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Highly non-concurrent longest paths in lattices

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Abstract: In this paper we consider graphs in which any pair of vertices is missed by some longest path. We are proving the existence of such graphs in the infinite triangular, square and hexagonal lattices in the plane. Moreover, we extend our investigation to lattices on several surfaces such as the torus, the Möbius strip and the Klein bottle.

Key words: lattice graph, longest path, torus, Möbius strip, Klein bottle

1. Introduction

A graph is called *hypohamiltonian* if it is not hamiltonian but deleting any vertex gives a hamiltonian graph. The existence of hypohamiltonian graphs was the motivation of the following problem: "Do there exist graphs in which every vertex is missed by some longest path?", which was raised by Gallai in 1966 [5]. This problem received a positive answer in 1969, by a planar example with 25 vertices found by Walther [15]. Later a graph with only 12 vertices was found by Voss and Walther [16] and Zamfirescu [20], independently. In [20], it was conjectured that the order 12 is the smallest possible for such a graph. Recently, by using computers, Brinkmann and Van Cleemput proved that it is indeed minimal (see reference [5] in [12]). [12] is a survey that includes results regarding the intersection behavior of all longest paths (or cycles) and some other related topics.

In 1972, Zamfirescu asked a more general question: "Do we have graphs with the property that for any j vertices there is a longest path avoiding all of them?"[18]. In particular, the question was (also) asked for planar graphs. Several examples answering Zamfirescu's questions were published in the following years by Grünbaum [6], Schmitz [11], Zamfirescu [19, 20], Wiener and Araya [17], and others (see [12, 16, 21]).

We shall say that a finite graph G is a $\mathbf{P}_k^j$*-graph*, if it is k-connected and any set of j vertices is missed by some longest path.

The concept of fault tolerance is well known in computer sciences and engineering. It is associated with reliability, with the absence of breakdowns, and it is one of the key criteria in deciding the structures of interconnection networks for parallel and distributed systems. A fault-tolerant system follows a pattern or design (along with a set of instructions), which is usually modeled as a graph (see, e.g., [1, 7]), in which vertices and edges correspond to processing units and communication links, respectively. Any $\mathbf{P}_k^j$-graph is a

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fault-tolerant design, in which n processing units are interlinked, and l of these n units forming a chain of maximal length are used to execute some task. Any such graph can tolerate the failure of up to j components or communication links, keeping constant performance. Notice the difference with respect to the concept of j-fault hamiltonicity (see, e.g., [9, 10]), for which the (maximal) length of the used circuit changes when failure occurs.

These graphs have applications in networking and lattice networks are widely used in regular settings like grid computing, distributed parallel computation, distributed control, and wired circuits. Thus, in 2001, Zamfirescu asked these question with respect to graphs embeddable in various lattices [21].

In [8], the last two authors and Zamfirescu proved the existence of $\mathbf{P}_1^1$- and $\mathbf{P}_2^1$-graphs in the infinite square lattice $\mathcal{L}$ and hexagonal lattice $\mathcal{H}$. In [4], Dino and Zamfirescu presented $\mathbf{P}_1^1$- and $\mathbf{P}_2^1$-graphs in the infinite equilateral triangular lattice $\mathcal{T}$, while the first author and Zamfirescu found a $\mathbf{P}_1^1$-graph embeddable in the 3-dimensional cubic lattice [3].

In this paper, we are proving the existence of $\mathbf{P}_1^2$-graphs in various lattices. Moreover, it is worth mentioning that no $\mathbf{P}_1^3$-graph at all (whether planar or not) is known (see [14], p. 79).

The rest of the paper is organized as follows. In the next section we prove the existence of $\mathbf{P}_1^2$-graphs in planar lattices, and in the last section results about nonplanar lattices are discussed. Each section begins with a lemma, due to which the proofs of our other results are reduced to finding appropriate embeddings in the various lattices considered here.

2. $\mathbf{P}_1^2$-graphs in planar lattices

Let G and K be graphs homeomorphic to the graphs G' and K' in Figure 1a and Figure 11, respectively. Each of these respectively contain thirteen and nine subgraphs isomorphic to the graph H' (called *house*) of Figure 1b. The variables s, t, u, v, w, x, y, and z denote the number of vertices of degree 1 or 2 on paths corresponding to edges shown on the respective figures as well.

