 \longrightarrow G_1^n \xrightarrow{\partial_1^n} \cdots \longrightarrow G_1^0 \xrightarrow{\partial_1^0} M_1 \longrightarrow 0$$

and a Gorenstein projective presentation $G_2^0 \overset{\partial_2^0}{\to} M_2$ of M_2. Then $\mathbf{p}(\partial_1^0, \partial_2^0)$ is a Gorenstein projective presentation of M in T-Mod and, if $K^0 = \binom{K_1^0}{K_2^0}$ is its kernel, there exist short exact sequences

$$0 \longrightarrow K_1^0 \longrightarrow G_1 \longrightarrow M_1 \longrightarrow 0$$

and

$$0 \longrightarrow K_2^0 \longrightarrow G_2 \oplus (U \otimes_A G_1) \longrightarrow M_2 \longrightarrow 0.$$

Then K_1^0 and K_2^0 have finite Gorenstein projective dimension by [18, Theorem 2.24]. Now take $\partial_2^1 : G_2^1 \to K_2^0$ a projective presentation. Since K_1^0 is the kernel of ∂_1^0, $\mathbf{p}(\partial_1^1, \partial_2^1)$ is a projective presentation of K^0 and, reasoning as before, its kernel, say $K^1 = \binom{K_1^1}{K_2^1}$, satisfies that K_1^1 and K_2^1 have finite Gorenstein projective dimension. Repeating this procedure, we construct a Gorenstein projective resolution of M

$$\cdots \longrightarrow \mathbf{p}(G_1^1, G_2^1) \longrightarrow \mathbf{p}(G_1^0, G_2^0) \longrightarrow M \longrightarrow 0,$$

such that $K^m = \binom{K_1^m}{K_2^m}$ is the kernel of $\mathbf{p}(\partial_1^m, \partial_2^m)$ for each $m \in \mathbb{N}$, both K_1^m and K_2^m have finite Gorenstein dimensions. Since $G_1^{n+1} = 0$, $\mathrm{Ker}\mathbf{p}(\partial_1^{n+1}, \partial_2^{n+1}) = \binom{0}{K_2^{n+1}}$. As K_2^{n+1} has finite Gorenstein projective dimension in B-Mod, we have the following exact sequence

$$0 \longrightarrow Q_2^{n+m} \longrightarrow \cdots \longrightarrow Q_2^{n+2} \longrightarrow K_2^{n+1} \longrightarrow 0,$$

which induces the finite resolution in T-Mod

$$0 \longrightarrow \mathbf{p}(0, Q_2^{n+m}) \longrightarrow \cdots \longrightarrow \mathbf{p}(0, Q_2^{n+2}) \longrightarrow \begin{pmatrix} 0 \\ K_2^{n+1} \end{pmatrix} \longrightarrow 0.$$

This means that $\binom{0}{K_2^{n+1}}$ has finite Gorenstein projective dimension, which implies that $\mathrm{Gpd}(M) < \infty$.

The proof of (2) is similar to the way we do it in (1); we can prove $\mathrm{Gid}(M) < \infty$ if and only if $\mathrm{Gid}(M_1) < \infty$, $\mathrm{Gid}(M_2) < \infty$. □

It follows from [2, Lemma 3.4] that if ${}_BU$ is projective, $\mathrm{pd}(M_1) \leq n$, and $\mathrm{pd}(M_2) \leq n$, then $\mathrm{pd}\binom{M_1}{M_2} \leq n+1$. Moreover, we have the following result, which is a more explicit version of Theorem 2.7 under some additional conditions.

Lemma 2.8 *Let ${}_BU$ have finite projective dimension, U_A have finite flat dimension, and B be a left Gorenstein regular ring. Let $M = \binom{M_1}{M_2}_{\varphi^M} \in \mathcal{GP}(T)$, $U \otimes_A M_1$ have finite projective dimension. Then there exists $P \in \mathcal{GP}(A)$ and $Q \in \mathcal{GP}(B)$ such that $M = \mathbf{p}(P, Q)$.*

Proof There exists an exact sequence

$$0 \longrightarrow U \otimes M_1 \xrightarrow{\varphi^M} M_2 \longrightarrow \frac{M_2}{\mathrm{Im}\varphi^M} \longrightarrow 0$$

in which M_1 and $\frac{M_2}{\mathrm{Im}\varphi^M}$ are Gorenstein projective. Since $U \otimes_A M_1$ has finite projective dimension, we know by [18, Theorem 2.20] that $\mathrm{Ext}_B^1(\frac{M_2}{\mathrm{Im}\varphi^M}, U \otimes M_1) = 0$, which means the above sequence splits, and $M_2 = (U \otimes M_1) \oplus \frac{M_2}{\mathrm{Im}\varphi^M}$. Let $P = M_1$, $Q = \frac{M_2}{\mathrm{Im}\varphi^M}$, then

$$M = \begin{pmatrix} M_1 \\ M_2 \end{pmatrix} = \begin{pmatrix} M_1 \\ U \otimes M_1 \end{pmatrix} \oplus \begin{pmatrix} 0 \\ \frac{M_2}{\mathrm{Im}\varphi^M} \end{pmatrix} = \begin{pmatrix} P \\ U \otimes P \end{pmatrix} \oplus \begin{pmatrix} 0 \\ Q \end{pmatrix} = \mathbf{p}(P, Q).$$

□

Dually, we have the following result for Gorenstein injective T-modules.

