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Extension of refinement rings

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Abstract: In this paper we prove that a ring R in which every finitely generated projective R-module lifts modulo $J(R)$ is a refinement ring if and only if $\frac{R}{J(R)}$ is a refinement ring. We also prove that the refinement property for rings is Morita invariant. Several examples are constructed as well.

Key words: Refinement rings, projective modules, exchange rings

1. Introduction

The study of refinement monoids has a rich history. Dobbertin [8] in 1982 defined the monoid $(M,+,0)$ to be a refinement monoid if the following conditions are satisfied :

(1) There are no nonzero inverse elements, i.e. if $x+y=0$, then $x=y=0$.

(2) M has the refinement property; that is, given $x_i, y_j \in M$ with $\sum_i x_i = \sum_j y_j$, there are $z_{ij} \in M$ ($i<n, j<m$, where $n,m \in \mathbb{N}$ and $n,m \geqslant 2$) such that $x_i = \sum_j z_{ij}$ and $y_j = \sum_i z_{ij}$.

Note that we need only to show the above property for $m=n=2$. After him many authors, like Ara and Pardo [1], Brookfield [4], and Moreira [11], studied refinement monoids. Huang [9] in 2011 defined a ring R to be a refinement ring if the monoid of finitely generated projective R-modules is a refinement monoid. Following Chen [5], a ring R is an exchange ring if for any R-module M and any two decompositions $M = A \oplus B = \bigoplus_{i\in I} A_i$, where $A_R \cong R$ and index set I is finite, there exist $A_i' \subseteq A_i$ such that $M = A \oplus (\bigoplus_{i\in I} A_i')$.

In this article we investigate some elementary properties of refinement rings. In Section 2 it is shown that every projective-free ring is a refinement ring and we make an example of a refinement ring that is not projective-free. We also construct an example of a ring that is not a refinement ring. We prove that if R is a refinement ring then every finitely generated projective left R-module is isomorphic to a direct sum of left ideals generated by idempotents. We show that for any ring R such that finitely generated projective R-modules lift modulo $J(R)$, R is a refinement ring if and only if $\frac{R}{J(R)}$ is a refinement ring. Finally, we prove that the refinement property is closed under the finite direct product and Morita equivalent.

Throughout, all rings are associative with identity and all modules are unitary left R-modules. For any ring R, $FP(R)$ denotes the set of all finitely generated projective left R-modules, $V(R)$ the monoid of

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isomorphic classes $[P]$ of finitely generated projective left R-modules whose addition is defined by $[P]+[Q]=[P\oplus Q]$, and $M_n(R)$ the ring of all $n\times n$ matrices over R.

2. Refinement rings

We begin this section by recalling the concept of projective-free rings that is central to our work (for more information, see [5]).

Following Cohn [6], a ring R with invariant basis property is said to be projective-free if every finitely generated projective R-module is free.

Definition 2.1 *A ring R is said to be right (left) refinement if the monoid of finitely generated projective right (left) R-modules is refinement, i.e. $A_1\oplus A_2\cong B_1\oplus B_2$ implies that there exist R-modules $C_{ij}, 1\leqslant i,j\leqslant 2$ such that $A_i\cong C_{i1}\oplus C_{i2}$ and $B_j\cong C_{1j}\oplus C_{2j}$.*

As there exists a duality functor $Hom_R(-,R)$ between the category of finitely generated projective left R-modules and the category of finitely generated projective right R-modules, every right refinement ring is left refinement and conversely, so we use refinement instead of right or left refinement.

In 1964, Crawley and Jonsson [7] proved that the monoid of finitely generated projective modules of every exchange ring has the refinement property. It is also easy to see that every projective-free ring is a refinement ring. However, in the following example we see that every refinement ring is not always projective-free or exchange.

Example 2.2 *If $R=M_2(\mathbb{R})$, where $\mathbb{R}$ is the ring of real numbers, then R is a refinement ring, but it is not projective-free. Also, $\mathbb{Z}$ (the ring of integer numbers) is a refinement ring that is not exchange.*

Proof As $\mathbb{R}$ is a field, it is also a refinement ring. We will see later that the refinement property is Morita invariant, so R is a refinement ring. Also, $\mathbb{R}^2$ is a projective left R-module that it is not free, so R is not projective-free. On the other hand, $\mathbb{Z}$ is projective-free and so refining, but it cannot be an exchange ring. To see this, notice that $\mathbb{Z}$ is commutative with no nontrivial idempotents. If it is an exchange ring, then by [15, Theorem 1], it must be a local ring. □

We see that the class of refinement rings is very large, so it is not easy to find a ring that does not have the refinement property. Here we give an example of a monoid that is not refined.

Example 2.3 *If $M=\{0,1,\frac{1}{2}\}$, where $1+1=1+0=1+\frac{1}{2}=\frac{1}{2}+1=\frac{1}{2}+\frac{1}{2}=1$, then $(M,+,0)$ is a commutative monoid that is not refined, because the equation $1+1=\frac{1}{2}+\frac{1}{2}$ cannot be refined.*

To find a ring that is not a refinement ring, we use Bergman's Theorem [3].