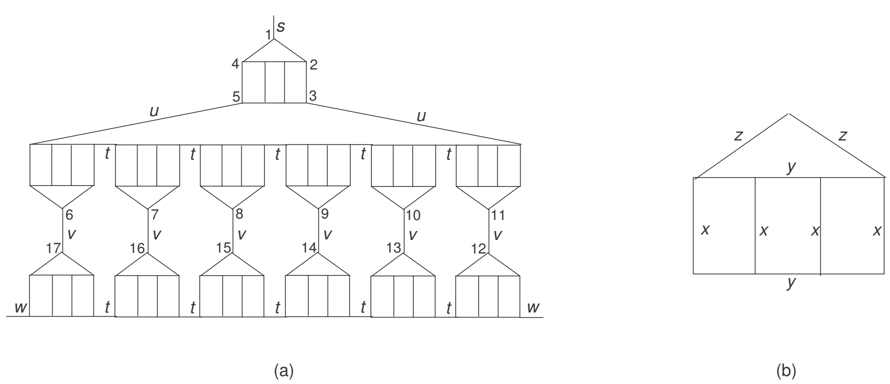

Figure 1.

Lemma 2.1 The graph G is a $\mathbf{P}_1^2$-graph if $2x \geq y + 2z + 1$, $u = 2x + \frac{1}{2}(y + 3t) + 4$, $v = 2x - 2z + u$, $w = 9x + 2y + 3t + 17$, and $s = x - z + w$.

Proof The proving technique is similar to the proof of the lemma in [13].

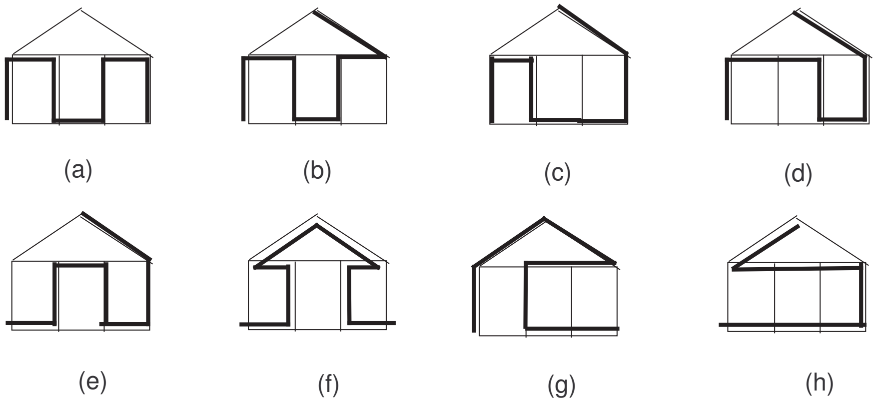

Figure 2.

Figure 2 shows all possible ways in which any longest path of G can go through a house of G. The graph G'' of Figure 3 stands for the graph obtained by contracting all houses in G. Every vertex of a house is avoided by a path of one of the types shown in Figures 2a–2e and paths symmetric to them.

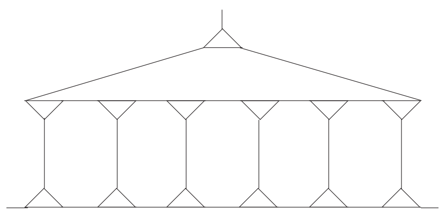

Figure 3. The graph $'G'$.

The graph G has the desired property if the following two conditions are satisfied:

(i) The paths of Figures 2f and 2g are not longer than the path of Figure 2a, and the path of Figure 2h is not longer than the path of Figure 2b.

(ii) Every pair of edges in G'' should be avoided by some path corresponding to a longest path of G.

The paths in Figures 2b–2e are of the same length. The respective lengths of the paths in Figures 2a, 2b, 2f, 2g, and 2h are:

$$a = 4x + y + 7,$$

$$b = 3x + y + z + 7,$$

$$f = 2x + 2z + 8,$$

$$g = 2x + 2y + 2z + 8,$$

$$h = x + 2y + z + 8.$$

The condition (i) is equivalent to the inequalities $a \geq \max\{f, g\} = g$ and $b \geq h$. The inequality $a \geq g$ means that $2x \geq y + 2z + 1$, which is the first condition of the lemma, while $b \geq h$ means $2x \geq y + 1$, which follows from the preceding inequality.

