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Zero-divisor graph of matrix rings and Hurwitz rings

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Abstract: Let R be ring a with identity $1 \neq 0$, $S_n(R)$ be a subring of the ring $T_n(R)$ of $n \times n$ upper triangular matrices over R, and $H_n(R)$ be the ring defined in the next section using HR, the ring of the Hurwitz series over R. In this paper, we introduce the zero-divisor graph $\overrightarrow{\Gamma}(S_n(R))$ and its underlying undirected graph $\Gamma(S_n(R))$ of $S_n(R)$. We give some basic graph theory properties of $\overrightarrow{\Gamma}(S_n(R))$. Moreover, we obtain some results of the zero-divisor directed graph of $\overrightarrow{\Gamma}(H_n(R))$.

Key words: Zero-divisor graph, matrix ring, Hurwitz ring

1. Introduction

Zero-divisor graphs were first defined for commutative rings by Beck in [2]. However, he let all elements of a ring R be vertices of the graph and was mainly interested in colorings. In [1], Anderson and Livingston introduced and studied the zero-divisor graph whose vertices are the nonzero zero-divisors of R. They studied the interplay between the ring-theoretic properties of a commutative ring and the graph theoretic properties of its zero-divisor graph. In [7], Li and Tucci studied the zero-divisor graphs of upper triangular matrix rings over commutative rings with identity. We extend their results to some special matrix rings.

Let R be a commutative ring with identity $1 \neq 0$. Let $Z(R)$ denote the set of all zero-divisors of R, and $Z(R)^* = Z(R) \setminus \{0\}$ the nonzero zero-divisors of R. The *zero-divisor graph* of R, denoted by $\Gamma(R)$, is the undirected graph whose vertices are the elements of $Z(R)^*$, and two distinct vertices r and s are adjacent if and only if $rs = 0$.

The zero-divisor graph of a noncommutative ring R is a directed graph, which is denoted by $\overrightarrow{\Gamma}(R)$. We denote the underlying undirected graph of $\overrightarrow{\Gamma}(R)$ by $\Gamma(R)$. An element $r \in R$ is a *left (resp., right) zero-divisor* if there exists $0 \neq s \in R$ such that $rs = 0$ (*resp.*, $sr = 0$). In R, the sets of nonzero left and right zero-divisors are denoted by $ZD_l(R)^*$ and $ZD_r(R)^*$, respectively. The vertex set of $\overrightarrow{\Gamma}(R)$ is $V(\overrightarrow{\Gamma}(R)) = ZD_l(R)^* \cup ZD_r(R)^*$, and there is an edge from r to s, denoted by $r \to s$, if and only if $rs = 0$. For general background on graph theory, please see [3].

In Section 2, we study the zero-divisor graphs of $S_n(R)$. Assume that R is a commutative ring with nonzero identity. Let $T_n(R)$ denote the $n \times n$ upper triangular matrix ring over R and $S_n(R)$, $T(R, n)$ be two

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subrings of $T_n(R)$ defined as follows for any $n \geq 2$ respectively.

$$S_n(R) = \left\{ \begin{pmatrix} a & a_{12} & a_{13} & \cdots & a_{1n} \\ 0 & a & a_{23} & \cdots & a_{2n} \\ 0 & 0 & a & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & a \end{pmatrix} : a, a_{ij} \in R \right\},$$

$$T(R,n) = \left\{ \begin{pmatrix} a_1 & a_2 & a_3 & \cdots & a_n \\ 0 & a_1 & a_2 & \cdots & a_{n-1} \\ 0 & 0 & a_1 & \cdots & a_{n-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & a_1 \end{pmatrix} : a_i \in R \right\}.$$

If there is no confusion, we write S and T instead of $S_n(R)$ and $T_n(R)$. In this section we determine the girth of $\overrightarrow{\Gamma}(S)$ and get some conditions for $\Gamma(S)$ to be planar. We also extend some of the results from [4] and [7] to $S_n(R)$.

In Section 3, we study the zero-divisor graphs of Hurwitz rings. Let R be any ring. We denote $H(R)$, or simply HR, the ring of Hurwitz series over R, defined as follows. The elements of HR are sequences of the form $a = (a_n) = (a_0, a_1, a_2, ...)$, where $a_n \in R$ for each $n \in \mathbb{N}$. An element in HR can be thought of as a function from $\mathbb{N}$ to R.

