Shellability of simplicial complexes and simplicial complexes with the free vertex property

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Abstract: To a simplicial complex Δ, we associate a square-free monomial ideal $\mathcal{F}(\Delta)$ in the polynomial ring generated by its facet over a field. Furthermore, we could consider $\mathcal{F}(\Delta)$ as the Stanley–Reisner ideal of another simplicial complex $\delta_N(\mathcal{F}(\Delta))$ from facet ideal theory and Stanley–Reisner theory. In this paper, we determine what families of simplicial complexes Δ have the property that their Stanley–Reisner complexes $\delta_N(\mathcal{F}(\Delta))$ are shellable. Furthermore, we show that the simplicial complex with the free vertex property is sequentially Cohen–Macaulay. This result gives a new proof for a result of Faridi on the sequentially Cohen–Macaulayness of simplicial forests.

Key words: Simplicial complex, Stanley–Reisner ring, shellability, sequentially Cohen–Macaulay

1. Introduction

From the point of view of commutative algebra, the focus of this paper is on finding squarefree monomial ideals that have sequentially Cohen–Macaulay quotients. Sequentially Cohen–Macaulay modules were introduced by Stanley [13] in connection with the work of Björner and Wachs [1] on nonpure shellability. Pure shellable simplicial complexes are Cohen–Macaulay, and Stanley identified sequentially Cohen–Macaulay as the appropriate analog in the nonpure setting; that is, all nonpure shellable simplicial complexes are sequentially Cohen–Macaulay. Recently, a number of authors have been interested in classifying or identifying (sequentially) Cohen–Macaulay graphs or Cohen–Macaulay simplicial complexes in terms of the combinatorial properties of graphs or simplicial complexes. For example, Duval in [3] showed that algebraic shifting preserves the h-triangle of a simplicial complex Δ if and only if Δ is sequentially Cohen–Macaulay; Herzog et al. [11] proved that a chordal graph is Cohen–Macaulay if and only if it is unmixed; Francisco and Tuyl [8] showed that all chordal graphs G are sequentially Cohen–Macaulay; Tuyl and Villarreal [14] classified some of the sequentially Cohen–Macaulay bipartite graphs; Faridi [5] showed that simplicial trees are sequentially Cohen–Macaulay; and Francisco and Hà [7] gave various sufficient and necessary conditions on a subset S of the vertices of G such that the graph $G \cup W(S)$, obtained from G by adding a whisker to each vertex in S, is a sequentially Cohen–Macaulay graph. Because a shellable simplicial complex has the property that its associated Stanley–Reisner ring is sequentially Cohen–Macaulay, our goal is, by identifying shellable simplicial complexes, to characterize some of the sequentially Cohen–Macaulay simplicial complexes.

The paper is organized as follows: Section 2 reviews some definitions and basic facts about simplicial complexes. In Section 3, we introduce the definition of shellable simplicial complexes and identify some families of shellable Stanley–Reisner complexes of the facet ideals of simplicial complexes. In Section 4, we introduce the notion of a simplicial complex with the free vertex property and show that such a simplicial complex is sequentially Cohen–Macaulay. We recover as a corollary the fact that all simplicial forests are sequentially Cohen–Macaulay. This result was first proved by Faridi [5].

2. Preliminaries

We first recall some definitions and basic facts about simplicial complexes and their edge ideal to make this paper self-contained. However, for more details of the notions, we refer the reader to [2, 4, 5, 6, 9, 10, 13, 14].

Definition 2.1 *A simplicial complex Δ on a set of vertices V is a collection of subsets of V, with the property that for any $x \in V$, $\{x\} \in \Delta$, and if $F \in \Delta$, then all subsets of F are also in Δ (including the empty set). An element of Δ is called a face of Δ, and the dimension of a face F of Δ is defined as $|F| - 1$, where $|F|$ is the number of vertices of F (by convention, $\dim \emptyset = -1$). The maximal faces of Δ under inclusion are called facets of Δ. The dimension of the simplicial complex Δ is the maximal dimension of its facets. Δ is called pure if all of its facets have the same dimension; otherwise, Δ is nonpure. If $F_1, \ldots, F_q$ is a complete list of the facets of Δ, we sometimes write Δ as $\Delta = \langle F_1, \ldots, F_q \rangle$. A simplicial complex with only one facet is called a simplex. A subcollection of Δ is a simplicial complex whose facets are also facets of Δ.*

Definition 2.2 *A simplicial complex $\Delta = \langle F_1, \ldots, F_q \rangle$ is connected if for every pair i, j, $1 \le i < j \le q$, there exists a sequence of facets $F_{t_1}, \cdots, F_{t_r}$ of Δ such that $F_{t_1} = F_i, F_{t_r} = F_j$ and $F_{t_s} \cap F_{t_{s+1}} \ne \emptyset$ for $s = 1, \ldots, r - 1$.*

Definition 2.3 *A facet F of a simplicial complex Δ is called a leaf if either F is the only facet of Δ or there exists a facet $G \in \Delta \setminus \langle F \rangle$, such that $F \cap F' \subseteq F \cap G$ for every facet $F' \in \Delta \setminus \langle F \rangle$. Such a facet G is called a joint of F.*

A simplicial complex Δ is a simplicial forest if every nonempty subcollection of Δ has a leaf. A connected simplicial forest is called a simplicial tree.