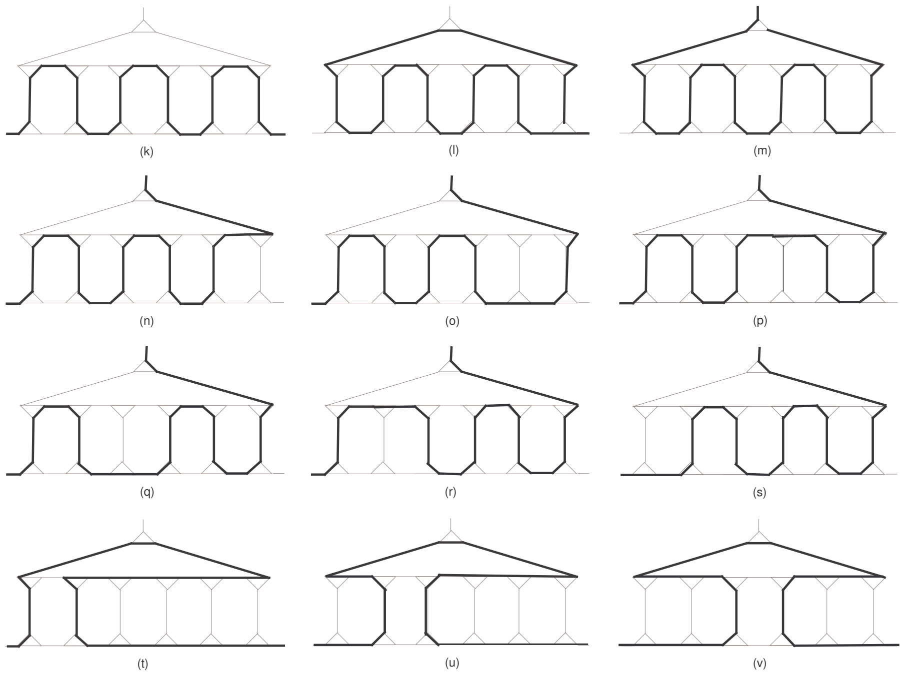

Figure 4.

Under the condition $2x \geq y + 2z + 1$, all possible paths (including paths symmetric to them) in G'' whose corresponding paths in G are candidates to be longest in G are shown in Figure 4. With respect to their lengths in G, these paths can be divided into five types, because the paths in Figures 4n–4s are of the same length and those in Figures 4t–4v are of same length, too. The lengths of paths in Figures 4k, 4l, 4m, 4n, and 4t are:

$$k = 36x + 12y + 12z + 5t + 6v + 2w + 95,$$

$$l = 41x + 13y + 12z + 5t + 2u + 6v + w + 104,$$

$$m = 40x + 13y + 13z + s + 5t + 2u + 6v + 104,$$

$$n = 37x + 12y + 11z + s + 5t + u + 5v + w + 95,$$

$$t = 48x + 13y + 4z + 8t + 2u + 2v + 2w + 103,$$

respectively. The reader can observe that all these five types of paths are needed to fulfill condition (ii). Hence, $k = l = m = n = t$ must hold. By solving these equalities we get the remaining four conditions of the lemma. □

Theorem 2.2 *There exist $\mathbf{P}_1^2$-graphs of order 483, 617, and 1366 in $\mathcal{T}$, $\mathcal{L}$, and $\mathcal{H}$, respectively.*

Proof The conditions of Lemma 1 are satisfied if we take $x = 2$, $y = 1$, $z = 0$, $t = 1$, $u = 10$, $v = 14$, $w = 40$, and $s = 42$, and the resulting graph G is a $\mathbf{P}_1^2$-graph of order 483, which has a realization in $\mathcal{T}$. Figure 5 shows such an embedding of G in $\mathcal{T}$. For a $\mathbf{P}_1^2$-graph in $\mathcal{L}$, see Figure 6; it is of order 617 and is obtained under the conditions of Lemma 1, for $x = 3$, $y = 1$, $z = 2$, $t = 1$, $u = 12$, $v = 14$, $w = 49$, and $s = 50$. In $\mathcal{H}$, we get such a graph for $x = 8$, $y = z = 3$, $t = 5$, $u = 29$, $v = 39$, $w = 110$, and $s = 115$. It has order 1366; see Figure 7. □