Two elements (a_n) and (b_n) in HR are equal if they are equal as functions from $\mathbb{N}$ to R, i.e. if $a_n = b_n$ for all $n \in \mathbb{N}$. The element $a_m \in R$ will be called the mth term of (a_n). Addition in HR is defined termwise, so that $(a_n) + (b_n) = (c_n)$, where $c_n = a_n + b_n$ for all $n \in \mathbb{N}$. If one identifies a formal power series $\sum_{i=0}^{\infty} a_n x^n \in R[[x]]$ with the sequence of its coefficients (a_n), then multiplication in HR is similar to the usual product of formal power series, except that binomial coefficients are introduced at each term in the product as follows by [5]. The (Hurwitz) product of (a_n) and (b_n) is given by $(a_n)(b_n) = (c_n)$, where

$$c_n = \sum_{k=0}^{n} C_k^n a_k b_{n-k}.$$

Hence,

$$(a_0, a_1, a_2, a_3, ...)(b_0, b_1, b_2, b_3, ...) =$$

$$(a_0b_0, a_0b_1 + a_1b_0, a_0b_2 + 2a_1b_1 + a_2b_0, a_0b_3 + 3a_1b_2 + 3a_2b_1 + a_3b_0, ...).$$

Set

$$H(R,n) = \left\{ \begin{pmatrix} a_0 & a_1 & \cdots & a_n \\ 0 & a_0 & \cdots & a_{n-1} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_0 \end{pmatrix} : a_i \in R \quad for \quad 0 \leq i \leq n \right\}.$$

We can identify $H(R,n)$ with the set

$$\{(a_0, a_1, ..., a_n) : a_i \in R \quad for \quad 0 \leq i \leq n\}.$$

Then $H(R,n)$ is a ring with addition defined componentwise and multiplication is given by

$$(a_0, a_1, ..., a_n)(b_0, b_1, ..., b_n) = (c_0, c_1, ..., c_n),$$

where $c_0 = a_0 b_0$ and $c_m = \sum_{k=0}^{m} C_k^m a_k b_{m-k}$ for each $1 \leq m \leq n$. Note that sometimes the ring $H(R,n)$ is shown by $H_n(R)$. In this note, from now on we will use H_n instead of $H(R,n) = H_n(R)$.

2. Zero-divisor graph of $S_n(R)$

Suppose that $S_n(R) = S$ is the ring consisting of upper triangular matrices defined in the previous section. In this section, we will give some results of the graph $\Gamma(S)$ and the underlying graph $\overrightarrow{\Gamma}(S)$ such as the girth, diameter, vertices, edges, etc. For all undefined notions we refer to [2] and [7]. We begin with a known result, which will be used throughout the paper.

Theorem 2.1 *[6, Theorem 2.1] Let R be a commutative ring with identity $1 \neq 0$, and let $Q(R)$ be the total quotient ring of R. Then $\overrightarrow{\Gamma}(T_n(R)) \cong \overrightarrow{\Gamma}(T_n(Q(R)))$.*

Because of Theorem 2.1, we can assume throughout this paper that every element of R is either a unit or a zero-divisor.

Lemma 2.2 *Let $A = [a, a_{ij}] \in S = S_n(R)$. Then $detA$ is a zero-divisor in R if and only if $a_{jj} = a$ is a zero divisor in R for all $i = 1, 2, ..., n$.*

Proof $'' \Rightarrow ''$ Let $detA \in ZD(R)$. Then $a^n \in ZD(R) \Rightarrow a^n r = 0$ for some $r \in R$. We want to show that $as = 0$ for some nonzero $s \in R$. If $a = 0$, then $a \in ZD(R)$. If $a \neq 0$, then $aa^{n-1}r = 0$. Now there are two possibilities: $a^{n-1}r = 0$ or $a^{n-1}r \neq 0$. If $a^{n-1}r = 0$, then the proof goes as above. If $a^{n-1}r \neq 0$, then a is a zero divisor.
$'' \Leftarrow ''$ Let a be a zero divisor. Then $\exists 0 \neq r \in R : ar = 0 \Rightarrow aar = 0 \Rightarrow a^3 r = 0 \Rightarrow \cdots \Rightarrow a^n r = 0$. Then $a^n = detA$ is a zero divisor since $r \neq 0$. □