Throughout this paper, let k be any field, $x_1, \ldots, x_n$ be indeterminates, and R be the polynomial ring $k[x_1, \ldots, x_n]$. By abuse of notation, we often use $x_1, \ldots, x_n$ to denote both the vertices of Δ and the variables appearing in $\mathcal{F}(\Delta)$, and $x_{i_1} \cdots x_{i_s}$ to denote a facet of Δ as well as a monomial generator of $\mathcal{F}(\Delta)$.

To a squarefree monomial ideal I in R, one can associate two simplicial complexes $\delta_{\mathcal{F}}(I)$ and $\delta_{\mathcal{N}}(I)$ on the vertex set $\{x_1, \ldots, x_n\}$. Conversely, given a simplicial complex Δ with vertex set $\{x_1, \ldots, x_n\}$, one can associate two squarefree monomial ideals $\mathcal{F}(\Delta)$ and $\mathcal{N}(\Delta)$ in the polynomial ring $k[x_1, \ldots, x_n]$, and these are all defined below:

Facet complex of I:

$$\delta_{\mathcal{F}}(I) = \langle \{x_{i_1}, \ldots, x_{i_s}\} \mid x_{i_1} \cdots x_{i_s} \text{ is a minimal generator of } I \rangle,$$

Stanley–Reisner complex of I:

$$\delta_{\mathcal{N}}(I) = \langle \{x_{i_1}, \ldots, x_{i_s}\} \mid x_{i_1} \cdots x_{i_s} \notin I \rangle,$$

Facet ideal of Δ:

$$\mathcal{F}(\Delta) = (x_{i_1} \cdots x_{i_s} \mid \{x_{i_1}, \ldots, x_{i_s}\} \text{ is a facet of } \Delta),$$

Stanley–Reisner ideal of Δ

$$\mathcal{N}(\Delta) = (x_{i_1} \cdots x_{i_s} \mid \{x_{i_1}, \ldots, x_{i_s}\} \notin \Delta).$$

Definition 2.4 *Suppose that Δ is a simplicial complex and $\sigma \in \Delta$; the deletion of σ from Δ is the simplicial complex defined by*

$$\Delta \setminus \sigma = \{\tau \in \Delta \mid \sigma \nsubseteq \tau\},$$

and when $\sigma = \{v\}$, we shall abuse notation and write $\Delta \setminus v$ for $\Delta \setminus \{v\}$.

The link of σ is defined to be

$$lk_\Delta(\sigma) = \{\tau \in \Delta \mid \sigma \cup \tau \in \Delta, \sigma \cap \tau = \emptyset\},$$

and when $\sigma = \{v\}$, then we shall abuse notation and write $lk_\Delta(v)$ for $lk_\Delta(\{v\})$.

If $\Delta = \langle F_1, \ldots, F_q \rangle$, $\mathcal{F}(\Delta) = (M_1, \ldots, M_q)$ is its facet ideal in $R = k[x_1, \ldots, x_n]$. The simplicial complex obtained by removing the facet F_i from Δ is the simplicial complex

$$\Delta \setminus \langle F_i \rangle = \langle F_1, \ldots, \hat{F}_i, \ldots, F_q \rangle.$$

Note that $\mathcal{F}(\Delta \setminus \langle F_i \rangle) = \langle M_1, \ldots, \hat{M}_i, \ldots, M_q \rangle$ and the vertex set of $\Delta \setminus \langle F_i \rangle$ is a subset of the vertex set of Δ.

Definition 2.5 *Let Δ be a simplicial complex with vertex set V. A vertex cover for Δ is a subset A of V that intersects every facet of Δ. If A is a vertex cover, and no proper subset of A is a vertex cover for Δ, it is called a minimal vertex cover of Δ.*

Using the minimal vertex covers of a given simplicial complex, we can construct a new simplicial complex.

Definition 2.6 *Given a simplicial complex Δ, the simplicial complex Δ_M whose facets are the minimal vertex covers of Δ is called the cover complex of Δ.*

Remark 2.7 *Suppose that Δ is a simplicial complex; by Proposition 2.4 of [5], we obtain that F is a facet of the Stanley–Reisner complex $\delta_\mathcal{N}(\mathcal{F}(\Delta))$ of the facet ideal $\mathcal{F}(\Delta)$ of Δ if and only if $V \setminus F$ is a facet of the cover complex Δ_M of Δ.*

3. Shellable simplicial complexes

In this section, we introduce the notions of shellable simplicial complexes. Given a simplicial complex Δ, we describe some properties of the Stanley–Reisner complex $\delta_\mathcal{N}(\mathcal{F}(\Delta))$ of its facet ideal $\mathcal{F}(\Delta)$, and we identify some families of shellable simplicial complexes.

Definition 3.1 *A simplicial complex Δ is shellable if the facets of Δ can be ordered $F_1, \ldots, F_s$ such that for all $1 \leq i < j \leq s$, there exists some $v \in F_j \setminus F_i$ and some $l \in \{1, \ldots, j-1\}$ with $F_j \setminus F_l = \{v\}$. We call $F_1, \ldots, F_s$ a shelling of Δ when the facets have been ordered with respect to the shellable definition. For a fixed shelling of Δ, if $F, F' \in \Delta$ then we write $F < F'$ to mean that F appears before F' in the ordering.*

Remark 3.2 *The above definition of shellable is due to Björner and Wachs [1] and is usually referred to as nonpure shellable; in this paper, we will drop the adjective "nonpure". If the simplicial complex Δ is pure and satisfies the above definition of shellable, we will say Δ is pure shellable.*

To prove that the Stanley–Reisner complex $\delta_N(\mathcal{F}(\Delta))$ of the facet ideal $\mathcal{F}(\Delta)$ of a simplicial complex Δ is shellable, it suffices to prove that each connected component of Δ is shellable, as demonstrated below.