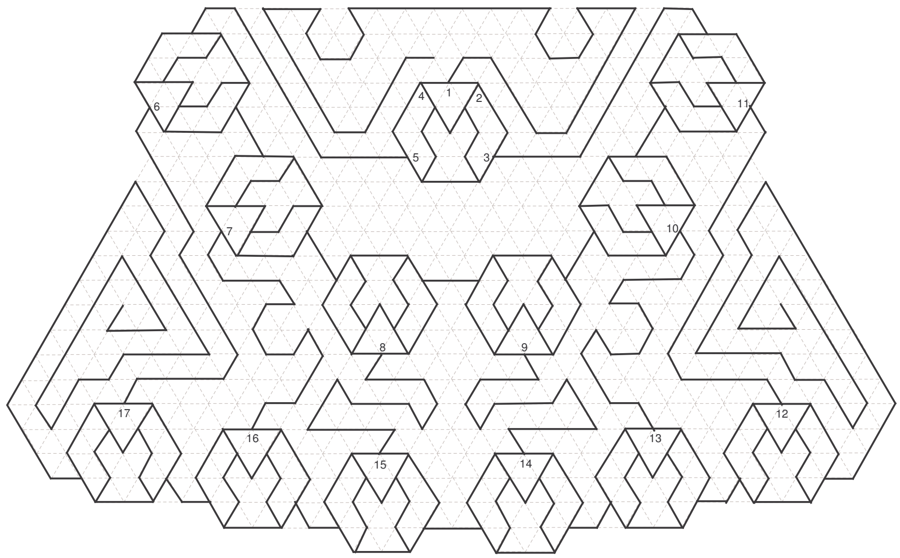

Figure 5.

3. $\mathbf{P_1^2}$-graphs in other lattices

The lattices that are under consideration here are lattices on the torus, the Möbius strip, and the Klein bottle. Each of these lattices is defined by embedding a finite subgraph of the triangular, square, or hexagonal lattice on some flat polygon representation of the considered surface. More precisely, we define them as follows.

We consider an $(m+1)\times(n+1)$ parallelogram (with $(m+1)(n+1)$ vertices) in $\mathcal{T}$ and by identifying opposite vertices on the boundary as indicated in Figure 8a we obtain the *toroidal triangular lattice* $\mathcal{T}^T_{m,n}$. It has mn vertices. The *toroidal square lattice* $\mathcal{L}^T_{m,n}$ and the *toroidal hexagonal lattice* $\mathcal{H}^T_{m,n}$ are defined similarly; see Figures 8b and 8c, respectively. Note that $\mathcal{H}^T_{m,n}$ is defined only for even values of m and n.

In a similar way lattices on the Möbius strip and Klein bottle are defined.

A *triangular lattice on the Möbius strip* $\mathcal{T}^M_{m,n}$ is defined according to Figures 9a and 9b, with respective

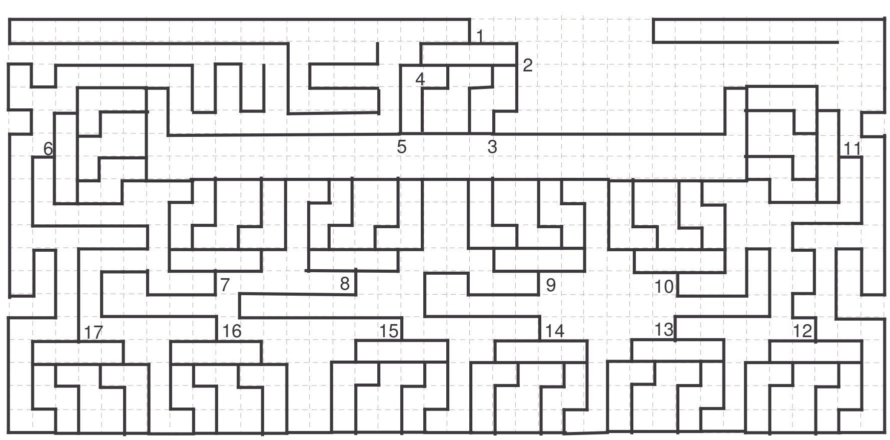

Figure 6.