Theorem 2.3 *[4, Theorem 9.1] Let $M_n(R)$ be the ring of $n \times n$ matrices over a commutative ring R with identity, and let $A \in M_n(R)$. Then*

$$A \in ZD_l(M_n(R)) \Longleftrightarrow \det A \in ZD(R) \Longleftrightarrow A \in ZD_r(M_n(R)).$$

Since $S_n(R) \subseteq M_n(R)$, Theorem 2.2 holds for any matrix in $S_n(R)$. Since we will use the results of this fact in the paper, we give a simple proof here.

Lemma 2.4 *Let $A \in S = S_n(R)$. Then*

$$A \in ZD_l(S) \Longleftrightarrow \det A \in ZD(R) \Longleftrightarrow A \in ZD_r(S).$$

Proof Since $S \subseteq M_n(R)$ we have the implication $A \in ZD_l(S) \Rightarrow A \in ZD_l(M_n(R))$. Thus, $detA \in ZD(R)$ and $A \in ZD_r(T)$ by Theorem 2.2. Thus, there exists a $0 \neq B \in M_n(R)$ such that $BA = 0$. Let $\overrightarrow{b}$

be any nonzero row of B and let $B' = [\vec{b}, \vec{0}, ..., \vec{0}]^t \in M_n(R)$ whose first row is $\vec{b}$ and whose other rows are all $\vec{0}$. Then $B' \neq 0 \in S_n$ and $B'A$. Thus, $A \in ZD_r(S)$. Similarly, it can be shown that $A \in ZD_r(S) \Longrightarrow detA \in ZD(R) \Longrightarrow A \in ZD_l(S)$. □

Theorem 2.5 *Let* $A = [a, aij] \in S$*.*
(a) The matrix A *is a left and right zero-divisor in* S *if and only if* $a = a_{ii}$ *is a zero-divisor in* R *for all* $i = 1, 2, ...n$
(b) If every element of R *is a unit or a zero-divisor, then every element of* T *is either a unit or a zero-divisor.*

Proof (a) This follows from the lemmas above.
(b) This follows because for $A \in S$, $detA \in R$, and hence $detA$ is either a zero-divisor or a unit. □

Proposition 2.6 *Let* $n \geq 3$ *and* $S = S_n(R)$*. The following statements hold.*
(a) The girth of $\vec{\Gamma}(S)$ *is 3.*
(b) If R *is a commutative ring, then* $\mathrm{diam}\Gamma(R) \leq \mathrm{diam}\vec{\Gamma}(S) \in \{2, 3\}$*.*

Proof (a) Let $A = \begin{pmatrix} 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}$, $B = \begin{pmatrix} 0 & 0 & 0 & \cdots & 1 & 1 \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 0 \end{pmatrix}$, and $C = \begin{pmatrix} 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}$.

A, B, C are distinct matrices in $Z(S_n(R))^*$ such that $AB = BC = CA = 0$. Thus, $A \to B \to C \to A$ is a directed cycle of length 3, as asserted.

(b) We define $\phi : Z(R)^* \to Z(S_n(R))^*$ by $\phi(a) = \begin{pmatrix} a & 0 & 0 & \cdots & 0 & 0 \\ 0 & a & 0 & \cdots & 0 & 0 \\ 0 & 0 & \ddots & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & a \end{pmatrix}$ for any $a \in Z(R)^*$. Since $\phi(a) = \phi(b)$ implies $a = b$, ϕ is an injection from $\Gamma(R)$ to $\vec{\Gamma}(S)$. It is clear that $ab = 0$ if and only if $\phi(a)\phi(b) = 0$. Hence, $\Gamma(R)$ is isomorphic to a subgraph of $\vec{\Gamma}(S)^*$. Since $\Gamma(R)$ is connected, we conclude that $\mathrm{diam}\Gamma(R) \leq \mathrm{diam}\vec{\Gamma}(S)$. □

We denote the underlying graph of $\vec{\Gamma}(S)$ by $\Gamma(S)$.