Proposition 3.3 *Let $\Delta_1 = \langle F_1, \ldots, F_p \rangle$, $\Delta_2 = \langle F_{p+1}, \ldots, F_q \rangle$ be two simplicial complexes with sets of vertices V_1 and V_2 respectively such that $V_1 \cap V_2 = \emptyset$, and let Δ be the simplicial complex whose facets are $F_1, \ldots, F_p, F_{p+1}, \ldots, F_q$. Then $\delta_N(\mathcal{F}(\Delta_1))$ and $\delta_N(\mathcal{F}(\Delta_2))$ are shellable if and only if $\delta_N(\mathcal{F}(\Delta))$ is shellable.*

Proof $(\Rightarrow)$ Let $G_1, \ldots, G_r$ and $H_1, \ldots, H_s$ be the shellings of $\delta_N(\mathcal{F}(\Delta_1))$ and $\delta_N(\mathcal{F}(\Delta_2))$ respectively. We first prove that the facets of $\delta_N(\mathcal{F}(\Delta))$ are $G_1 \cup H_1$, $\ldots, G_1 \cup H_s, G_2 \cup H_1, \ldots, G_2 \cup H_s, \ldots, G_r \cup H_1, \ldots, G_r \cup H_s$. Indeed, $G_1, \ldots, G_r$ are all the facets of $\delta_N(\mathcal{F}(\Delta_1))$, and by Remark 2.7, we have that $V_1 \setminus G_1, \ldots, V_1 \setminus G_r$ are all of the minimal vertex covers of Δ_1. Similarly, $V_2 \setminus H_1, \ldots, V_2 \setminus H_s$ are also all of the minimal vertex covers of Δ_2. Thus, we get that $(V_1 \cup V_2) \setminus (G_i \cup H_j)$, $i = 1, \ldots, r$, $j = 1, \ldots, s$ are all of the minimal vertex covers of Δ. Again by Remark 2.7, this claim is true.

If we order the facets of $\delta_N(\mathcal{F}(\Delta))$ as

$$G_1 \cup H_1, \ldots, G_1 \cup H_s, G_2 \cup H_1, \ldots, G_2 \cup H_s, \ldots, G_r \cup H_1, \ldots, G_r \cup H_s,$$

then we get a shelling of $\delta_N(\mathcal{F}(\Delta))$. Indeed, if $F' < F$ are two facets of $\delta_N(\mathcal{F}(\Delta))$, we have two cases to consider. Case (i): $F' = G_i \bigcup H_k$, and $F = G_j \cup H_t$, where $i < j$. Because $\delta_N(\mathcal{F}(\Delta_1))$ is shellable there is some $x \in G_j \setminus G_i$ and some $l < j$ with $G_j \setminus G_l = \{x\}$. Hence, $x \in F \setminus F', G_l \cup H_t < F$, and $F \setminus (G_l \cup H_t) = \{x\}$. Case (ii): $F' = G_i \cup H_k$, and $F = G_i \cup H_t$, where $k < t$. This case follows from the shellability of $\delta_N(\mathcal{F}(\Delta_2))$.

$(\Leftarrow)$ Note that if F is a facet of $\delta_N(\mathcal{F}(\Delta))$, then $F' = F \cap V_1$, respectively, $F'' = F \cap V_2$ is the facet of $\delta_N(\mathcal{F}(\Delta_1))$, respectively, $\delta_N(\mathcal{F}(\Delta_2))$. We now show that $\delta_N(\mathcal{F}(\Delta_1))$ is shellable and omit the similar proof for the shellability of $\delta_N(\mathcal{F}(\Delta_2))$. Let $F_1, \ldots, F_t$ be a shelling of $\delta_N(\mathcal{F}(\Delta))$, and consider the subsequence

$$F_{i_1}, \ldots, F_{i_s} \text{ with } 1 \leq i_1 < i_2 < \cdots < i_s \leq t$$

where $F_1 \cap V_2 = F_{i_j} \cap V_2$, for $\forall i_j \in \{i_1, \ldots, i_s\}$, but $F_1 \cap V_2 \neq F_k \cap V_2$ for any $k \in \{1, 2, \ldots, t\} \setminus \{i_1, \ldots, i_s\}$.

We then claim that
$$F_1' = F_{i_1} \cap V_1, F_2' = F_{i_2} \cap V_1, \ldots, F_s' = F_{i_s} \cap V_1$$

is a shelling of $\delta_N(\mathcal{F}(\Delta_1))$. We first show that this is a complete list of facets of $\delta_N(\mathcal{F}(\Delta_1))$. Indeed, every $F_j' = F_{i_j} \cap V_1$ is a facet of $\delta_N(\mathcal{F}(\Delta_1))$, and furthermore, for any facet $F' \in \delta_N(\mathcal{F}(\Delta_1))$, by the argument of necessity above, we have that $F' \cup (F_1 \cap V_2)$ is a facet of $\delta_N(\mathcal{F}(\Delta))$, and hence $F' \cup (F_1 \cap V_2) = F_{i_j}$ for some $i_j \in \{i_1, \ldots, i_s\}$.

