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An extension of Cline's formula for a generalized Drazin inverse

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Abstract: In this note we give an answer to a question recently posed by Zeng and Zhong, to note that Cline's formula for a generalized Drazin inverse extends to the case when $aba = aca$. Cline's formula for a pseudo Drazin inverse is also presented in this case.

Key words: Cline's formula, generalized Drazin inverse, pseudo Drazin inverse, quasinilpotent

1. Introduction

Let R be an associative ring with identity 1, and R^{-1} and $J(R)$ denote, respectively, *invertible group* and *Jacobson radical* of R. For $a \in R$, the *commutant* and *double commutant* of a are defined by

$$comm(a) = \{x \in R, ax = xa\}$$

and

$$comm^2(a) = \{x \in R,\ xy = yx \text{ for all } y \in comm(a)\}.$$

Drazin [3] introduced the notion of a Drazin inverse in a ring in 1958. An element $a \in R$ is said to be *Drazin invertible* if there exist $b \in R$ and $k \in \mathbb{N}$ such that

$$b \in comm(a),\ bab = b \text{ and } a^k ba = a^k.$$

In this case b is unique if it exists and is called a *Drazin inverse* of a, denoted by $b = a^D$, and the least nonnegative integer k satisfying $a^k ba^k = a^k$ is called the *Drazin index* $i(a)$ of a. According to Drazin [[3], Theorem 1], $a^D \in comm^2(a)$.

Following Harte [4], an element $a \in R$ is said to be *quasinilpotent* if $1 + ax$ is invertible for all $x \in comm(a)$. Using this concept, Koliha and Patrício [6] introduced the notion of a generalized Drazin inverse in a ring in 2002. An element $a \in R$ is said to be *generalized Drazin invertible* if there exists $b \in R$ such that

$$b \in comm^2(a),\ bab = b \text{ and } aba - a \text{ is quasinilpotent.} \tag{1.1}$$

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In this case b is unique if it exists and is called a *generalized Drazin inverse* of a, denoted by $b = a^{gD}$. Moreover, in the Banach algebra case, the condition $b \in comm^2(a)$ in (1.1) can be weaken as $b \in comm(a)$ (see [[5], Theorem 4.4]).

In 2012, Wang and Chen [9] introduced an intermedium between Drazin inverse and generalized Drazin inverse. An element $a \in R$ is said to be *pseudo Drazin invertible* if there exist $b \in R$ and $k \in \mathbb{N}$ such that

$$b \in comm^2(a), \; bab = b \text{ and } a^k ba - a^k \in J(R). \tag{1.2}$$

In this case b is unique if it exists and is called a *pseudo Drazin inverse* of a, denoted by $b = a^{pD}$, and the least nonnegative integer k satisfying $a^k ba - a^k \in J(R)$ is called the *pseudo Drazin index* $i(a)$ of a. Moreover, in the Banach algebra case, the condition $b \in comm^2(a)$ in (1.2) can be weakened as $b \in comm(a)$ (see [[9], Remark 5.1]).

In 1965, Cline [1] showed that if ab is Drazin invertible then so is ba and in this case

$$(ba)^D = b((ab)^D)^2 a.$$

This equation is now known as Cline's formula. It plays an important role in finding the Drazin inverse of a sum of two elements and that of a block matrix (see [8]). Generalizations of Cline's formula for generalized Drazin inverse and pseudo Drazin inverse were recently proved in [7] and [9], respectively. Their proof relied on the bridge "quasipolar" and "pseudopolar", respectively.

As extensions of Jacobson's lemma, in 2013 Corach et al. [2]] firstly investigated common properties of $ac - 1$ and $ba - 1$ in the algebraic viewpoint and also obtained some interesting topological analogues under the assumption

$$aba = aca,$$

where $a, b, c \in R$. Recently, Zeng and Zhong [10] extended Cline's formula for the Drazin inverse in a ring to the case when $aba = aca$. However, Cline's formula for a generalized Drazin inverse in this case was established only in the setting of Banach algebra, and in a ring it left open at that time.