Proposition 2.7 *If* $n \geq 3$*, then* $\Gamma(S)$ *is not planar.*

Proof Consider the ring R consisting of the elements $\{0, 1\}$. To draw the graph $\Gamma(S_3(R))$ we first need to determine the nonzero zero-divisors of $S_3(R)$. The nonzero zero divisors are as follows: (1) $\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$,

(2) $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$, (3) $\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, (4) $\begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, (5) $\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$, (6) $\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$, (7) $\begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$.

Now the vertex set is $\{(1),(2),(3),(4),(5),(6),(7)\}$. From these vertices, the vertices $(1),(2),(3),(4),(5)$, and (6) are connected with each other. Therefore, when we start to draw the graph, in one step we reach the complete graph K_6, which is not planar. Hence, the graph $\Gamma(S_3(R))$ is not planar. Hence, for any $n \geq 3$ $\Gamma(S_n(R))$ is not planar. □

Proposition 2.8 *If $\Gamma(R)$ is not planar, then $\Gamma(S)$ is also not planar.*

Proof Assume that $\Gamma(R)$ is not planar. Then $\Gamma(R)$ contains the subgraphs $K_{3,3}$ or K_5, and so there are at least 5 zero divisors, say r_1, r_2, r_3, r_4, r_5. On the other hand, since R is a ring with nonzero identity it has at least 7 elements. We start to prove assuming $n = 2$. There are 6 nonzero zero divisors in $S_2(R)$: (1) $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, (2) $\begin{pmatrix} 0 & r_1 \\ 0 & 0 \end{pmatrix}$, (3) $\begin{pmatrix} 0 & r_2 \\ 0 & 0 \end{pmatrix}$, (4) $\begin{pmatrix} 0 & r_3 \\ 0 & 0 \end{pmatrix}$, (5) $\begin{pmatrix} 0 & r_4 \\ 0 & 0 \end{pmatrix}$, (6) $\begin{pmatrix} 0 & r_5 \\ 0 & 0 \end{pmatrix}$, and these vertices construct the graph K_6 since there are vertices between each pair of them. Thus, $\Gamma(S_2(R))$ contains K_6 and so $\Gamma(S_2(R))$ is not planar.

Proving the same result for the $n \times n$ case is quite easy. Take only

(1) $\begin{pmatrix} 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}$, (2) $\begin{pmatrix} 0 & 0 & 0 & \cdots & r_1 \\ 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}$, (3) $\begin{pmatrix} 0 & 0 & 0 & \cdots & r_2 \\ 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}$, (4) $\begin{pmatrix} 0 & 0 & 0 & \cdots & r_3 \\ 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}$,

(5) $\begin{pmatrix} 0 & 0 & 0 & \cdots & r_4 \\ 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}$, (6) $\begin{pmatrix} 0 & 0 & 0 & \cdots & r_5 \\ 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}$. Then the proof follows as the previous part. Hence, $\Gamma(S_n(R)) = \Gamma(S)$ is not planar. □

The following example shows us that the inverse of the proposition need not be true.

Example 2.9 *Let $R = \mathbb{Z}_7$ and consider the graph $\Gamma(S_2(R))$. As we can see above there are 6 nonzero zero divisors like in the first part of the previous proposition, and the only difference is taking 2,3,... instead of $r_1, r_2, \ldots$. Thus, $\Gamma(S_2(R))$ is not planar. On the other hand, since there is no nonzero zero divisor in $\mathbb{Z}_7$ the graph $\Gamma(R) = \Gamma(\mathbb{Z}_7)$ is planar.*