From the fact that $F_1, \ldots, F_t$ is a shelling of $\delta_N(\mathcal{F}(\Delta))$ and for $1 \leq k < j \leq s$, $F_{i_j} \cap V_2 = F_{i_k} \cap V_2 = F_1 \cap V_2$, there exists some $x \in F_{i_j} \setminus F_{i_k} = (F_{i_j} \cap V_1) \setminus (F_{i_k} \cap V_1) = F_j' \setminus F_k'$ such that $\{x\} = F_{i_j} \setminus F_l$ for some $1 \leq l < i_j$. It suffices to show that F_l is among $F_{i_1}, \ldots, F_{i_s}$. Now because $F_{i_j} \cap V_2 \subseteq F_{i_j}$ and $x \notin F_{i_j} \cap V_2$, we must have $F_{i_j} \cap V_2 \subseteq F_l$. Thus, $F_{i_j} \cap V_2 \subseteq F_l \cap V_2$, but both $F_{i_j} \cap V_2$ and $F_l \cap V_2$ are facets of $\delta_N(\mathcal{F}(\Delta_2))$, and we must have $F_{i_j} \cap V_2 = F_l \cap V_2$. Therefore, $F_l = F_{i_r}$ for some $r < j$, and hence, $\{x\} = F_j' \setminus F_r'$. $\qquad\square$

Proposition 3.4 *Let Δ be a connected simplicial complex on a vertex set V and $x \in V$, and let $F_1, \ldots, F_t$ be all the facets of Δ containing x. Set $\Delta_1 = \Delta \setminus x$ and $\Delta_2 = \Delta \setminus \langle F_1, \ldots, F_t \rangle$. Then:*

(1) $\delta_N(\mathcal{F}(\Delta_1)) = lk_{\delta_N(\mathcal{F}(\Delta))}(x)$;   (2) $\delta_N(\mathcal{F}(\Delta)) \setminus x = \delta_N(\mathcal{F}(\Delta_2)) \setminus x$.

Proof

(1) Let F be any facet of $\delta_N(\mathcal{F}(\Delta_1))$, and then by Remark 2.7, we have that $V \setminus F$ is a minimal vertex cover of Δ_1. We claim that $V \setminus F$ is also a minimal vertex cover of Δ. Indeed, for $i = 1, \ldots, t$, set $F_i = \{x\} \cup B_i$, we have that $B_i \in \Delta_1$. Thus, $(V \setminus F) \cap B_i \neq \emptyset$, and furthermore $(V \setminus F) \cap F_i \neq \emptyset$. Note that x is not the vertex of Δ_1, and hence $x \notin V \setminus F$, i.e. $x \in F$. It is obvious that $(V \setminus F) \cup \{x\} = V \setminus (F \setminus \{x\})$ is a vertex cover of Δ, and then $F \setminus \{x\} \in \delta_N(\mathcal{F}(\Delta))$. Hence, $F = (F \setminus \{x\}) \cup \{x\} \in lk_{\delta_N(\mathcal{F}(\Delta))}(x)$. Conversely, let F' be any facet of $lk_{\delta_N(\mathcal{F}(\Delta))}(x)$, and then F' is a face of $\delta_N(\mathcal{F}(\Delta))$ and $F' \cup \{x\}$ is a facet of $\delta_N(\mathcal{F}(\Delta))$. Then, by Remark 2.7, $V \setminus F'$ is a vertex cover of Δ and $V \setminus (F' \cup \{x\})$ is a minimal vertex cover of Δ. However, $x \notin V \setminus (F' \cup \{x\})$, and we have that $V \setminus (F' \cup \{x\})$ is a vertex cover of Δ_1. Again by Remark 2.7, $F' \cup \{x\} \in \delta_N(\mathcal{F}(\Delta_1))$. Hence, $F' \in \delta_N(\mathcal{F}(\Delta_1))$.

(2) Take any facet $F \in \delta_N(\mathcal{F}(\Delta)) \setminus x$, and then, by Remark 2.7, we get that $V \setminus F$ is a minimal vertex cover of Δ and $x \in V \setminus F$. Thus, $(V \setminus F) \setminus x = V \setminus (F \cup \{x\})$ is a minimal vertex cover of Δ_2. Hence, $F \cup \{x\} \in \delta_N(\mathcal{F}(\Delta_2))$. Furthermore, we have that $F \in \delta_N(\mathcal{F}(\Delta_2))$ and hence $F \in \delta_N(\mathcal{F}(\Delta_2)) \setminus x$. Conversely, let $F \in \delta_N(\mathcal{F}(\Delta_2)) \setminus x$, and then $F \in \delta_N(\mathcal{F}(\Delta_2))$ and $x \notin F$, and by Remark 2.7 we get that $V \setminus F$ is a vertex cover of Δ_2 and $x \in V \setminus F$. Thus, $V \setminus F$ is also a vertex cover of Δ. Again by Remark 2.7, we have that $F \in \delta_N(\mathcal{F}(\Delta))$, and hence $x \in \delta_N(\mathcal{F}(\Delta)) \setminus x$.

$\square$

The following proposition shows that the property of shellability is preserved when we remove a vertex x from each facet containing it.

