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# Abundance of $E$-order-preserving transformation semigroups

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**Abstract:** Let $\mathcal{T}_X$ be the full transformation semigroup on a finite totally ordered set $X = \{1 < 2 < \ldots < n\}$ $(n \geq 3)$ and $E$ be a nontrivial equivalence relation on $X$. In this paper, we consider a subsemigroup of $\mathcal{T}_X$ defined by

$$EOP_X = \{f \in \mathcal{T}_X : \forall\, x, y \in X,\ (x, y) \in E, x \leq y \Rightarrow (f(x), f(y)) \in E, f(x) \leq f(y)\}$$

and present a necessary and sufficient condition under which the semigroup $EOP_X$ is abundant.

**Key words:** Transformation semigroup, $\mathcal{L}^*$-relation, $\mathcal{R}^*$-relation, idempotent, abundance

## 1. Introduction

Let $S$ be a semigroup. We say that $a, b \in S$ are $\mathcal{L}^*$-related in $S$ if they are $\mathcal{L}$-related in a semigroup $T$ such that $S$ is a subsemigroup of $T$ and write $(a, b) \in \mathcal{L}^*$. The relation $\mathcal{R}^*$ is defined in the dual way. The equivalence relations $\mathcal{L}^*$ and $\mathcal{R}^*$ have been intensely studied in semigroup theory and have been used to define some important classes of semigroups. For instance, Fountain [3] pointed out that a semigroup $S$ has the property that for every $a \in S$ the right ideal $aS^1$ is projective (as an $S$-act) if and only if every $\mathcal{L}^*$-class of $S$ contains an idempotent. We call such semigroups *right abundant*. *Left abundant* semigroups are defined dually. A semigroup is *abundant* if it is both left and right abundant; see Fountain [4]. The property of being abundant can be considered as a wide generalization of regularity. (Recall that in a regular semigroup $\mathcal{L}^* = \mathcal{L}$ and $\mathcal{R}^* = \mathcal{R}$.)

Many papers have been written describing the abundances of various transformation semigroups on the nonempty set $X$ ( see [1, 8–12]). For example, Umar [11] observed that the semigroup $S_n^-$ of nonbijective, order-decreasing transformations on a finite totally ordered set $X = \{1 < 2 < \ldots < n\}$ is abundant but not regular. Let $\mathcal{T}_X$ be the full transformation semigroup on a set $X$ and $E$ be an arbitrary equivalence relation on $X$. Araujo and Konieczny [1] proved that the semigroup

$$T_E(X, R) = \{f \in \mathcal{T}_X : f(R) \subseteq R \text{ and } \forall\, x, y \in X,\ (x, y) \in E \Rightarrow (f(x), f(y)) \in E\},$$

where $R$ is a cross-section of the partition $X/E$ of $X$ induced by $E$, is abundant if and only if it is regular. Pei and Zhou [8] gave a condition under which the semigroup

$$T_E(X) = \{f \in \mathcal{T}_X : \forall\, x, y \in X,\ (x, y) \in E \Rightarrow (f(x), f(y)) \in E\}$$

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is abundant. Sun [9] proved that the semigroup

$$T(X,Y) = \{f \in \mathcal{T}_X : f(X) \subseteq Y\}\,(Y \subseteq X)$$

is left abundant but not right abundant if $|Y| \geq 2$ and $Y \neq X$. Sun and Wang [10] showed that the semigroup

$$T_\exists(X) = \{f \in \mathcal{T}_X : \forall\, x,\, y \in X,\, (f(x), f(y)) \in E \Rightarrow (x,y) \in E\}$$

is also left abundant but not right abundant if the partition $X/E$ of $X$ is infinite.

Given an arbitrary equivalence relation $E$ on a finite totally ordered set $X = \{1 < 2 < \ldots < n\}$, the authors [6] introduced a new family of the subsemigroup of $\mathcal{T}_X$ defined by

$$EOP_X = \{f \in \mathcal{T}_X : \forall\, x, y \in X,\, (x,y) \in E, x \leq y \Rightarrow (f(x), f(y)) \in E, f(x) \leq f(y)\},$$

which is called an $E$-order-preserving transformation semigroup, and investigated the properties for $EOP_X$, such as Green's relations and the natural partial order on the semigroup $EOP_X$ in [6] and [7], respectively. In particular, the regularity of the semigroup $EOP_X$ was described as follows.