Corollary 2.10 *Let R be a finite ring. Then if $\Gamma(S)$ is planar, R is isomorphic to one of the following rings:*
$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$, $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3$, $\mathbb{Z}_2 \times \mathbb{Z}_8, \mathbb{Z}_2 \times \mathbb{Z}_2[x]/(x^3), \mathbb{Z}_2 \times \mathbb{Z}_4[x]/(2x, x^2-2)$,
$\mathbb{Z}_2 \times R_2, \mathbb{Z}_3 \times R_2$, *where* $|Z(R_2)| \leqslant 3$,
$\mathbb{Z}_4$, $\mathbb{Z}_2[x]/(x^2)$,
$\mathbb{Z}_8$, $\mathbb{Z}_2[x]/(x^3)$, $\mathbb{Z}_4[x]/(2x, x^2-2)$, $\mathbb{Z}_2[x,y]/(x,y)^2$, $\mathbb{Z}_4[x]/(2x, x^2)$,
$\mathbb{Z}_9$, $\mathbb{Z}_3[x]/(x^2)$,
$\mathbb{Z}_{16}$, $\mathbb{F}_4[x]/(x^2)$, $\mathbb{Z}_4[x]/(x^2-2)$, $\mathbb{Z}_4[x]/(x^2+2x+2)$, $\mathbb{Z}_4[x]/(x^2+x+1)$,
$\mathbb{Z}_2[x]/(x^4)$, $\mathbb{Z}_2[x,y]/(x^2-y^2, xy)$, $\mathbb{Z}_2[x,y]/(x^2,y^2)$, $\mathbb{Z}_4[x]/(2x, x^3-2)$,

$\mathbb{Z}_4[x,y]/(x^2-2,xy,y^2-2,2x)$, $\mathbb{Z}_4[x,y]/(x^2,xy-2,y^2)$, $\mathbb{Z}_4[x]/(x^2)$,
$\mathbb{Z}_4[x]/(x^2-2x)$, $\mathbb{Z}_8[x]/(2x,x^2-4)$,
$\mathbb{Z}_{25}$, $\mathbb{Z}_5[x]/(x^2)$,
$\mathbb{Z}_{27}$, $\mathbb{Z}_9[x]/(x^2-3,3x)$, $\mathbb{Z}_9[x]/(x^2-6,3x)$, $\mathbb{Z}_3[x]/(x^3)$.

3. Zero-divisor graph of $H_n(R)$

Suppose that $H_n(R) = H_n$ is the ring consisting of upper triangular matrices defined in the first section. In this section, we will give some properties of the graph $\overrightarrow{\Gamma}(H_n)$.

Lemma 3.1 *Suppose that $|ZD_r(H_n)| < \infty$ and $\overrightarrow{\Gamma}(H_n)$ is a nonempty graph. Then $|ZD_r(R)| < \infty$.*

Proof Since $\overrightarrow{\Gamma}(H_n)$ is a nonempty graph, there exist two nonzero elements $A_0 = (a_0, a_1, ..., a_n)$ and $B_0 = (b_0, b_1, ..., b_n)$ of H_n such that $A_0B_0 = 0$. Let $I_r := Ann^r_{H_n}(A_0) = \{C \in H_n : A_0C = 0\}$. Then I_r is a right ideal of H_n, which is nonzero since $B_0 \in I_r$. On the other hand, $I_r \subseteq ZD_r(H_n)$, so $|I_r| < \infty$. Now suppose that $|ZD_r(R)| = \infty$. Thus, for each $r \in ZD_r(R)$, $A_r = (r, 0, ..., 0) \in H_n$ and $B_0A_r \in I_r$. Since $|I_r| < \infty$, there exists an element $M \in I_r$ such that $J = \{r \in ZD_r(R) : B_0A_r = M\}$ is a finite set. For each $r, s \in J$, we have $B_0A_r = M = B_0A_s$, and thus $A_r - A_s \in Ann^r_{H_n}(B_0)$. This implies that $Ann^r_{H_n}(B_0)$ is an infinite subset of $ZD_r(H_n)$, which is a contradiction. Therefore, $|ZD_r(R)| < \infty$. □

Theorem 3.2 *$\overrightarrow{\Gamma}(H_n)$ is a finite graph and has at least two vertices as $A_0 = (a_0, a_1, ..., a_n)$ and $B_0 = (b_0, b_1, ..., b_n)$ such that $a_0b_0 \neq 0$ and $A_0B_0 = 0$ if and only if R is finite and not an integral domain.*

Proof Assume that R is finite and not an integral domain. Since R is finite, it is clear that $\overrightarrow{\Gamma}(H_n)$ is finite. On the other hand, since R is not an integral domain, there exist two nonzero elements a and b of R such that $ab = 0$. Now if we take $A_0 = (a, 0, ..., 0)$ and $B_0 = (b, 0, ..., 0)$, then it is clear that these are two nonzero elements of H_n such that $A_0B_0 = 0$.