Proposition 3.5 *Let Δ be a connected simplicial complex on a vertex set V and $x \in V$. Set $\Delta_1 = \Delta \setminus x$. If $\delta_N(\mathcal{F}(\Delta))$ is shellable, then $\delta_N(\mathcal{F}(\Delta_1))$ is shellable.*

Proof   Assume that $G_1, \ldots, G_q$ is a shelling of $\delta_N(\mathcal{F}(\Delta))$, and the subsequence

$$G_{i_1}, \ldots, G_{i_t} \quad \text{with} \ 1 \leq i_1 < \cdots < i_t \leq q$$

is the list of all the facets with $x \in G_{i_j}$. Set $H_j = G_{i_j} \setminus \{x\}$, for each $j = 1, \ldots, t$. Proposition 3.4 (1) implies that $H_1, \ldots, H_t$ are all the facets of $\delta_N(\mathcal{F}(\Delta_1))$.

We claim that $H_1, \ldots, H_t$ is a shelling of $\delta_N(\mathcal{F}(\Delta_1))$. Because $G_1, \ldots, G_q$ form a shelling of $\delta_N(\mathcal{F}(\Delta))$, for all $1 \leq j < k \leq t$, there exists some $v \in G_{i_k} \setminus G_{i_j} = (G_{i_k} \setminus \{x\}) \setminus (G_{i_j} \setminus \{x\}) = H_k \setminus H_j$ such that $\{v\} = G_{i_k} \setminus G_l$ for some $1 \leq l < i_k$. It suffices to show that $G_l \in \{G_{i_1}, \ldots, G_{i_t}\}$. However, because $x \in G_{i_k}$ and $x \neq v$, we must have $x \in G_l$. Thus, $G_l = G_{i_s}$ for some $s < k$, but then $\{v\} = G_{i_j} \setminus G_l = G_{i_j} \setminus G_{i_s} = H_k \setminus H_s$. Thus, $H_1, \ldots, H_t$ form a shelling of $\delta_N(\mathcal{F}(\Delta_1))$. $\square$

A vertex of a simplicial complex Δ is called a free vertex if it belongs to exactly one facet of Δ.

Theorem 3.6 *Let Δ be a connected simplicial complex with a free vertex x and F_1 be the only facet of Δ containing x. Let $y \in F_1$ and $y \neq x$. Set $\Delta_1 = \Delta \setminus x$ and $\Delta_2 = \Delta \setminus y$, and then $\delta_N(\mathcal{F}(\Delta))$ is shellable if and only if $\delta_N(\mathcal{F}(\Delta_1))$ and $\delta_N(\mathcal{F}(\Delta_2))$ are shellable.*

Proof If $\delta_N(\mathcal{F}(\Delta))$ is shellable, then $\delta_N(\mathcal{F}(\Delta_1))$ and $\delta_N(\mathcal{F}(\Delta_2))$ are shellable by Proposition 3.5. It suffices to prove the reverse direction. Let $G_1, \ldots, G_r$ and $H_1, \ldots, H_s$ be the shellings of $\delta_N(\mathcal{F}(\Delta_1))$ and $\delta_N(\mathcal{F}(\Delta_2))$ respectively. We first prove that $G_1 \cup \{x\}, \ldots, G_r \cup \{x\}, H_1 \cup \{y\}, \ldots, H_s \cup \{y\}$ is a complete list of facets of $\delta_N(\mathcal{F}(\Delta))$. Indeed, take any facet F of $\delta_N(\mathcal{F}(\Delta))$, and then $V \setminus F$ is a minimal vertex cover of Δ by Remark 2.7. We consider two cases:

(1) If $y \notin F$, we must have $x \in F$, because if not, then $x, y \in V \setminus F$, contradicting with $V \setminus F$ being a minimal vertex cover of Δ. Thus, $V \setminus (F \setminus \{x\}) = (V \setminus F) \cup \{x\}$ is a vertex cover of Δ, and we have that $F \setminus \{x\} \in \delta_N(\mathcal{F}(\Delta))$. Hence, $F \setminus \{x\} = G_i$ for some i.

(2) If $y \in F$, then $x \notin F$. Indeed, if $x \in F$, then $x \notin V \setminus F$. As $V \setminus F$ is a minimal vertex cover of Δ, then $V \setminus F$ is also a vertex cover of Δ_1. We claim that this vertex cover is minimal. If not, then there exists a nonempty subset $T \subseteq V \setminus F$, such that $(V \setminus F) \setminus T = V \setminus (F \cup T)$ is a minimal vertex cover of Δ_1. From the proof of Proposition 3.4 (1), we obtain that $(V \setminus F) \setminus T$ is also a vertex cover of Δ. It is not possible. Using the same arguments as in (1), we have $F \setminus \{y\} = H_i$ for some i.