In this note, we establish Cline's formula for the generalized Drazin inverse in a ring in the case when $aba = aca$, answering a question posed in [[10], Question 2.15]. We also present Cline's formula for the pseudo Drazin inverse in this case. The proofs for the general case are more direct and slightly more technical than those in [7] and/or [9] for the special case $b = c$.

2. Main results

We start with the following well-known Jacobson's lemma.

Lemma 2.1 (Jacobson's lemma) *Let $a, b \in R$. Then*

$$1 + ab \text{ is invertible} \Longleftrightarrow 1 + ba \text{ is invertible.}$$

In this case, we have $(1 + ba)^{-1} = 1 - b(1 + ab)^{-1}a$.

Lemma 2.2 *Suppose that $a, b, c \in R$ satisfy $aba = aca$. Then*

$$ac \text{ is quasinilpotent} \Longleftrightarrow ba \text{ is quasinilpotent.}$$

Proof Suppose that ac is quasinilpotent. Then for all $x \in comm(ac)$, $1+xac$ is invertible. Let $y \in comm(ba)$. Since

$$(ay^3bac)(ac) = (ay^3bab)(ac) = (abay^3b)(ac) = (ac)(ay^3bac),$$

$1 + (ay^3bac)(ac)$ is invertible. Therefore, by Lemma 2.1, we have

$$(1 + yba)(1 - yba + y^2baba) = 1 + y^3bababa = 1 + y^3bacaca$$

is invertible. Noting that $1+yba$ commutes with $1-yba+y^2baba$, we see that $1+yba$ is invertible. Consequently, ba is quasinilpotent.

For the special case "$b = c$", the previous paragraph shows that if ab is quasinilpotent, then ba is quasinilpotent. Then interchanging a and b, we find an equivalence statement: ab is quasinilpotent if and only if ba is quasinilpotent, for any $a, b \in R$.

Now suppose that ba is quasinilpotent. Then ab is quasinilpotent. Interchanging b and c in the first paragraph of this proof, we infer that ca is quasinilpotent. Therefore, ac is quasinilpotent. □

We are now ready to answer Question 2.15 in [10] affirmatively.

Theorem 2.3 *Suppose that $a, b, c \in R$ satisfy $aba = aca$. Then*

$$ac \text{ is generalized Drazin invertible} \Longleftrightarrow ba \text{ is generalized Drazin invertible.}$$

In this case, we have $(ba)^{gD} = b((ac)^{gD})^2a$ and $(ac)^{gD} = a((ba)^{gD})^2c$.

Proof Suppose that ac is generalized Drazin invertible and let $d = (ac)^{gD}$. Then

$$d \in comm^2(ac),\ d(ac)d = d \text{ and } (ac)d(ac) - ac \text{ is quasinilpotent.}$$

Put

$$e = bd^2a.$$

In order to prove that $e = (ba)^{gD}$, it needs to be shown that

$$\text{(i) } e \in comm^2(ba),\ \text{(ii) } e(ba)e = e \text{ and (iii) } (ba)e(ba) - ba \text{ is quasinilpotent.}$$

(i) Let $f \in comm(ba)$. Then we have

$$fe = fbd^2a = fb(acacd^4)a = fbabacd^4a = babafcd^4a = b(acafc)d^4a. \tag{2.1}$$

Since

$$ac(acafc) = abab afc = afbabac = afbacac = abafcac = (acafc)ac$$

and $d \in comm^2(ac)$,

$$(acafc)d = d(acafc). \tag{2.2}$$

Therefore, putting (2.2) into (2.1), we get

$$\begin{aligned} fe &= b(acafc)d^4a = bd^4(acafc)a \\ &= bd^4abafca = bd^4afbaca \\ &= bd^4afbaba = bd^4ababaf \\ &= bd^4acacaf = bd^2af \\ &= ef. \end{aligned}$$

(ii) We have $e(ba)e = bd^2a(ba)bd^2a = bd^2abababacd^3a = bd^2(ac)^3d^3a = bd^2a = e$.