Lemma 2.9 *Let ${}_BU$ have finite projective dimension, U_A have finite flat dimension, A be a left Gorenstein regular ring. Let $M = \binom{M_1}{M_2}_{\varphi^M} \in \mathcal{GI}(T)$, $\mathrm{Hom}_B(U, M_2)$ have finite injective dimension. Then there exists $E \in \mathcal{GI}(A)$ and $I \in \mathcal{GP}(B)$ such that $M = \mathbf{h}(E, I)$.*

Proposition 2.10 *Let U_A have finite flat dimension, B be a left Gorenstein regular, $M = \binom{M_1}{M_2}$ a T-module, $Gpd(M_1) \leq n$, $Gpd(M_2) \leq n$, $U \otimes_A M_1$ has finite projective dimension. If ${}_BU$ is projective, then* $\mathrm{Gpd}(M) \leq n+1$.

Proof By Lemma 2.8, a Gorenstein projective resolution of the T-module M can be written in the following form

$$\cdots \to \mathbf{p}(P_n, Q_n) \to \cdots \to \mathbf{p}(P_1, Q_1) \to \mathbf{p}(P_0, Q_0) \to M \to 0,$$

where $P_i \in \mathcal{GP}(A)$ and $Q_i \in \mathcal{GP}(B)$ for $i > 0$. Let $\binom{K_1}{K_2}$ be the n-th syzygy of M. We show that $\binom{K_1}{K_2}$ has Gorenstein projective dimension at most one. The above resolution induces the following exact sequence

$$0 \to K_1 \to P_{n-1} \to \cdots \to P_1 \to P_0 \to M_1 \to 0$$

of A-modules and an exact sequence

$$0 \to K_2 \to (U \otimes P_{n-1}) \oplus Q_{n-1} \to \cdots \to (U \otimes P_1) \oplus Q_1 \to (U \otimes P_0) \oplus Q_0 \to M_2 \to 0$$

of B-module. We know that $U \otimes P_{n-1} \in \mathcal{GP}(B)$ by Proposition 2.2 for $i = 0, \cdots, n-1$. The projectivity of ${}_BU$ in conjunction with our assumption on the Gorenstein projective dimensions of M_1 and M_2 implies that K_1 and K_2 are Gorenstein projective. Now consider the following short exact sequence of T-modules

$$0 \longrightarrow \begin{pmatrix} 0 \\ U \otimes K_1 \end{pmatrix} \longrightarrow \begin{pmatrix} K_1 \\ U \otimes K_1 \end{pmatrix} \oplus \begin{pmatrix} 0 \\ K_2 \end{pmatrix} \longrightarrow \begin{pmatrix} K_1 \\ K_2 \end{pmatrix} \longrightarrow 0$$

Since ${}_BU$ is projective, we deduce that the first two left terms are Gorenstein projective, and so $\mathrm{Gpd}\binom{K_1}{K_2} \leq 1$. Therefore $\mathrm{Gpd}(M) \leq n + 1$. □

3. Gorenstein flat dimensions

In the following, we describe Gorenstein flat T-module over a triangular matrix ring.

Lemma 3.1 *([4, Theorem 3.1])Let U_A and ${}_BU$ be finitely generated and have finite projective dimensions. Then T is Gorenstein if and only if A and B are Gorenstein.*

Lemma 3.2 *([11, Proposition 1.14]) Let $F = \binom{F_1}{F_2}_{\varphi^F}$ be a left T-module. Then F is flat if and only if F_1 and $\frac{F_2}{Im\varphi^F}$ are flat and morphism φ^F is injective.*

Proposition 3.3 *Let n be a nonnegative integer, $M = \binom{M_1}{M_2}_{\varphi^M}$ a left T-module. Then $\mathrm{fd}(M) \leq n$ if and only if $\mathrm{fd}(M_1) \leq n$, $\mathrm{fd}(\frac{M_2}{\mathrm{Im}\varphi^M}) \leq n$, and if $\binom{K_1}{K_2}_{\varphi^K}$ is a n-th syzygy of M, then φ^K is injective.*