**Lemma 1.1** ([6]) *The $E$-order-preserving transformation semigroup $EOP_X$ is regular if and only if either $E = X \times X$ or $E = \{(x,x) : x \in X\}$.*

In this paper our aim is to investigate the abundance of the semigroup $EOP_X$. Note that if $E = X \times X$ or $E = \{(x,x) : x \in X\}$ then $EOP_X$ is abundant. Thus, for the remainder of the paper, we assume that $E$ is nontrivial on the finite totally ordered set $X = \{1 < 2 < \ldots < n\}\,(n \geq 3)$; that is, both $E \neq X \times X$ and $E \neq \{(x,x) : x \in X\}$. Under the assumption, we first characterize the relations $\mathcal{L}^*$ and $\mathcal{R}^*$ on the semigroup $EOP_X$ and then present a necessary and sufficient condition under which the semigroup $EOP_X$ is abundant. Throughout this paper, we apply transformations on the left so that for $f,\, g \in EOP_X$, their product $fg$ is the transformation obtained by performing first $g$ and then $f$.

## 2. The main result

The following lemma gives a characterization of $\mathcal{L}^*$ and $\mathcal{R}^*$ that can be found, for instance, in [5, Sect. X.1].

**Lemma 2.1** *Let $S$ be a semigroup. Then*

$$\mathcal{L}^* = \{(a,b) \in S \times S : (\forall\, s,t \in S^1)\, as = at \Leftrightarrow bs = bt\}$$

*and*

$$\mathcal{R}^* = \{(a,b) \in S \times S : (\forall\, s,t \in S^1)\, sa = ta \Leftrightarrow sb = tb\}.$$

We begin with the $\mathcal{L}^*$-relation.

**Lemma 2.2** *Let $f,\, g \in EOP_X$. Then $(f,g) \in \mathcal{L}^*$ if and only if $\ker f = \ker g$.*

**Proof** For the 'if' part, suppose that $\ker f = \ker g$, and then $f$ and $g$ are known to be $\mathcal{L}$-related in the full transformation semigroup $\mathcal{T}_X$; see, for instance, [2, Sect. 2.2]1. Hence, $f$ and $g$ are $\mathcal{L}^*$-related in $EOP_X$.

[1] In order to prevent any chance of confusion, recall that in [2] transformations are written on the right of their arguments, while the description of Green's relations in [2, Section 2.2] should be left-right dualized to be applied in the present paper's setting.

For the 'only if' part, suppose that $(f,g) \in \mathcal{L}^*$. For $x \in X$, let $\langle x \rangle$ be the constant transformation with the range $\{x\}$; this transformation clearly belongs to $EOP_X$. Take $(x,y) \in \ker f$ for $x,y \in X$. Then $f\langle x \rangle = \{f(x)\} = \{f(y)\} = f\langle y \rangle$. Applying the characterization of $\mathcal{L}^*$ from Lemma 2.1, we have $g\langle x \rangle = g\langle y \rangle$. This means $g(x) = g(y)$ and $(x,y) \in \ker g$. Thus, $\ker f \subseteq \ker g$ and by symmetry $\ker g \subseteq \ker f$. Hence, $\ker f = \ker g$. □

In what follows we consider the $\mathcal{R}^*$-relation.

**Lemma 2.3** *Let $f, g \in EOP_X$. Then $(f,g) \in \mathcal{R}^*$ if and only if $f(X) = g(X)$.*

**Proof** For the 'if' part, suppose that $f(X) = g(X)$, and then $f$ and $g$ are known to be $\mathcal{R}$-related in the full transformation semigroup $\mathcal{T}_X$. Hence, $f$ and $g$ are $\mathcal{R}^*$-related in $EOP_X$.

For the 'only if' part, suppose that $(f,g) \in \mathcal{R}^*$ and $a \notin f(X)$. Let

$$\mathcal{A} = \{A \in X/E : A \cap f(X) \neq \emptyset\}.$$

For each $A \in \mathcal{A}$, let $A \cap f(X) = \{a_1 < a_2 < \ldots < a_s\}$. Write $a_0 = \min A$ and $a_* = \max A$. Define $h_* : A \longrightarrow A$ by

$$h_*(x) = \begin{cases} a_1 & \text{if} \quad x \in [a_0, a_1] \\ a_t & \text{if} \quad x \in (a_{t-1}, a_t] \, (2 \leq t \leq s) \\ a_s & \text{if} \quad x \in (a_s, a_*]. \end{cases}$$

Clearly, $h_*(A) = \{a_1, a_2, \ldots, a_s\} = A \cap f(X)$. Now we define $h : X \to X$. There are two cases to consider.