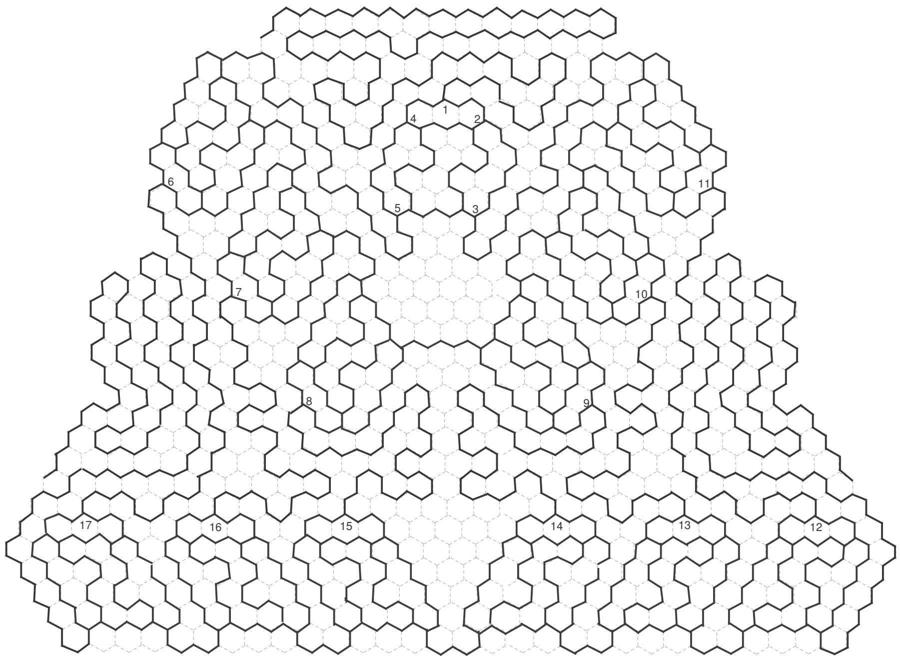

Figure 7.

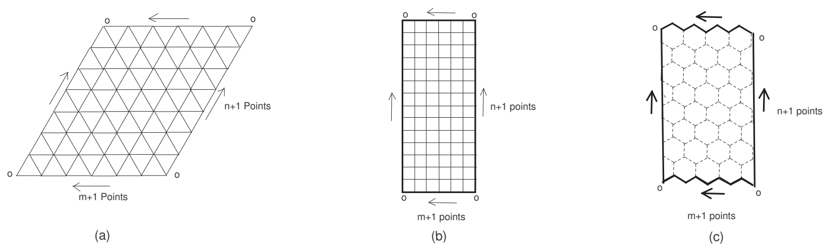

Figure 8. Toroidal lattices.

orders mn and $\frac{n}{2}(2m+1)$, while the *square lattice on the Möbius strip* $\mathcal{L}^M_{m,n}$ with mn vertices is defined as indicated in Figure 9c, and the *hexagonal lattice on the Möbius strip* $\mathcal{H}^M_{m,n}$ of order mn is defined according to Figures 9d and 9e.

The lattices on Klein bottles, $\mathcal{T}^K_{m,n}$, $\mathcal{L}^K_{m,n}$, and $\mathcal{H}^K_{m,n}$, are defined according to Figures 10a, 10b, and 10c, respectively. Each of these has order mn.

Lemma 3.1 The graph K is a $\mathbf{P}^2_1$-graph if $2x \geq y + 2z + 1$, $2x + u = 2z + v$ and $w = 2x + y + 3z + 2v + 9$.

A homeomorphic copy of the graph K is also embeddable in higher dimensional cubic lattices with 207 vertices (see [2]).

Theorem 3.2 *Each of the lattices* $\mathcal{T}^T_{21,16}$, $\mathcal{T}^M_{18,17}$, *and* $\mathcal{T}^K_{18,16}$ *contain a* $\mathbf{P}^2_1$*-graph of order* 267.

Proof The conditions of Lemma 2 are satisfied if we take $x = 2$, $y = 1$, $z = 0$, $u = 1$, $v = 5$, and $w = 24$,

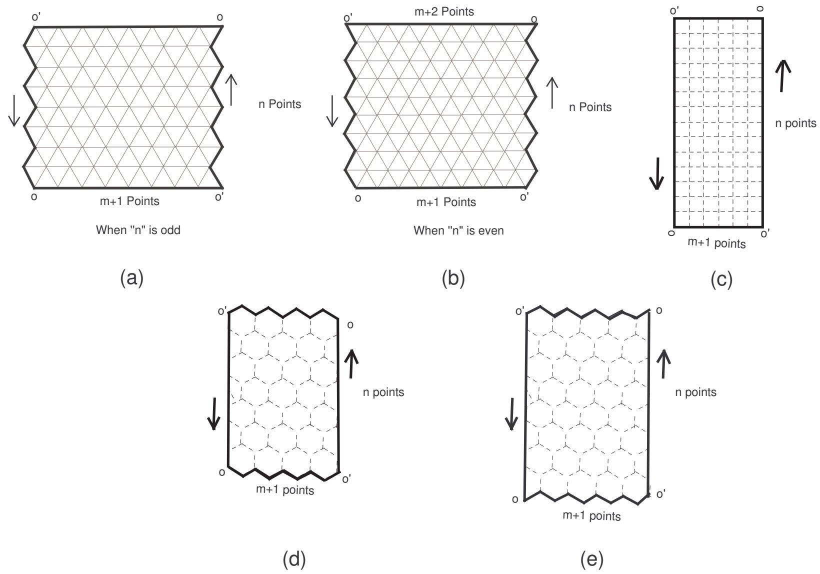

Figure 9. Lattices on Möbius strips.