Proof ($\Rightarrow$) Let $\mathrm{fd}(M) \leq n$. There exists an exact sequence of T-modules

$$\mathbf{F}: \ 0 \longrightarrow \begin{pmatrix} F_1^n \\ F_2^n \end{pmatrix} \xrightarrow{\binom{\partial_1^n}{\partial_2^n}} \cdots \longrightarrow \begin{pmatrix} F_1^1 \\ F_2^1 \end{pmatrix} \xrightarrow{\binom{\partial_1^1}{\partial_2^1}} \begin{pmatrix} F_1^0 \\ F_2^0 \end{pmatrix} \xrightarrow{\binom{\partial_1^0}{\partial_2^0}} \begin{pmatrix} M_1 \\ M_2 \end{pmatrix} \longrightarrow 0,$$

where $\binom{F_1^i}{F_2^i}$ is flat for $i = 0, \cdots, n$, which induces the exact sequence

$$\mathbf{F}_1 : 0 \longrightarrow F_1^n \longrightarrow \cdots \longrightarrow F_1^1 \longrightarrow M_1 \longrightarrow 0.$$

Since $\binom{F_1^i}{F_2^i}$ is flat, F_1^i is flat by Lemma 3.2, and so $\mathrm{fd}(M_1) \leq n$. By Lemma 2.4, we apply the functor $\frac{T}{J} \otimes -$ on $\mathbf{F}$ to get the following exact sequence

$$\overline{\mathbf{F}_2}: \quad 0 \longrightarrow \frac{F_2^n}{\mathrm{Im}\varphi^n} \longrightarrow \cdots \longrightarrow \frac{F_2^0}{\mathrm{Im}\varphi^0} \longrightarrow \frac{M_2}{\mathrm{Im}\varphi^M} \longrightarrow 0$$

where $\frac{F_2^i}{\mathrm{Im}\varphi^i}$ is flat for $i = 0, \cdots, n$. Thus $\mathrm{fd}(\frac{M_2}{\mathrm{Im}\varphi^M}) \leq n$. If $\binom{K_1}{K_2}_{\varphi^K}$ is n-th syzygy of M, $\binom{K_1}{K_2}_{\varphi^K}$ is flat since $\mathrm{fd}(M) \leq n$, and morphism φ^K is injective by Lemma 3.2.

($\Leftarrow$) Suppose $\mathrm{fd}(M_1) \leq n$, $\mathrm{fd}(\frac{M_2}{\mathrm{Im}\varphi^M}) \leq n$, and $\binom{K_1}{K_2}_{\varphi^K}$ is n-th syzygy of M, φ^K is injective. Then there exists an exact sequence

$$\mathbf{F}': \quad 0 \longrightarrow \begin{pmatrix} K_1^n \\ K_2^n \end{pmatrix} \longrightarrow \begin{pmatrix} F_1^{n-1} \\ F_2^{n-1} \end{pmatrix} \longrightarrow \cdots \longrightarrow \begin{pmatrix} F_1^1 \\ F_2^1 \end{pmatrix} \longrightarrow \begin{pmatrix} F_1^0 \\ F_2^0 \end{pmatrix} \longrightarrow \begin{pmatrix} M_1 \\ M_2 \end{pmatrix} \longrightarrow 0.$$

Then we only need to verify that $\binom{K_1^n}{K_2^n}$ is flat. We have an exact sequence

$$0 \longrightarrow K_1^n \longrightarrow F_1^{n-1} \longrightarrow \cdots \longrightarrow F_1^0 \longrightarrow M_1 \longrightarrow 0.$$

As $\mathrm{fd}(M_1) \leq n$, we know that K_1^n is flat. Applying the functor $\frac{T}{J} \otimes -$ on $\mathbf{F}'$, by Lemma 2.4 we get the following exact sequence

$$0 \longrightarrow \frac{K_2^n}{\mathrm{Im}\varphi^K} \longrightarrow \cdots \longrightarrow \frac{P_2^0}{\mathrm{Im}\varphi^0} \longrightarrow \frac{M_2}{\mathrm{Im}\varphi^M} \longrightarrow 0,$$

we obtain that $\frac{K_2^n}{\mathrm{Im}\varphi^K}$ is flat since $\mathrm{fd}(\frac{M_2}{\mathrm{Im}\varphi^M}) \leq n$. The previous Lemma implies that $\binom{K_1^n}{K_2^n}$ is flat. Thus $\mathrm{fd}(M) \leq n$. □

Proposition 3.4 *Let $M = \binom{M_1}{M_2}_{\varphi^M}$ be a left T-module. Suppose that ${}_BU$ has finite flat dimension, Then M has finite flat dimension if and only if M_1 and M_2 have finite flat dimension.*

Proof It is true by combining Proposition 3.3 with [6, Proposition 2.8(1)]. □

Proposition 3.5 *Let U_A and ${}_BU$ have finite projective dimension, T be a right Gorenstein regular ring, $M = \binom{M_1}{M_2}_{\varphi^M}$ a left T-module. If M is Gorenstein flat, then M_1 and $\frac{M_2}{\mathrm{Im}\varphi^M}$ are Gorenstein flat and the morphism φ^M is injective.*