Case 1. $\overline{a} \notin \mathcal{A}$ where $\overline{a}$ is the $E$-class containing $a$. Fix $A_0 \in \mathcal{A}$ and $b \in A_0 \cap f(X)$. For each $A \in X/E$, define $h : X \to X$ by

$$h(x) = \begin{cases} h_*(x) & \text{if} \quad x \in A \text{ where } A \in \mathcal{A} \\ x & \text{if} \quad x \in A \text{ where } A \notin \mathcal{A} \text{ and } A \neq \overline{a} \\ b & \text{if} \quad x \in \overline{a}. \end{cases}$$

Case 2. $\overline{a} \in \mathcal{A}$. For each $A \in X/E$, define $h : X \to X$ by

$$h(x) = \begin{cases} h_*(x) & \text{if} \quad x \in A \text{ where } A \in \mathcal{A} \\ x & \text{if} \quad x \in A \text{ where } A \notin \mathcal{A}. \end{cases}$$

It is routine to show $h \in EOP_X$, $h \neq \mathrm{id}_X$, and $hf = \mathrm{id}_X f$, where $\mathrm{id}_X$ is the identity transformation on $X$. We assert that $a \notin g(X)$. Indeed, if $g(x') = a$ for some $x' \in X$, then applying the characterization of $\mathcal{R}^*$ from Lemma 2.1, we have $hg = \mathrm{id}_X g$ and $hg(x') = \mathrm{id}_X g(x')$. If $\overline{a} \notin \mathcal{A}$, then

$$b = h(\overline{a}) = hg(x') = \mathrm{id}_X g(x') = a,$$

a contradiction. If $\overline{a} \in \mathcal{A}$, then

$$h_* g(x') = hg(x') = \mathrm{id}_X g(x') = a \in f(X),$$

a contradiction. It follows readily that $a \notin g(X)$. This means that $g(X) \subseteq f(X)$. By symmetry, $f(X) \subseteq g(X)$. Consequently, $f(X) = g(X)$, as required. □

Let $Y, Z \subseteq X$ and $Y \cap Z = \emptyset$. $Y < Z$ means that $y < z$ for any $y \in Y$ and $z \in Z$.

**Lemma 2.4** *Let $f \in EOP_X$. Then $(f,e) \in \mathcal{R}^*$ for some idempotent $e \in EOP_X$. Consequently, the semigroup $EOP_X$ is left abundant.*

**Proof** Assume that

$$\{A \in X/E : A \cap f(X) \neq \emptyset\} = \{A_1 < A_2 < \ldots < A_t\}.$$

For each $A_i\,(1 \leq i \leq t)$, let $A_i \cap f(X) = \{a_{i1} < a_{i2} < \ldots < a_{is}\}$. Write $a_{i0} = \min A_i$ and $a_{i*} = \max A_i$ and then define $e_i : A_i \longrightarrow A_i$ by

$$e_i(x) = \begin{cases} a_{i1} & \text{if} \quad x \in [a_{i0}, a_{i1}] \\ a_{il} & \text{if} \quad x \in (a_{il-1}, a_{il}]\,(2 \leq l \leq s) \\ a_{is} & \text{if} \quad x \in (a_{is}, a_{i*}]. \end{cases}$$

For every $A \in X/E$, define $e : X \to X$ by

$$e(x) = \begin{cases} e_i(x) & \text{if} \quad x \in A_i\,(1 \leq i \leq t) \\ a_{11} & \text{if} \quad x \in A \text{ where } \overline{1} \leq A < A_1 \\ a_{i1} & \text{if} \quad x \in A \text{ where } A_{i-1} < A < A_i\,(2 \leq i \leq t) \\ a_{ts} & \text{if} \quad x \in A \text{ where } A_t < A \leq \overline{n}. \end{cases}$$

It is routine to show $e \in EOP_X$, $e^2 = e$, and $e(X) = f(X)$. By Lemma 2.3, we have $(e,f) \in \mathcal{R}^*$. □

In general, the semigroup $EOP_X$ is not right abundant; that is, there may be no idempotents in some $\mathcal{L}^*$-class of $EOP_X$. In what follows we pursue a necessary and sufficient condition under which the semigroup $EOP_X$ is abundant. For $f \in \mathcal{T}_X$, let $\pi(f)$ be the partition of $X$ induced by $\ker f$, namely

$$\pi(f) = \{f^{-1}(y) : y \in f(X)\},$$

and call $f^{-1}(y)$ a $\ker f$-class. For each $f \in T_E(X)$, let $E_f = E \vee \ker f$. Then $E_f$ is the smallest equivalence relation on $X$ containing both $E$ and $\ker f$ and each $E_f$-class is a union of $E$-classes as well as a union of $\ker f$-classes. Moreover, $f(F) \subseteq A \in X/E$ for each $E_f$-class $F$.

Recall that, in [1], a transformation $f$ is said to be *normal* if for each $E_f$ class $F$, there is some $E$-class $A \subseteq F$ such that $A \cap K \neq \emptyset$ for each $\ker f$-class $K \subseteq F$.