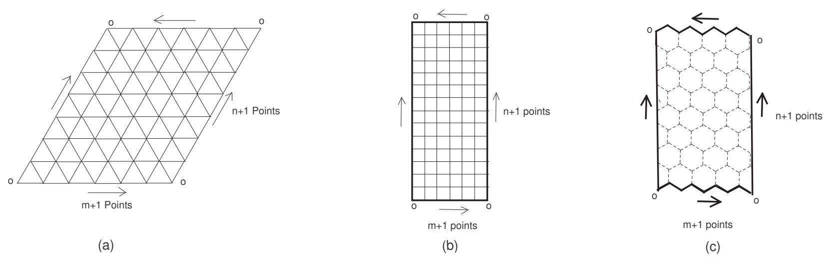

Figure 10. Lattices on Klein bottles.

and the resulting graph K is a $\mathbf{P}_1^2$-graph of order 267, which is embeddable in $\mathcal{T}_{21,16}^T$ and $\mathcal{T}_{18,17}^M$; see Figures 12 and 13, respectively. By identifying the two unmarked sides in Figure 13, in the same direction, we get an embedding of K in $\mathcal{T}_{18,16}^K$. □

Theorem 3.3 *$\mathcal{L}_{29,17}^T$ contains a $\mathbf{P}_1^2$-graph of order 363, while in $\mathcal{L}_{30,19}^M$, and $\mathcal{L}_{30,16}^K$ we have a $\mathbf{P}_1^2$-graph of order 345.*

Proof The conditions of Lemma 2 are satisfied if we take $x = 3$, $y = 1$, $z = 2$, $u = 2$, $v = 4$, and $w = 30$ and the resulting graph K is a $\mathbf{P}_1^2$-graph of order 363, which is embeddable in $\mathcal{L}_{29,17}^T$; see Figure 14.

The conditions of Lemma 2 are also satisfied if we take $x = 3$, $y = 1$, $z = 2$, $u = 1$, $v = 3$, and $w = 28$ and the resulting graph K is a $\mathbf{P}_1^2$-graph of order 345, which is embeddable in $\mathcal{L}_{30,19}^M$; see Figure 15. The graph K is realizable in $\mathcal{L}_{30,18}^K$, but we found an embedding of K in $\mathcal{L}_{30,16}^K$, which is given in Figure 16. □

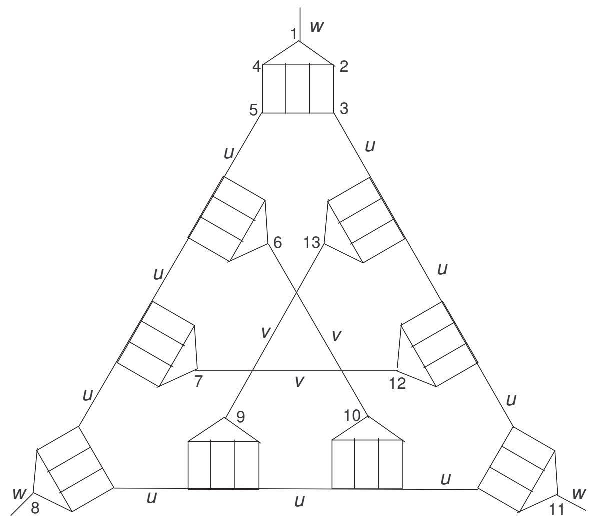

Figure 11.

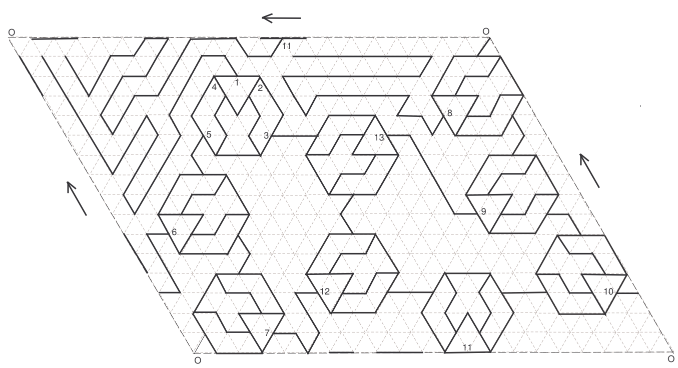

Figure 12.