Proof Suppose M is Gorenstein flat in T-Mod and let

$$\mathbf{F}: \ \cdots \longrightarrow \begin{pmatrix} F_1^{n-1} \\ F_2^{n-1} \end{pmatrix} \xrightarrow{\binom{\partial_1^{n-1}}{\partial_2^{n-1}}} \begin{pmatrix} F_1^n \\ F_2^n \end{pmatrix} \xrightarrow{\binom{\partial_1^n}{\partial_2^n}} \begin{pmatrix} F_1^{n+1} \\ F_2^{n+1} \end{pmatrix} \longrightarrow \cdots$$

be an exact sequence consisting of flat left T-modules that is $E \otimes -$-exact for each injective right T-module E and such that $\mathrm{Ker}\binom{\partial_1^0}{\partial_2^0} \cong M$. Then

$$\mathbf{F}_1: \quad \cdots \longrightarrow F_1^{n-1} \longrightarrow F_1^n \longrightarrow F_{n+1} \longrightarrow \cdots$$

is an exact sequence consisting of flat left A-modules with $\mathrm{Ker}\partial_1^0 \cong M_1$. Moreover, A is right Gorenstein regular; then for each injective right A-module I, I has finite flat dimension, and we get that $I \otimes \mathbf{F}_1$ is exact by [6, Lemma 2.3]. This means that M_1 is a Gorenstein flat left A-module.

Now note that for every morphism $\binom{\partial_1^n}{\partial_2^n} : \binom{F_1^n}{F_2^n} \longrightarrow \binom{F_1^{n+1}}{F_2^{n+1}}$, we can construct the following commutative diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & U \otimes F_1^n & \xrightarrow{\varphi^n} & F_2^n & \xrightarrow{\pi^n} & \frac{F_2^n}{\mathrm{Im}\varphi^n} & \longrightarrow & 0 \\
 & & {\scriptstyle 1_U \otimes \partial_1^n}\downarrow & & {\scriptstyle \partial_2^n}\downarrow & & {\scriptstyle \overline{\partial_2^n}}\downarrow & & \\
0 & \longrightarrow & U \otimes F_1^{n+1} & \xrightarrow{\varphi^{n+1}} & F_2^{n+1} & \xrightarrow{\pi^{n+1}} & \frac{F_2^{n+1}}{\mathrm{Im}\varphi^{n+1}} & \longrightarrow & 0,
\end{array}$$

where π^n and π^{n+1} are the canonical projections. Using this fact, the complex $\mathbf{F}$ induces the complex

$$\overline{\mathbf{F}_2}: \qquad \cdots \longrightarrow \frac{F_2^{n-1}}{\mathrm{Im}\varphi^{n-1}} \xrightarrow{\overline{\partial_2^{n-1}}} \frac{F_2^n}{\mathrm{Im}\varphi^n} \xrightarrow{\overline{\partial_2^n}} \frac{F_2^{n+1}}{\mathrm{Im}\varphi^{n+1}} \longrightarrow \cdots,$$

where φ^i is the structural map of the T-module $\binom{F_1^i}{F_2^i}$ for each $i \in \mathbb{Z}$. We get that each $\frac{F_2^n}{\mathrm{Im}\varphi^n}$ is Gorenstein flat in B-Mod, since $\binom{F_1^i}{F_2^i}$ is a Gorenstein flat left T-module for each $i \in \mathbb{Z}$. Moreover, the complex $\overline{\mathbf{F}_2}$ is exact, since there exists a short exact sequence of complexes

$$0 \longrightarrow U \otimes \mathbf{F}_1 \longrightarrow \mathbf{F}_2 \longrightarrow \overline{\mathbf{F}_2} \longrightarrow 0,$$

in which $U \otimes \mathbf{F}_1$ is exact by [6, Lemma 2.3], and $\mathbf{F}_2$ is exact. It is easy to see that $\mathrm{Ker}\overline{\partial_2^0} \cong \frac{M_2}{Im\varphi^M}$. Since B is right Gorenstein regular, for each injective right B-module E, E has finite flat dimensions; then we get that $\overline{\mathbf{F}_2}$ is $E \otimes -$-exact by [6, Lemma 2.3]. Thus $\frac{M_2}{\mathrm{Im}\varphi^M}$ is Gorenstein flat.