**Lemma 2.5** *Let $e \in EOP_X$ be an idempotent. Then $e$ is normal.*

**Proof** The proof is similar to that of [8, Lemma 2.8] and it is omitted. □

**Lemma 2.6** *Let $f \in EOP_X$. Then the following statements hold.*

*(1) $f$ is normal if and only if there is an idempotent $e \in EOP_X$ such that $\ker e = \ker f$.*

*(2) The semigroup $EOP_X$ is abundant if and only if $f$ is normal.*

**Proof** (1) For the 'if' part, suppose that $\ker e = \ker f$ for some idempotent $e \in EOP_X$. It is clear that $E_f = E_e$ and $f$ is normal.

For the 'only if' part, suppose that $f$ is normal. For each $E_f$-class $F$, there is some $E$-class $A$ such that $A \cap K \neq \emptyset$ for each $\ker f$-class contained in $F$. Take $k \in A \cap K$ and define $e : K \to K$ by $e(K) = k$. To see $e \in EOP_X$, take $E$-class $B \subseteq F$ and $x, y \in B$, $x \leq y$. Obviously, $e(B) \subseteq e(F) \subseteq A$, which implies that $(e(x), e(y)) \in E$. Now assume that $x \in K_x$ and $y \in K_y$ where $K_x, K_y \in \pi(f)$. If $K_x = K_y$, then

$e(x) = e(y)$. If $K_x \neq K_y$, then $x \neq y$ and $f(x) < f(y)$. By the definition of $e$, we have $e(x) = k_x$ and $e(y) = k_y$ where $k_x \in A \cap K_x$ and $k_y \in A \cap K_y$. Now we assert that $k_x < k_y$. Indeed, if $k_x > k_y$, then $f(x) = f(k_x) > f(k_y) = f(y)$, which leads to a contradiction. Hence, $k_x < k_y$ and $e \in EOP_X$. It is routine to show that $e^2 = e$ and $\ker e = \ker f$.

(2) The proof is similar to that of [8, Theorem 2.10] and it is also omitted. □

Recall that, in [1], an equivalence relation $E$ on $X$ is said to be *simple* if there is exactly one $E$-class ($\neq X$) containing more than one point and the other $E$-classes are singletons, and $E$ is said to be $n$-*bounded* if the cardinality of each $E$-class is not more than $n$.

**Lemma 2.7** *Let $E$ be an equivalence relation on $X$. Then the following statements hold.*

*(1) If $E$ is either simple or 2-bounded, then each $f \in EOP_X$ is normal.*

*(2) If $E$ is neither simple nor 2-bounded, then $EOP_X$ is not abundant.*

**Proof** (1) The proof is to similar to that of Lemmas 2.12 and 2.13 of [8].

(2) Assume that $A = \{a_1 < a_2 < \ldots < a_s\} \in X/E$ and $B = \{b_1 < b_2 < \ldots < b_t\} \in X/E$ for $s \geq 3, t \geq 2$. Now define $f : X \to X$ by

$$f(x) = \begin{cases} a_1 & \text{if } x = a_1 \\ a_2 & \text{if } x \in \{a_2, a_3, \ldots, a_s, b_1\} \\ a_3 & \text{if } x \in \{b_2, b_3, \ldots, b_t\} \\ x & \text{otherwise.} \end{cases}$$

It is clear that $f \in EOP_X$ and all $E_f$-class are $F = A \cup B$ and $C \in X/E$ with $C \neq A, C \neq B$. Moreover, there are exactly three $\ker f$-classes $K_1, K_2$, and $K_3$ contained in $F$, where

$$K_1 = \{a_1\}, K_2 = \{a_2, a_3, \ldots, a_s, b_1\}, K_3 = \{b_2, b_3, \ldots, b_t\}.$$

However, there is no $E$-class $D \subseteq F$ such that $D \cap K_i \neq \emptyset$ for $i = 1, 2, 3$, so $f$ is not normal. Therefore, $EOP_X$ is not abundant. □

Clearly, if $|X| = 3$, then $E$ is both simple and 2-bounded, so the semigroup $EOP_X$ is abundant. If $|X| = 4$, then $E$ is either simple or 2-bounded and the semigroup $EOP_X$ is also abundant. Thus, we have the main result in this paper.

**Theorem 2.8** *Let $E$ be a nontrivial equivalence on the finite totally ordered set $X = \{1 < 2 < \ldots < n\}$ $(n \geq 3)$. Then the following statements hold.*

*(1) If $|X| = 3$ or $|X| = 4$, then the semigroup $EOP_X$ is abundant.*

*(2) If $|X| \geq 5$, then the semigroup $EOP_X$ is abundant if and only if $E$ is either simple or 2-bounded.*

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