Bergman's Theorem

Let K be a field and M be a commutative monoid with a distinguished element $1\neq 0$ such that:

(1) $\forall x,y\in M, x+y=0\Rightarrow x=y=0$.

(2) $\forall x\in M, \exists y\in M$ such that $x+y=n\cdot 1$ for some $n\in\mathbb{N}$.

Then there exists a hereditary K-algebra R such that $V(R)\cong M$ as monoid [3].

It is straightforward to verify that the monoid $M = \{0, 1, \frac{1}{2}\}$ of Example 2.3 satisfies the hypotheses of Bergman's Theorem. So we have the following:

Proposition 2.4 *There is a ring that is not a refinement ring.*

Proof It is enough to consider the ring that is obtained by applying Bergman's Theorem to the monoid constructed in Example 2.3. □

Now we give an example from [8] of a refinement monoid that has a nonrefinement submonoid.

Example 2.5 *Let* $A = \{0, 1, 2, \cdots, n, n+1, \cdots\}$ *(the set of all nonnegative integers), and* $B = \{0, 2, 3, \cdots, n, n+1, \cdots\}$*, and* $(A, 0, +)$ *and* $(B, 0, +)$ *are two monoids where* B *is a submonoid of* A *and* A *is a refinement monoid, while* B *is not, since* $2 + 4 = 3 + 3$ *cannot be refined.*

The previous example shows that the submonoid of a refinement monoid is not necessarily a refinement one.

In 1972 Warfield [16] proved that over any exchange ring every projective module is isomorphic to a direct sum of cyclic left ideals that are generated by idempotents; we have the same statement for projective modules over refinement rings.

Theorem 2.6 *Let* R *be a refinement ring; then every finitely generated projective left* R*-module is isomorphic to a direct sum of cyclic left ideals that are generated by idempotent elements.*

Proof Let P be a finitely generated projective R-module; then $P \oplus Q \cong R^n$ for some positive integer n and some R-module Q. As R is a refinement ring, there are projective left R-modules $I_1, I_2, \cdots, I_n$ and $J_1, J_2, \cdots, J_n$, such that $I_1 \oplus I_2 \oplus \cdots \oplus I_n \cong P$, $J_1 \oplus J_2 \oplus \cdots \oplus J_n \cong Q$ and $I_1 \oplus J_1 \cong R, I_2 \oplus J_2 \cong R \cdots I_n \oplus J_n \cong R$. Thus, there are $e_i = e_i^2 \in R$ such that $Re_i \cong I_i$ for all $i = 1, 2, \cdots, n$. Hence, we have $P = \sum_{i=1}^{n} Re_i$. □

Corollary 2.7 *Let* $A_1, \cdots, A_n$ *be finitely projective right modules over a refinement ring* R*. Then there exist orthogonal idempotents* $e_1, \cdots, e_k \in R$ *and nonnegative integers* t_{ij} *such that each* $A_i \cong t_{i1}(e_1R) \oplus \cdots \oplus t_{ik}(e_kR)$*.*

Proof By an application of Theorem 2.6, the proof is exactly similar to the proof of [13, Theorem 2.1] □

Lemma 2.8 *Let* B *and* C *be finitely generated projective* R*-modules over a commutative refinement ring* R*. If* $B/BP \cong C/CP$ *for all prime ideal* P *of* R*, then* $B \cong C$*.*

Proof By Corollary 2.7, there exist orthogonal idempotents $e_1, \cdots, e_k \in R$ such that $B \cong t_{11}(e_1R) \oplus \cdots \oplus t_{1k}(e_kR)$ and $C \cong t_{21}(e_1R) \oplus \cdots \oplus t_{2k}(e_kR)$. If $B \ncong C$, then we have an index j such that $e_j \neq 0$ and $t_{1j} \neq t_{2j}$. As the intersection of all prime ideals of R (i.e. prime radical) is nil, we can find a prime ideal P of R such that $e_j \notin P$; otherwise, $e_j \in R$ is nilpotent, and hence $e_j = 0$. On the other hand, for each $i \neq j$, $e_ie_j = 0 \in P$, and then $e_i \in P$. This shows that $B/BP \cong t_{1j}(R/P)$ and $C/CP \cong t_{2j}(R/P)$. As R is commutative, we easily get $t_{1j} = t_{2j}$, which is absurd. This completes the proof. □

Theorem 2.9 *Let* R *be a commutative refinement ring. For any* A, B, C *in* $FP(R)$ *and any* $n \in \mathbb{N}$*, we have:*