Finally, we prove that morphism φ^M is injective. By [6, Lemma 2.3], $U \otimes \mathbf{F}_1$ is an exact sequence. This means that if $\iota_1 : M_1 \to F_1^0$ is the inclusion, $1_U \otimes \iota_1$ is injective. However, since $\binom{M_1}{M_2}_{\varphi^M}$ is a submodule of $\binom{F_1^0}{F_2^0}_{\varphi^F}$, the following diagram commutes:

$$\begin{array}{ccc}
U \otimes M_1 & \xrightarrow{1_U \otimes \iota_1} & U \otimes F_1^0 \\
{\scriptstyle \varphi^M}\downarrow & & {\scriptstyle \varphi^0}\downarrow \\
M_2 & \xrightarrow{\iota_2} & F_1^0,
\end{array}$$

where ι_2 is the inclusion. Since φ^0 is injective and $\binom{F_1^0}{F_2^0}_{\varphi^F}$ is Gorenstein flat, we conclude that φ^M is injective. □

In the following we will show that the converse of Proposition 3.5 is true under some additional conditions.

Lemma 3.6 *([13]) Let* $M = \binom{M_1}{M_2}_{\varphi^M}$ *be a left* T*-module. Then* M *is finitely generated if and only if* M_1 *and* $\frac{M_2}{\mathrm{Im}\varphi^M}$ *are finitely generated.*

Lemma 3.7 *Let U_A and ${}_BU$ be finitely generated and have finite projective dimension, T be a Gorenstein ring, $F = \binom{F_1}{F_2}_{\varphi^F}$ a left T-module. If M_1 and M_2 are Gorenstein flat left A-module and left B-module, respectively, then $\mathbf{p}(M_1, M_2)$ is a Gorenstein flat left T-module.*

Proof We first note that A and B are Gorenstein by Lemma 3.1. Then it follows from [6, Theorem 10.3.8] that both M_1 and M_2 are direct limits of finitely generated Gorenstein projective modules. However, if P_1 is a finitely generated Gorenstein projective A-module, then $\binom{P_1}{U\otimes P_1}$ is a finitely generated Gorenstein projective T-module. Suppose that $M_1 = \varinjlim P_1^i$, where each module P_1^i, $i \in \mathbb{Z}$ is finitely generated Gorenstein projective; we get that $\binom{M_1}{U\otimes M_1} = \varinjlim \binom{P_1^i}{U\otimes P_1^i}$. Hence $\binom{M_1}{U\otimes M_1} \in \mathcal{GF}(T)$, reasoning as before, so is $\binom{0}{M_2}$. Then $\mathbf{p}(M_1, M_2) = \binom{M_1}{U\otimes M_1} \oplus \binom{0}{M_2}$ is a Gorenstein flat left T-module. □

Theorem 3.8 *Let U_A and ${}_BU$ be finitely generated and have finite projective dimension, T be a Gorenstein ring, $F = \binom{F_1}{F_2}_{\varphi^F}$ a left T-module. Then $F \in \mathcal{GF}(T)$ if and only if $F_1 \in \mathcal{GF}(A)$, $\frac{F_2}{\mathrm{Im}\varphi^F} \in \mathcal{GF}(B)$ and φ^F is injective.*

Proof By Proposition 3.5, it is easy to get $\frac{F_2}{\mathrm{Im}\varphi^F} \in \mathcal{GF}(B)$ and φ^F is injective.

Conversely, we assume that $F = \binom{F_1}{F_2}_{\varphi^F}$ satisfies that F_1 and $\frac{F_2}{\mathrm{Im}\varphi^F}$ are Gorenstein flat, φ^F is injective, and we want to argue that F is Gorenstein flat. By Lemma 3.8, both $\binom{F_1}{U\otimes F_1}$ and $\binom{0}{\frac{F_2}{\mathrm{Im}\varphi^F}}$ are Gorenstein flat, and there exists an short exact sequence in T-Mod

$$0 \longrightarrow \begin{pmatrix} F_1 \\ U\otimes F_1 \end{pmatrix} \longrightarrow \begin{pmatrix} F_1 \\ F_2 \end{pmatrix} \longrightarrow \begin{pmatrix} 0 \\ \frac{F_2}{Im\varphi^F} \end{pmatrix} \longrightarrow 0.$$

T is coherent since T is a Gorenstein ring, and we know, by [18, Theorem 3.7], that $F = \binom{F_1}{F_2} \in \mathcal{GF}(T)$. □

In the following we want to characterize the Gorenstein flat dimension of a module over triangular matrix rings. Using a similar way as we do in the proof of Theorem 2.5, we have the following result.