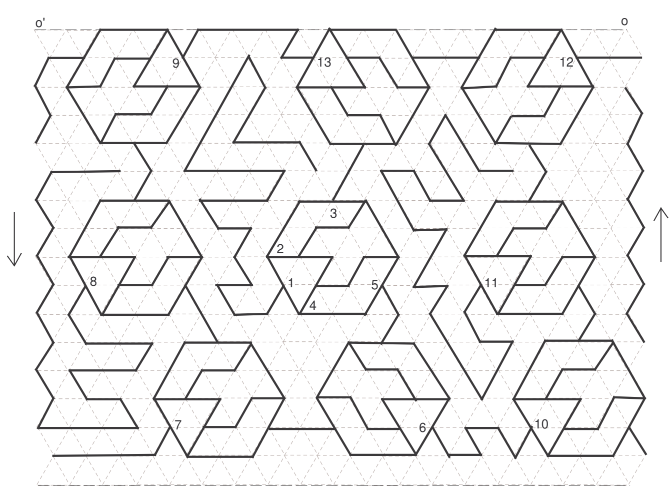

Figure 13.

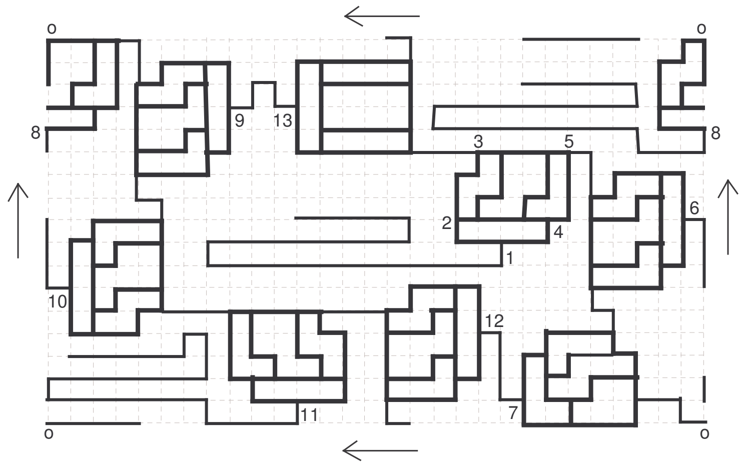

Figure 14.

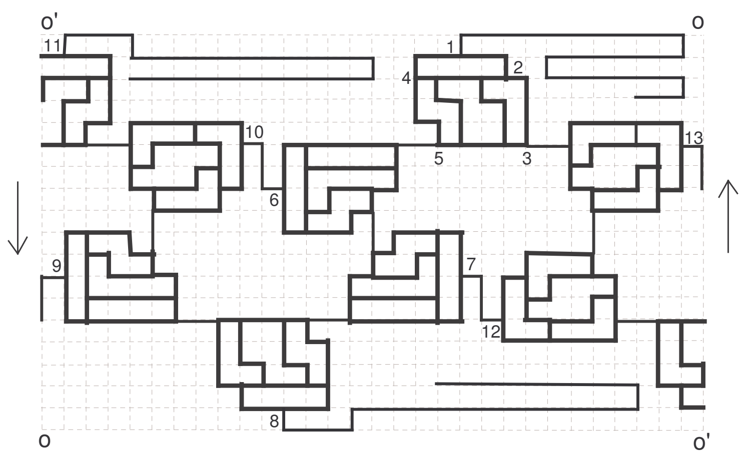

Figure 15.

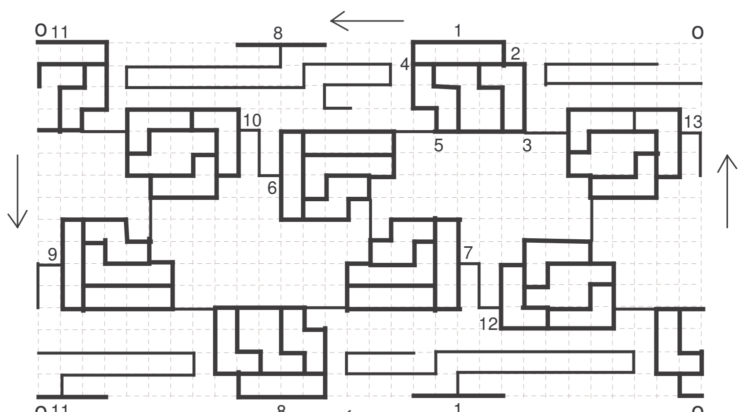

Figure 16.