For the other direction of the proof, suppose that $\overrightarrow{\Gamma}(H_n)$ is finite and there exist two nonzero elements $A_0 = (a_0, a_1, ..., a_n)$ and $B_0 = (b_0, b_1, ..., b_n)$ of H_n such that $a_0, b_0 \neq 0$ and $A_0B_0 = 0$. Thus, we have $a_0b_0 = 0$ and $0 = \sum_{k=0}^{m} C^m_k a_k b_{m-k}$ for each $1 \leq m \leq n$. Let $I = Ann^r_R(a_0)$. Since $\overrightarrow{\Gamma}(R)$ is a subgraph of $\overrightarrow{\Gamma}(H_n)$ and $\overrightarrow{\Gamma}(H_n)$ is finite, we conclude that $|ZD(R)| < \infty$ and especially $|ZD_r(R)| < \infty$. Thus, $|I| < \infty$. Now by a similar proof, we can see that if R is infinite, then $Ann^r_R(b_0)$ is an infinite subset of $ZD_r(R)$, which contradicts Lemma 3.1. □

Lemma 3.3 *Let $A = (a_0, a_1, ..., a_n) \in H_n$. Then $detA$ is a zero divisor of R if and only if a_0 is a zero divisor of R.*

Proof If $det(A) = a_0^n$ is a zero-divisor of R, then there is an $r \in R$ such that $a_0^n r = 0$. If $a_0 = 0$, then $a_0 \in ZD(R)$; otherwise, if $a_0 \neq 0$, then either $a_0^{n-1}r = 0$ or $a_0^{n-1}r \neq 0$. In the first case, if we continue by induction we can catch the result. In the second case, let $p = a_0^{n-1}r$, since $a_0p = 0$, we conclude that

$a_0 \in ZD(R)$. Conversely, assume that $a_0 \in ZD(R)$, and then there is an $0 \neq r \in R$ such that $a_0 r = 0$; thus, $a_0^n r = 0$, which implies that $det(A) = a_0^n$ is a zero-divisor. □

Lemma 3.4 *If $\overrightarrow{\Gamma}(H_n)$ is a complete graph (i.e. there are two sided arcs between any two vertices), then $\overrightarrow{\Gamma}(R)$ is also complete.*

Proof Let $a, b \in V(\overrightarrow{\Gamma}(R))$. Then there exist some nonzero elements c, d of R such that $ac = bd = 0$. Let $A = (a, 0, 0, ..., 0), B = (b, 0, 0, ..., 0), C = (c, 0, 0, ..., 0), D = (d, 0, 0, ..., 0)$. It is clear that $AC = 0 = BD$. Therefore, $A, B \in V(\overrightarrow{\Gamma}(H_n))$. Since $\overrightarrow{\Gamma}(H_n)$ is a complete graph, we have $AB = 0$, which implies that $ab = 0$, so $d(a, b) = 1$ in $\overrightarrow{\Gamma}(R)$. Hence, $\overrightarrow{\Gamma}(R)$ is complete. □

Theorem 3.5 *Suppose that $\overrightarrow{\Gamma}(H_n)$ is the triangle $A \leftrightarrow B \leftrightarrow C \leftrightarrow A$. Then the following hold:*

(i) $|R| < 16$.

(ii) If $|ZD(R)| = 3$, then R is not reduced.

(iii) If $|ZD(R)| = 3$, then C is an n-tuples consisting of the elements $c_0, c_1, ..., c_n$, where $c_i \in \{0, d^{m-1}\}$ such that $0 \neq d$ is a nilpotent element of R, $C \neq B$ and not all c_is are zero.

Proof

(i) By Lemma 3.3, it is clear that $\overrightarrow{\Gamma}(R)$ is a complete graph. Thus, either it is a triangle or a path of length two. Thus, $|ZD(R)| \leq 4$ and so $|R| \leq |ZD(R)|^2 \leq 16$.