Theorem 3.9 *Let n be a nonnegative integer, U_A and ${}_BU$ be finitely generated and have finite projective dimension, T be a Gorenstein ring, and $M = \binom{M_1}{M_2}_{\varphi^M}$ be a left T-module. Then $\mathrm{Gfd}(M) \leq n$ if and only if $\mathrm{Gfd}(M_1) \leq n$, $\mathrm{Gfd}(\frac{M_2}{\mathrm{Im}\varphi^M}) \leq n$, and if $\binom{K_1}{K_2}_{\varphi^K}$ is a n-th syzygy of M, then φ^K is injective.*

Proposition 3.10 *Let U_A and ${}_BU$ be finitely generated and have finite projective dimension, and T be Gorenstein. Let $M = \binom{M_1}{M_2}_{\varphi^M}$ be a left T-module. Then M, M_1, and M_2 have finite Gorenstein flat dimension.*

Proof Note that if T is Gorenstein, every Gorenstein projective module is Gorenstein flat, and so we have $\mathrm{Gfd}(M) \leq \mathrm{Gpd}(M)$. Since T is Gorenstein, T is left Gorenstein regular and $\mathrm{glGpd}(T) < \infty$. Thus $\mathrm{Gfd}(M) < \infty$. By Lemma 3.1, A and B are Gorenstein, reasoning as before, M_1 and M_2 have finite Gorenstein flat dimension. □

4. Strongly Gorenstein homological dimensions

In this section, we present some characterizations of strongly Gorenstein projective (resp. injective, flat) modules over triangular matrix rings.

Recall that a left R-module M is *strongly Gorenstein projective* if there exists an exact sequence of projective left R-modules $\cdots \to P \xrightarrow{f} P \xrightarrow{f} P \to \cdots$ with $M \cong \mathrm{Ker} f$ such that $\mathrm{Hom}_R(-, Q)$ leaves the sequence exact for any projective left R-module Q. The strongly Gorenstein injective modules are defined dually. A left R-module M is said to be *strongly Gorenstein flat* if there exists an exact sequence $\cdots \to F \xrightarrow{g} F \xrightarrow{g} F \to \cdots$ of flat left R-modules with $M \cong \mathrm{Ker} g$ such that $I \otimes_R -$ leaves the sequence exact for any injective right R-module I. As usual, $\mathrm{SGpd}(M)$ and $\mathrm{SGid}(M)$ denote the strongly Gorenstein projective and injective dimensions of a left R-module M, respectively. Since every strongly Gorenstein projective left R-module is Gorenstein projective, by the proof of [6, Proposition 3.4] and [6, Theorem 3.5], we have the following conclusions.

Proposition 4.1 *Suppose that U_A has finite flat dimension.*

(1) If M_1 is a strongly Gorenstein projective left A-module, then $\mathbf{p}(M_1, 0)$ is a strongly Gorenstein projective left T-module.

(2) If B is left Gorenstein regular, ${}_BU$ has finite projective dimension and $(M_1, M_2) \in A\text{-Mod} \times B\text{-Mod}$ is a strongly Gorenstein projective object, then $\mathbf{p}(M_1, M_2)$ is a strongly Gorenstein projective left T-module.

Proof (1) Suppose that M_1 is strongly Gorenstein projective and let $\mathbf{P} : \cdots \to P \xrightarrow{f} P \xrightarrow{f} P \to \cdots$ be an exact sequence consisting of projective left A-modules, which is $\mathrm{Hom}_B(-, C)$-exact for each projective left A-module C and such that $\mathrm{Ker}\partial^0 \cong M_1$. By [6, Lemma 2.3], we get that the complex $U \otimes_A \mathbf{P}$ is exact in A-Mod, which implies that the complex $\mathbf{p}(\mathbf{P})$ is exact in T-Mod. Moreover, it clearly verifies that $\mathrm{Ker}\partial^0_{\mathbf{p}(\mathbf{P})} = \mathbf{p}(M_1, 0)$. Finally, if $P = \binom{P_1}{P_2}$ is a projective left T-module, then the complex $\mathrm{Hom}_T(\mathbf{p}(\mathbf{P}), P)$ is isomorphic, by adjointness, to the complex $\mathrm{Hom}_A(\mathbf{P}, P_1)$, which is exact. This means that $\mathrm{Hom}_T(\mathbf{p}(\mathbf{P}), P)$ is exact, and so $\mathbf{p}(M_1)$ is strongly Gorenstein projective.

(2) We only need to prove that both modules $\mathbf{p}(M_1, 0)$ and $\mathbf{p}(0, M_2)$ are strongly Gorenstein projective when M_1 and M_2 are strongly Gorenstein projective. By (1), $\mathbf{p}(M_1, 0)$ is strongly Gorenstein projective.