Now we prove that $G_1 \cup \{x\}, \ldots, G_r \cup \{x\}, H_1 \cup \{y\}, \ldots, H_s \cup \{y\}$ is a shelling of $\delta_N(\mathcal{F}(\Delta))$. Let $G < H$ be two facets of $\delta_N(\mathcal{F}(\Delta))$. We need only consider the case $G = G_i \cup \{x\}$, $H = H_j \cup \{y\}$. In this case, we can obtain that $H_j \cup \{x\}$ is contained in some facet of $\delta_N(\mathcal{F}(\Delta))$. Because $H = H_j \cup \{y\}$, by the above argument, we get $x \notin H$, and thus $x \in V \setminus H$ and $V \setminus H = V \setminus (H_j \cup \{y\})$ is a minimal vertex cover of Δ by Remark 2.7. Therefore, $(V \setminus H) \cup \{y\} = V \setminus H_j$ is a vertex cover of Δ, but $x, y \in V \setminus H_j$, $(V \setminus H_j) \setminus \{x\} = V \setminus (H_j \cup \{x\})$ is also a vertex cover of Δ. In particular, $V \setminus (H_j \cup \{x\})$ is a vertex cover of Δ_1, and then $H_j \cup \{x\}$ is a face of $\delta_N(\mathcal{F}(\Delta_1))$ by the fact that $\delta_N(\mathcal{F}(\Delta_1)) = lk_{\delta_N(\mathcal{F}(\Delta))}(x)$. Thus, $H_j \cup \{x\} \subseteq G_l \cup \{x\}$ for some l. Hence, $(H_j \cup \{y\}) \setminus (G_l \cup \{x\}) \subseteq (H_j \cup \{y\}) \setminus (H_j \cup \{x\}) = \{y\}$. As $H = H_j \cup \{y\}$ and $G = G_i \cup \{x\}$, we have that $y \in H \setminus G$, and $G_l \cup \{x\} < H$. The remaining two cases follow readily from the shellability of $\delta_N(\mathcal{F}(\Delta_1))$ and $\delta_N(\mathcal{F}(\Delta_2))$. □

4. Simplicial complexes with the free vertex property are sequentially Cohen–Macaulay

In this section, we introduce the notion of simplicial complex with the free vertex property and show that such a simplicial complex is sequentially Cohen–Macaulay. The results of this section allow us to give a new proof for a result of Faridi [5] on the sequentially Cohen–Macaulayness of simplicial forests.

Definition 4.1 *(free vertex property) Let Δ be a simplicial complex, if it satisfies the following conditions:*

(1) Δ is a simplex, or

(2) Δ has a free vertex x such that both $\Delta \setminus \langle F \rangle$ and $\Delta \setminus x$ also have free vertices, where F is the only facet of Δ containing x.

We call such a simplicial complex Δ as having the free vertex property.

Remark 4.2 *It is easy to prove that a leaf must contain a free vertex, so both simplicial trees and simplicial forests have the free vertex property.*

The following example shows that a simplicial complex with the free vertex property is not always a simplicial tree or a simplicial forest.

Example 4.3 *The simplicial complex on the left has the free vertex property, but it is not a simplicial tree, because although all of its facets* $\{x_1, x_2, x_3, x_4\}, \{x_1, x_2, x_5\}$, $\{x_2, x_3, x_6\}$, *and* $\{x_1, x_3, x_7\}$ *are leaves, if one removes the facet* $\{x_1, x_2, x_3, x_4\}$, *the remaining simplicial complex (on the right) has no leaf.*
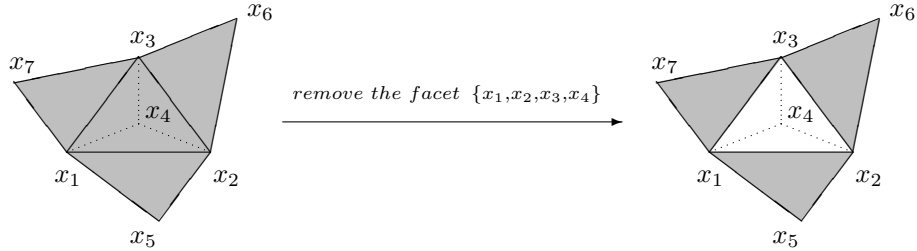
Figure. Simplicial complex with the free vertex property is not always a simplicial tree or a simplicial forest.

Definition 4.4 *(sequentially Cohen–Macaulay) Let* M *be a graded module over* $R = k[x_1, \ldots, x_n]$. *We say that* M *is sequentially Cohen–Macaulay if there exists a filtration of graded* R-*submodules of* M

$$0 = M_0 \subset M_1 \subset \cdots \subset M_r = M$$

such that each quotient M_i/M_{i-1} *is Cohen–Macaulay and the Krull dimensions of the quotients are increasing, i.e.* $dim\,(M_1/M_0) < dim\,(M_2/M_1) < \cdots < dim\,(M_r/M_{r-1})$.

   A simplicial complex is said to be sequentially Cohen–Macaulay if its Stanley–Reisner ideal has a sequentially Cohen–Macaulay quotient.

   As first shown by Stanley [13], shellability implies sequentially Cohen–Macaulay.

Theorem 4.5 *Let* Δ *be a simplicial complex, and suppose that* $R/\mathcal{N}(\Delta)$ *is the associated Stanley–Reisner ring. If* Δ *is shellable, then* $R/\mathcal{N}(\Delta)$ *is sequentially Cohen–Macaulay.*

Proposition 4.6 *Let* Δ *be a simplicial complex over a vertex set* $V = \{x_1, \ldots, x_n\}$, $A \subset V$ *be a set of vertices, and* $\Delta_M = \langle G_1, \ldots, G_p \rangle$ *be the cover complex of* Δ. *If the Stanley–Reisner complex* $\delta_{\mathcal{N}}(\mathcal{F}(\Delta))$ *of the facet ideal* $\mathcal{F}(\Delta)$ *of* Δ *is shellable, then the Stanley–Reisner complex* $\delta_{\mathcal{N}}(I)$ *of the ideal*

$$I = \bigcap_{A \cap G_i = \emptyset} (G_i)$$

is shellable with respect to the linear ordering of the facets of $\delta_{\mathcal{N}}(I)$ *induced by the shelling of the simplicial complex* $\delta_{\mathcal{N}}(\mathcal{F}(\Delta))$.