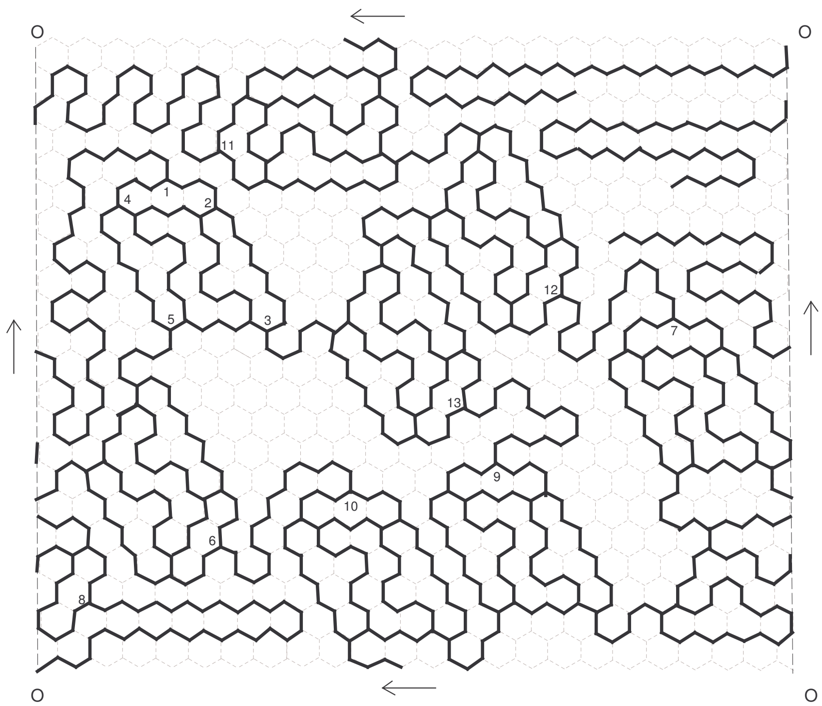

Figure 17.

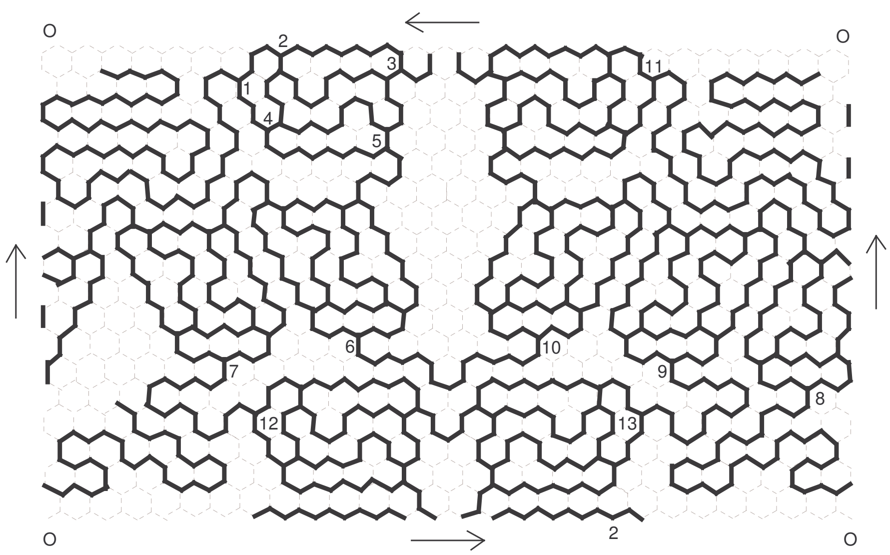

Figure 18.

Theorem 3.4 *$\mathcal{H}^T_{46,22}$ and $\mathcal{H}^K_{54,18}$ contain $\mathbf{P}^2_1$-graphs of order 768 each.*

Proof The conditions of Lemma 2 are also satisfied if we take $x = 8$, $y = z = 3$, $u = 5$, $v = 15$, and $w = 67$, and the resulting graph K is a $\mathbf{P}^2_1$-graph of order 768, which is embeddable in $\mathcal{H}^T_{46,22}$ and $\mathcal{H}^K_{54,18}$; see Figures 17 and 18, respectively. □

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