Assume that M_2 is strongly Gorenstein projective and let $\mathbf{P} : \cdots \to P \xrightarrow{f} P \xrightarrow{f} P \to \cdots$ be an exact sequence consisting of projective left B-modules that is $\mathrm{Hom}_B(,C)$-exact for each projective left B-module and such that $\mathrm{Ker}\partial^0 \cong M_2$. Then $\mathbf{p}(\mathbf{P})$ is an exact sequence of left T-modules such that $\mathrm{Ker}\partial^0 \cong \mathbf{p}(0, M_2)$. It remains to see that it is $\mathrm{Hom}_T(,C)$-exact for each projective left T-module C. Let C be a projective left T-module, and note that, as a consequence of [6, Corollary 2.3], there exists a projective object (C_1, C_2) in $A\text{-Mod} \times B\text{-Mod}$ such that $\mathbf{p}(C_1, C_2) = C$. Then $C = \binom{C_1}{(U \otimes C_1) \oplus C_2}$. Now, using adjointness, we get that the complex $\mathrm{Hom}_T(\mathbf{p}(\mathbf{P}), C)$ is isomorphic to the complex $\mathrm{Hom}_B(\mathbf{P}, U \otimes C_1) \oplus \mathrm{Hom}_B(\mathbf{P}, C_2)$. However, $\mathrm{Hom}_B(\mathbf{P}, C_2)$ is exact, since C_2 is projective. In order to see that $\mathrm{Hom}_B(\mathbf{P}, U \otimes_A C_1)$ is exact, note that $U \otimes_A C_1$ has finite projective dimension in B-Mod, since it is isomorphic to a direct sum of copies of U, and using the condition that B is left Gorenstein regular, it has finite injective dimension. Hence the exactness follows from [6, Lemma 2.4]. Consequently, $\mathrm{Hom}_T(\mathbf{P}, C)$ is exact and the result is proved. $\square$

Theorem 4.2 *Let U_A have finite flat dimension, ${}_BU$ have finite projective dimension, and B be left Gorenstein regular. Let $M = \binom{M_1}{M_2}_{\varphi^M}$ be a left T-module. Then the following are equivalent:*

(1) M is strongly Gorenstein projective;

(2) M_1 and $\frac{M_2}{\mathrm{Im}\varphi^M}$ are strongly Gorenstein projective and the φ^M is injective.

Proof (1)$\Rightarrow$ (2) It is similar to the proof of Proposition 3.5.

(2)$\Rightarrow$ (1) Assume M_1 and $\frac{M_2}{\mathrm{Im}\varphi^M}$ are strongly Gorenstein projective and φ^M is injective. Then we have an exact sequence $\mathbf{Q} : \cdots \to Q \stackrel{d'}{\to} Q \stackrel{d'}{\to} Q \to \cdots$ of projective A-modules, which is $\mathrm{Hom}_A(-, C)$-exact for any projective left A-module C and such that $M_1 \cong \mathrm{Ker}d'$. By [6, Lemma 2.3], $U \otimes \mathbf{Q}$ is exact, and so $0 \to U \otimes M_1 \to U \otimes Q \stackrel{1_U \otimes d'}{\to} U \otimes Q \to \cdots$ is exact. Since $\frac{M_2}{\mathrm{Im}\varphi^M}$ is strongly Gorenstein projective, we have an exact sequence $\mathbf{P} : \cdots \to P \stackrel{d}{\to} P \stackrel{d}{\to} P \to \cdots$ of projective B-modules, which is $\mathrm{Hom}_B(-, C)$-exact for any projective left B-module C and such that $\frac{M_2}{\mathrm{Im}\varphi^M} \cong \mathrm{Ker}d$. Since ${}_BU$ has finite projective dimension, it follows from [3, Proposition 2.9] that $\mathrm{Ext}^1_B(\mathrm{Ker}d, U) = 0$. Since Q is a projective A-module, $\mathrm{Ext}^1_B(\mathrm{Ker}d, U \otimes Q) = 0$. Applying [20, Lemma 1.6(1)] to exact sequence $0 \longrightarrow U \otimes M_1 \longrightarrow M_2 \longrightarrow \frac{M_2}{\mathrm{Im}\varphi^M} \longrightarrow 0$, we obtain an exact sequence

$$0 \longrightarrow M_2 \longrightarrow (U \otimes Q) \oplus P \stackrel{\partial}{\longrightarrow} (U \otimes Q) \oplus P \longrightarrow \cdots \tag{1}$$

with $\partial = \begin{pmatrix} d & 0 \\ \sigma & 1 \otimes d' \end{pmatrix}$, $\sigma : P \to U \otimes M$. Applying Generalized Horseshoe Lemma [20, Lemma 1.6] to exact sequence $0 \longrightarrow U \otimes M_1 \longrightarrow M_2 \longrightarrow \frac{M_2}{\mathrm{Im}\varphi^M} \longrightarrow 0$, the exact sequence $\cdots \to U \otimes Q \stackrel{1_U \otimes d'}{\to} U \otimes Q \to U \otimes M_1 \to 0$, and $\cdots \to P \stackrel{d}{\to} P \to \frac{M_2}{\mathrm{Im}\varphi^M} \to 0$, we obtain another exact sequence

$$\cdots \longrightarrow (U \otimes Q) \oplus P \stackrel{\partial}{\longrightarrow} (U \otimes Q) \oplus P \longrightarrow M_2 \longrightarrow 0. \tag{2}$$