Proof   By Remark 2.7, we may assume that $F_1, \ldots, F_p$ is a shelling of $\delta_{\mathcal{N}}(\mathcal{F}(\Delta))$ and $F_i = V \setminus G_i$ for $i = 1, \ldots, p$. Let F_i and F_j be two facets of $\delta_{\mathcal{N}}(I)$ with $i < j$, and by Remark 2.7, we have that $A \cap G_i = \emptyset$ and $A \cap G_j = \emptyset$. By the shellability of $\delta_{\mathcal{N}}(\mathcal{F}(\Delta))$, there exists some $x \in F_j \setminus F_i$ and some $l \in \{1, \ldots, j-1\}$ with $F_j \setminus F_l = \{x\}$. It suffices to prove that $A \cap G_l = \emptyset$. If $A \cap G_l \neq \emptyset$, pick $y \in A \cap G_l \subseteq A$. By $A \cap G_i = \emptyset$ and $A \cap G_j = \emptyset$, we have that $y \notin G_i \cup G_j$ and $y \in F_i \cap F_j$. Since $y \notin F_l$ (otherwise $y \notin G_l$, a contradiction), we get $y \in F_j \setminus F_l = \{x\}$, i.e. $y = x$, a contradiction because $x \notin F_i$. $\square$

Proposition 4.7 *Let x_n be a free vertex of a simplicial complex $\Delta = \langle F_1, \ldots, F_q \rangle$, and let F_q be only the facet of Δ containing x_n and $F_q = \{x_n\} \cup A$.*

(1) *If $\Delta_1 = \Delta \backslash \langle F_q \rangle$ is the simplicial complex obtained by removing the facet F_q from Δ, then C is a minimal vertex cover of Δ containing x_n if and only if $C \cap A = \emptyset$ and $C = \{x_n\} \cup C'$ for some minimal vertex cover C' of Δ_1;*

(2) *If $\Delta_2 = \Delta \backslash x_n$ is the simplicial complex obtained by deleting the face $\{x_n\}$ from Δ, then C is a minimal vertex cover of Δ not containing x_n if and only if C is a minimal vertex cover of Δ_2.*

Proof

(1) Assume that C is a minimal vertex cover of Δ containing x_n, and then $C = \{x_n\} \cup C'$ for some set C' of vertices. If $C \cap A \neq \emptyset$, then $C' = C \setminus \{x_n\}$ is a vertex cover of Δ, a contradiction. Thus, $C \cap A = \emptyset$. Hence, it suffices to note that $C' = C \setminus \{x_n\}$ is a minimal vertex cover of Δ_1. The sufficiency is obvious.

(2) Assume that C is a minimal vertex cover of Δ not containing x_n; then clearly $C \cap A \neq \emptyset$ because $F_q = \{x_n\} \cup A$. Thus, C is a vertex cover of Δ_2. To prove that C is a minimal vertex cover of Δ_2, we can take $C' \subsetneq C$. We must show that there is a facet F of Δ_2 not covered by C'. As C' is not a vertex cover of Δ, there is a facet F of Δ such that $F \cap C' = \emptyset$. If $F = F_i$ with $1 \leq i \leq q-1$, there is nothing to prove; otherwise $F = A$, and then $A \cap C' = \emptyset$, and the facet A of Δ_2 is not covered by C'. The converse also follows readily.

$\square$

Theorem 4.8 *If the simplicial complex Δ on a vertex set $V = \{x_1, \ldots, x_n\}$ has the free vertex property, then the Stanley–Reisner complex $\delta_\mathcal{N}(\mathcal{F}(\Delta))$ of the facet ideal $\mathcal{F}(\Delta)$ of Δ is shellable.*