(iii) Let $p = 1 - acd$. Then pac is quasinilpotent. Note that

$$\begin{aligned} ba - (ba)^2e &= ba - bababd^2a = ba - babababacd^2da \\ &= ba - bacacacd^2da = b(1 - acd)a \\ &= bpa. \end{aligned}$$

Since $abpa = ab(1 - acd)a = ac(1 - acd)a = acpa$, $(pa)b(pa) = (pa)c(pa)$. Therefore, by Lemma 2.2, we conclude that $ba - (ba)^2e$ is quasinilpotent. Consequently,

$$(ba)^{gD} = b((ac)^{gD})^2a. \tag{2.3}$$

For the special case "$b = c$", (2.3) shows that

$$(ba)^{gD} = b((ab)^{gD})^2a \text{ and } (ca)^{gD} = c((ac)^{gD})^2a,$$

for any $a, b, c \in R$. Then interchanging a and b, and a and c in the above formulae, we find that

$$(ab)^{gD} = a((ba)^{gD})^2b \text{ and } (ac)^{gD} = a((ca)^{gD})^2c. \tag{2.4}$$

Now interchanging b and c in (2.3), we infer that

$$(ca)^{gD} = c((ab)^{gD})^2a. \tag{2.5}$$

Therefore, taking (2.4) and (2.5) together, we get

$$\begin{aligned} (ac)^{gD} &= a((ca)^{gD})^2c = ac((ab)^{gD})^2ac((ab)^{gD})^2ac \\ &= aca((ba)^{gD})^2ba((ba)^{gD})^2baca((ba)^{gD})^2ba((ba)^{gD})^2bac \\ &= a((ba)^{gD})^2c. \end{aligned}$$

This completes the proof. □

Theorem 2.4 *Suppose that $a, b, c \in R$ satisfy $aba = aca$. Then*

$$ac \text{ is pseudo Drazin invertible} \Longleftrightarrow ba \text{ is pseudo Drazin invertible.}$$

In this case, we have

(1) $|i(ac) - i(ba)| \leq 1$;

(2) $(ba)^{pD} = b((ac)^{pD})^2a$ *and* $(ac)^{pD} = a((ba)^{pD})^2c$.

Proof Let d be the pseudo Drazin inverse of ac and let $k = i(ac)$. Then

$$d \in comm^2(ac),\ d(ac)d = d \text{ and } (ac)^kd(ac) - (ac)^k \in J(R).$$

Put

$$e = bd^2a.$$

As in the proof of Theorem 2.3, we get $e \in comm^2(ba)$ and $e(ba)e = e$. Moreover, since $(ac)^k d(ac) - (ac)^k \in J(R)$,

$$\begin{aligned}(ba)^{k+1}e(ba) - (ba)^{k+1} &= (ba)^{k+1}bd^2aba - (ba)^{k+1} \\ &= (ba)^{k+1}bd^2aca - b(ac)^k a \\ &= (ba)^{k+1}bacd^2a - b(ac)^k a \\ &= b(ac)^{k+1}acd^2a - b(ac)^k a \\ &= b(ac)^{k+1}da - b(ac)^k a \\ &= b((ac)^k d(ac) - (ac)^k)a \in J(R).\end{aligned}$$

Therefore, ba is pseudo Drazin invertible, $(ba)^{pD} = b((ac)^{pD})^2 a$, and $i(ba) \leq i(ac) + 1$.

By similar arguments as above, one can show that if ba is pseudo Drazin invertible, then ac is pseudo Drazin invertible, $(ac)^{pD} = a((ba)^{pD})^2 c$, and $i(ac) \leq i(ba) + 1$. □

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