Putting (1) and (2) together we get the following exact sequence of projective T-modules

$$\mathbf{L} : \cdots \longrightarrow \begin{pmatrix} Q \\ (U \otimes Q) \oplus P \end{pmatrix} \xrightarrow{\binom{d'}{\partial}} \begin{pmatrix} Q \\ (U \otimes Q) \oplus P \end{pmatrix} \xrightarrow{\binom{d'}{\partial}} \begin{pmatrix} Q \\ (U \otimes Q) \oplus P \end{pmatrix} \longrightarrow \cdots$$

with $\mathrm{Ker}\binom{d'}{\partial} \cong M$. In fact, $\mathbf{L} = \mathbf{p}(\mathbf{Q}, \mathbf{P})$. Let C be a projective left T-module. As a consequence of [5, Corollary 2.3], there exists a projective object (C_1, C_2) in A-Mod$\times B$-Mod such that $\mathbf{p}(C_1, C_2) = C$. Then $C = \binom{C_1}{(U \otimes C_1) \oplus C_2}$. Now, using adjointness, we get that the complex $\mathrm{Hom}_T(\mathbf{p}(\mathbf{Q}, \mathbf{P}), C)$ is isomorphic to the complex $\mathrm{Hom}_B(\mathbf{Q}, C_1) \oplus \mathrm{Hom}_B(\mathbf{P}, U \otimes C_1) \oplus \mathrm{Hom}_B(\mathbf{P}, C_2)$. However, $\mathrm{Hom}_B(\mathbf{Q}, C_1)$ and $\mathrm{Hom}_B(\mathbf{P}, C_2)$ are exact, since C_1 and C_2 are projective. In order to see that $\mathrm{Hom}_B(\mathbf{P}, U \otimes_A C_1)$ is exact, note that $U \otimes_A C_1$ has finite projective dimension in B-Mod, since it is isomorphic to a direct sum of copies of U, and using the condition that B is left Gorenstein regular, it has finite injective dimension. Then the exactness follows from [6, Lemma 2.4]. Consequently, $\mathrm{Hom}_T(\mathbf{L}, C)$ is exact and the result is proved. $\square$

By analogous arguments, we have the following results.

Proposition 4.3 *Suppose that ${}_BU$ has finite flat dimension.*

(1) If M_2 is a strongly Gorenstein injective left B-module, then $\mathbf{h}(0, M_2)$ is a strongly Gorenstein injective left T-module.

(2) If A is left Gorenstein regular, U_A has finite flat dimension, and $(M_1, M_2) \in A\text{-Mod} \times B\text{-Mod}$ is a strongly Gorenstein injective object, then $\mathbf{h}(M_1, M_2)$ is a strongly Gorenstein projective left T-module.

Proposition 4.4 *Suppose that U_A has finite flat dimension, ${}_BU$ has finite projective dimension, and A is left Gorenstein regular. Let $M = \binom{M_1}{M_2}_{\varphi^M}$ be a left T-module. Then the following are equivalent:*

(1) *M is strongly Gorenstein injective;*

(2) *M_2 and $\mathrm{Ker}\widetilde{\varphi^M}$ are strongly Gorenstein injective and the $\widetilde{\varphi^M}$ is a surjection.*

Proposition 4.5 *Let U_A and ${}_BU$ have finite projective dimension, T be a right Gorenstein regular ring, and $M = \binom{M_1}{M_2}_{\varphi^M}$ a left T-module. If M is strongly Gorenstein flat, then M_1 and $\frac{M_2}{\mathrm{Im}\varphi^M}$ are strongly Gorenstein flat and the morphism φ^M is injective.*

Proof It is similar to the proof of Proposition 3.5. □

One can prove the following argument in a similar way as we do in the proof of Theorem 2.5.

Theorem 4.6 *Let n be a non-negative integer, U_A have finite flat dimension, ${}_BU$ have finite projective dimension and B be left Gorenstein regular. Let $M = \binom{M_1}{M_2}_{\varphi^M}$ be a left T-module. Then $\mathrm{SGpd}(M) \leq n$ if and only if $\mathrm{SGpd}(M_1) \leq n$, $\mathrm{SGpd}(\frac{M_2}{\mathrm{Im}\varphi^M}) \leq n$ and if $\binom{K_1}{K_2}_{\varphi^K}$ is a n-th syzygy of M, then φ^K is injective.*

Theorem 4.7 *Let n be a nonnegative integer, U_A have finite flat dimension, ${}_BU$ have finite projective dimension, and A be left Gorenstein regular. Let $M = \binom{M_1}{M_2}_{\varphi^M}$ be a left T-module. Then $\mathrm{SGid}(M) \leq n$ if and only if $\mathrm{SGid}(M_2) \leq n$, $\mathrm{SGid}(\mathrm{Ker}\widetilde{\varphi^M}) \leq n$ and if $\binom{L_1}{L_2}_{\varphi^L}$ is a n-th cosyzygy of M, then $\widetilde{\varphi^L}$ is surjective.*

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