Proof We proceed by induction on n. The case $n = 1$ is clear. Assume that x_n is a free vertex of a simplicial complex Δ and $F = \{x_n\} \cup A$ is only the facet of Δ containing x_n. Consider the simplicial complexes $\Delta_1 = \Delta \setminus \langle F \rangle$ and $\Delta_2 = \Delta \setminus x_n$. Set V_1 and V_2 as the sets of vertices of the simplicial complexes Δ_1 and Δ_2, respectively, and then both V_1 and V_2 have fewer than n vertices (because x_n is the vertex of neither Δ_1 nor Δ_2), and by the induction hypothesis, we have that $\delta_\mathcal{N}(\mathcal{F}(\Delta_1))$ and $\delta_\mathcal{N}(\mathcal{F}(\Delta_2))$ are shellable. Assume that $G_1, \ldots, G_r$ are the facets of $\delta_\mathcal{N}(\mathcal{F}(\Delta))$ containing x_n and $H_1, \ldots, H_s$ are the facets of $\delta_\mathcal{N}(\mathcal{F}(\Delta))$ not containing x_n. Set $C_i = V \setminus H_i$ and $C_i' = C_i \setminus \{x_n\}$ for $i = 1, \ldots, s$. Then, by Remark 2.7, $C_1, \ldots, C_s$ are the set of minimal vertex covers of Δ containing x_n, and by Proposition 4.7 (1), one has that $C_1', \ldots, C_s'$ are the set of minimal vertex covers of Δ_1 with $C_i' \cap A = \emptyset$ for $i = 1, \ldots, s$. We claim that $C_i' \subseteq V_1$. If not, there exists $a \in C_i' \setminus V_1$, and thus $a \notin A$ (because $A \cap C_i' = \emptyset$). Thus, $V_1 \setminus C_i'$ is the facet of $\delta_\mathcal{N}(\mathcal{F}(\Delta_1))$ from the fact that C_i' is the set of minimal vertex covers of Δ_1 and Remark 2.7, and hence $a \in V_1$, a contradiction with $a \in A \setminus V_1$. Thus, we have the equality $H_i = V \setminus C_i = (V_1 \cup \{x_n\}) \setminus (C_i' \cup \{x_n\}) = V_1 \setminus C_i'$ and $H_i = V_1 \setminus C_i'$ is the facet of $\delta_\mathcal{N}(\mathcal{F}(\Delta_1))$ for $i = 1, \ldots, s$. Hence, by the shellability of $\delta_\mathcal{N}(\mathcal{F}(\Delta_1))$ and Proposition 4.6, we may assume that $H_1, \ldots, H_s$ is a shelling for the simplicial complex generated by $H_1, \ldots, H_s$. By Proposition 4.7 (2), one has that C is a minimal vertex cover of Δ not containing x_n if and only if C is a minimal vertex cover of Δ_2. Thus, G is a facet of $\delta_\mathcal{N}(\mathcal{F}(\Delta))$ that contains x_n, i.e. $G = \{x_n\} \cup G'$ if and only if G' is a facet

of $\delta_{\mathcal{N}}(\mathcal{F}(\Delta_2))$. By induction we may also assume that $G'_1 = G_1 \setminus \{x_n\}, \ldots, G'_r = G_r \setminus \{x_n\}$ is a shelling of $\delta_{\mathcal{N}}(\mathcal{F}(\Delta_2))$. We now prove that

$$G_1, \ldots, G_r, H_1, \ldots, H_s \ \ \text{with} \ \ G_i = \{x_n\} \cup G'_i$$

is a shelling of $\delta_{\mathcal{N}}(\mathcal{F}(\Delta))$. We need only show that given H_j and G_i there is $x \in H_j \setminus G_i$ and G_l such that $H_j \setminus G_l = \{x\}$. We can write

$$H_j = V \setminus C_j \ \ \text{and} \ \ G_i = V \setminus D_i$$

where C_j (resp. D_i) is a minimal vertex cover of Δ containing x_n (resp. not containing x_n). By Proposition 4.7, we have that: (i) $C_j = \{x_n\} \cup C'_j$ for some minimal vertex cover C'_j of Δ_1 such that $C'_j \cap A = \emptyset$, and (ii) D_i is a minimal vertex cover of Δ_2. From (i) we get that $A \subseteq H_j$. Observe that $A \nsubseteq G_i$; otherwise, $A \cap D_i = \emptyset$, a contradiction, because D_i must cover some vertex of A. Hence, there is some $x \in A \setminus G_i \subseteq H_j \setminus G_i$. Since $\{x_n\} \cup C'_j$ is a vertex cover of Δ, there is a minimal vertex cover D_l of Δ contained in $\{x_n\} \cup C'_j$. Clearly $x \in D_l$ because D_l is a minimal vertex cover of Δ, $D_l \cap A \neq \emptyset$ and $A \cap C'_j = \emptyset$. Thus, $G_l = V \setminus D_l$ is a facet of $\delta_{\mathcal{N}}(\mathcal{F}(\Delta))$ containing x_n. To finish the proof we now prove that $H_j \setminus G_l = \{x\}$. We know that $x \in H_j$. If $x \in G_l$, then $x \notin D_l$, a contradiction. Thus, $x \in H_j \setminus G_l$, i.e. $\{x\} \subseteq H_j \setminus G_l$. Conversely take $y \in H_j \setminus G_l$, i.e. $y \in H_j = V \setminus (\{x_n\} \cup C'_j)$ and $y \notin G_l = V \setminus D_l$, and then $y \notin \{x_n\} \cup C'_j$ and $y \in D_l \subset \{x\} \cup C'_j$. Hence, $y = x$, as required. □

Theorem 4.9 *Assume that a simplicial complex Δ on a vertex set V has the free vertex property, and then $R/\mathcal{F}(\Delta)$ is sequentially Cohen–Macaulay.*

Proof By Theorem 4.8, we obtain that the simplicial complex $\delta_{\mathcal{N}}(\mathcal{F}(\Delta))$ is shellable, and thus $R/\mathcal{F}(\Delta)$ is sequentially Cohen–Macaulay by Theorem 4.5. □

As a consequence of the above theorem, we obtain [5, *Corollary 5.6*].

Corollary 4.10 *Let Δ be a simplicial forest. Then $R/\mathcal{F}(\Delta)$ is sequentially Cohen–Macaulay.*

Proof As simplicial forest Δ has the free vertex property, the simplicial complex $\delta_{\mathcal{N}}(\mathcal{F}(\Delta))$ is shellable by Theorem 4.8. Hence, $R/\mathcal{F}(\Delta)$ is sequentially Cohen–Macaulay by Theorem 4.9. □

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