(ii) Assume that $|ZD(R)| = 3$. Then there exist nonzero distinct elements a, b of $ZD(R)$ such that $ab = 0$. Let $M = (a, 0, 0, ..., 0)$ and $N = (b, 0, 0, ..., 0)$. Then $MN = 0$. Since $\overrightarrow{\Gamma}(H_n)$ is the triangle $A \leftrightarrow B \leftrightarrow C \leftrightarrow A$, without loss of generality, we may assume that $A = M$ and $B = N$. Now we have $AC = BC = 0$, and so $ac_0 = 0 = bc_0$ and $\sum_{k=0}^{m} C_k^m a_k c_{m-k} = 0 = \sum_{k=0}^{m} C_k^m b_k c_{m-k}$ for each $1 \leq m \leq n$. Suppose that R is reduced, which means that it does not have any nonzero nilpotent element. Since $ac_0 = 0 = bc_0$ and $|ZD(R)^*| = 2$, we have $c_0 \in \{0, a, b\}$. However, R is reduced, so $c_0 = 0$. On the other hand, $a_0c_1 + a_1c_0 = 0 = b_0c_1 + b_1c_0$ implies that $ac_1 = 0 = bc_1$ and so $c_1 = 0$. For the next step, we can see that $a_0c_2 + 2a_1c_1 + a_2c_0 = 0 = b_0c_2 + 2b_1c_1 + b_2c_0$, which implies that $ac_2 = 0 = bc_2$ and so $c_2 = 0$. Continuing this process gives $c_i = 0$ for each $0 \leq i \leq n$, which is a contradiction since $C \neq 0$. Hence, R is not reduced.

(iii) Suppose that $|ZD(R)| = 3$. Thus, R is not reduced by (ii), and so there exists a nonzero nilpotent element d of R. Thus, there is an integer m such that $d^{m-1} \neq 0$ but $dd^{m-1} = d^m = 0$. Without loss of generality we may assume that $a = d$ and $b = d^{m-1}$. As seen in the previous proof, $c_0 \in \{0, d, d^{m-1}\}$. Since $|ZD(R)^*| = 2$, then $m > 2$ and so $c_0 \in \{0, d^{m-1}\}$. Similarly, we can see that $c_i \in \{0, d^{m-1}\}$ for every i. Thus, $C = (c_0, c_1, ..., c_n)$, where $c_i \in \{0, d^{m-1}\}$ such that $0 \neq d$ is a nilpotent element of R, $C \neq B$ and not all c_is are zero. □

Theorem 3.6 *If $\vec{\Gamma}(H_n)$ is a directed path and $Z(H_n) \neq \emptyset$, then $\vec{\Gamma}(R)$ is a connected graph and the following hold:*

(i) Let $0 \neq M$ be an arbitrary element of H_n. If $\vec{\Gamma}(R) = P_3$, then for each $K \in ZD_r(H_n)$ and $L \in ZD_l(H_n)$, we have $MK \in ZD_r(H_n)$ and $LM \in ZD_l(H_n)$.

(ii) If $\vec{\Gamma}(R) = P_2 = a \to b \to c$ and R is reduced, then we can determine the forms of elements of $ZD(R)$.

(iii) If $\vec{\Gamma}(R) = P_1$ and R is reduced, then we can determine the forms of all elements of $ZD(R)$.

Proof Since $\vec{\Gamma}(H_n)$ is a path, then it is a connected graph and so $diam(\vec{\Gamma}(H_n)) \leq 3$. Without loss of generality we may assume that

$$\vec{\Gamma}(H_n) : A \to B \to C \to D.$$

It should be mentioned that $\vec{\Gamma}(R)$ is always a subgraph of $\vec{\Gamma}(H_n)$. We claim that $\vec{\Gamma}(R)$ is connected. Otherwise, if $\vec{\Gamma}(R)$ is not connected, without loss of generality we may assume that $a \to b$ and $c \to d$ are distinct connected components of $\vec{\Gamma}(R)$. Note that we do not have an isolated vertex. Let $A_0 = (a, 0, 0, ..., 0)$, $B_0 = (b, 0, 0, ..., 0)$, $C_0 = (c, 0, 0, ..., 0)$, and $D_0 = (d, 0, 0, ..., 0)$. Then $A_0 \to B_0$ and $C_0 \to D_0$ are two paths with length 1. Now it is clear that $A_0 \to B_0$ and $C_0 \to D_0$ are subgraphs of $\vec{\Gamma}(H_n)$, but $\vec{\Gamma}(H_n)$ is a path of length three, and so we have either $A_0 \to B_0 \to C_0 \to D_0$ or $C_0 \to D_0 \to A_0 \to B_0$. In the first case, we have $B_0C_0 = 0$. This implies that $bc = 0$, which is impossible since $\vec{\Gamma}(R)$ is disconnected. Similarly, in the second case, we have $da = 0$, which is a contradiction. Thus, $\vec{\Gamma}(R)$ is connected. □

(i) If $\vec{\Gamma}(R) = P_3 : a \to b \to c \to d$, then we conclude that $A = (a, 0, 0, ...), B = (b, 0, 0, ...), C = (c, 0, 0, ...)$, and $D = (d, 0, 0, ...)$. Thus, $ZD_r(H_n) = \{0, A, B, C\}$ and $ZD_l(H_n) = \{0, B, C, D\}$. Let $0 \neq M \in H_n$. We claim that $MA \in ZD_r(H_n)$. It is easy to check that $MA = (m_0a, m_1a, ..., m_na)$. Thus, $(MAB)_j = \sum_{k=0}^{j} C_k^j (MA)_k B_{j-k} = m_jab = 0$ for each $0 \leq j \leq n$, and so $MAB = 0$, which implies that $MA \in ZD_r(H_n)$. One can follow a similar proof for the other elements. Similarly, we can see that $LM \in ZD_l(H_n)$ for any element $L \in ZD_l(H_n)$.

(ii) Without loss of generality we may assume that $A = (a, 0, ..., 0), B = (b, 0, ..., 0), C = (c, 0, ..., 0)$. Since $cd_0 = 0$, then $d_0 \in \{0, a, b, c\}$. Since R is reduced we conclude that $d_0 \in \{0, a, b\}$. In the next step, we have $0 = cd_1 + c_1d_0$, so $cd_1 = 0$. Thus, $d_1 \in \{0, a, b\}$. Continuing in this way, similar to the previous part, we can conclude that $d_i \in \{0, a, b\}$ for each i.

(iii) **Step 1:** Suppose that $A = (a, 0, ..., 0), B = (b, 0, ..., 0)$. Since $bc_0 = 0$ and R is reduced, we have $c_0 \in \{0, a\}$. By similar argument, we can see that $c_i \in \{0, a\}$ for each $1 \leq i \leq n$. Since $C \neq 0$, there exists $0 \leq j \leq n$ such that $0 \neq c_j = a$ and for each $i < j$, $c_i = 0$. On the other hand $CD = 0$, so for each i, $(CD)_i = 0$, especially $\sum_{k=0}^{j} C_m c_k d_{k-j} = (CD)_j = 0$, which implies that $ad_0 = c_jd_0 = 0$

and so $d_0 \in \{0, b\}$. In the second step, $0 = (CD)_{j+1}$, so $ad_1 + c_{j+1}d_0 = 0$. Since $d_0 \in \{0, b\}$ and $c_{j+1} \in \{0, a\}$, we will have four cases for the above equation. However, it is easy to see that in each case we get $d_1 \in \{0, b\}$. In the third step, $0 = (CD)_{j+2}$, so we have $ad_2 + c_{j+1}d_1 + c_{j+2}d_0 = 0$. Since $d_0, d_1 \in \{0, b\}$, and $c_{j+1}, c_{j+2} \in \{0, a\}$, by a similar argument we can see that $d_2 \in \{0, b\}$. Continuing in this way, we can conclude that $d_i \in \{0, b\}$ for each i.

Step 2: Suppose that $C = (c, 0, ..., 0)$ and $D = (d, 0, ..., 0)$. The proof in this case is similar to the proof of the first step and we can see that for each $0 \leq j \leq n$, $b_j \in \{0, d\}$ and $a_j \in \{0, c\}$.

Step 3: Suppose that $B = (b, 0, ..., 0)$, $C = (c, 0, ..., 0)$. By a similar proof we can see that for each $0 \leq j \leq n$, $a_j \in \{0, c\}$ and $d_j \in \{0, b\}$.

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