(1) *If* $nA \cong nB$ *then,* $A \cong B$*.*

(2) *If* $A \oplus B \cong A \oplus C$ *then,* $B \cong C$*.*

Proof Suppose that $nB \cong nC$ $(n \geq 1)$. Let P be a prime ideal of R. Then $n(B/BP) \cong n(C/CP)$. In view of Corollary 2.7, there exist orthogonal idempotents $e_1, \cdots, e_m \in R$ and nonnegative integers t_{ij} such that $B \cong t_{11}(e_1R) \oplus \cdots \oplus t_{1m}(e_mR)$ and $C \cong t_{21}(e_1R) \oplus \cdots \oplus t_{2m}(e_mR)$. As $e_i(1-e_i) \in P$, we see that $\overline{e_i} = \overline{0}, \overline{1}$ in R/P. Let $\overline{e_i} = \overline{1}$ $(1 \leq i \leq k)$ and $\overline{e_i} = \overline{0}$ $(k+1 \leq i \leq m)$. Then $B/BP \cong B \otimes_R (R/P) \cong (t_{11} + \cdots + t_{1k})(R/P)$ and $C/CP \cong C \otimes_R (R/P) \cong (t_{21} + \cdots + t_{2k})(R/P)$. This shows that $n(t_{11} + \cdots + t_{1k})(R/P) \cong n(t_{21} + \cdots + t_{2k})(R/P)$. As R is commutative, we get $n(t_{11} + \cdots + t_{1k}) = n(t_{21} + \cdots + t_{2k})$. We infer that $t_{11} + \cdots + t_{1k} = t_{21} + \cdots + t_{2k}$, and so $B/BP \cong C/CP$. Therefore, $B \cong C$, in terms of Lemma 2.8.

(2) Suppose that $A \oplus B \cong A \oplus C$. Since R is commutative, it follows by [5, page 291] that $nB \cong nC$ for some $n \geq 1$. In light of (1), we get $B \cong C$. □

We recall that a module M is said to be stably free if there exist finitely generated free modules F, F' such that $M \oplus F' \cong F$.

Corollary 2.10 *Every stably free module over a commutative refinement ring is free.*

Proof Let M be a stable free module over a commutative ring R. Write $M \oplus mR \cong nR (m, n \geq 1)$. Then $n \geq m$ as R is commutative. Hence, $M \oplus mR \cong (n-m)R \oplus mR$. In view of Theorem 2.9, $M \cong (n-m)R$, as desired. □

By Theorem 2.9, to show that a commutative ring R is not a refinement ring, it is enough to show that the monoid of finitely generated projective R-modules does not have the n-cancellation property.

Example 2.11 *Let $R = \mathbb{Z}[\sqrt{-5}]$ and $P = 2R + 2(1+\sqrt{-5})$. Then we have $2R \cong 2P$ as R-module, but $R \ncong P$. (See, e.g., Example 2.19D of [10].)*

Example 2.12 *Let $R = \mathbb{Z}[\sqrt{-5}], S = \mathbb{Q}[\sqrt{-5}]$, and $T = \mathbb{Z}[X]$. It is easy to show that R is a quotient of T and a subring of S. Also, S, T are projective-free, so they are refinement rings, but R is not a refinement ring.*

As R is a commutative ring that does not have the n-cancellation property, it is not a refinement ring.

Definition 2.13 *Let I be a two-sided ideal of a ring R. We say that every finitely generated projective module lifts modulo I, if for every finitely generated projetive $\frac{R}{I}$-module Q, there exists a finitely generated projective R-module P such that $\frac{P}{IP} \cong Q$.*

In 1977 Nicholson proved that a ring R is an exchange ring if and only if idempotents can be lifted modulo $J(R)$ and $\frac{R}{J(R)}$ has the exchange property [12]. We extend this assertion with a small change for the refinement property.

Theorem 2.14 *Let R be a ring such that finitely generated projective R-modules lift modulo $J(R)$. Then R is a refinement ring if and only if $\frac{R}{J(R)}$ is a refinement ring.*

Proof Let R be a refinement ring. Suppose that $P_1' \oplus P_2' \cong Q_1' \oplus Q_2'$ as finitely generated projective left $\frac{R}{J(R)}$-modules. As every finitely generated projective R-module lifts modulo $J(R)$, then there are P_1, P_2, Q_1

and $Q_2 \in FP(R)$ such that $\frac{P_1}{J(R)P_1} \cong P_1'$, $\frac{P_2}{J(R)P_2} \cong P_2'$, $\frac{Q_1}{J(R)Q_1} \cong Q_1'$ and $\frac{Q_2}{J(R)Q_2} \cong Q_2'$. Thus, we have:

$$\frac{P_1}{J(R)P_1} \oplus \frac{P_2}{J(R)P_2} \cong \frac{Q_1}{J(R)Q_1} \oplus \frac{Q_2}{J(R)Q_2}$$
$$\Rightarrow \frac{P_1 \oplus P_2}{J(R)(P_1 \oplus P_2)}$$
$$\cong \frac{Q_1 \oplus Q_2}{J(R)(Q_1 \oplus Q_2)}.$$

Since $J(R)$ is a superfluous ideal, we have by [2, Theorem 5] $P_1 \oplus P_2 \cong Q_1 \oplus Q_2$. Since R is a refinement ring, there are P_{11}, P_{12}, Q_{11}, and Q_{12} such that

$$P_1 \cong P_{11} \oplus P_{12}, P_2 \cong Q_{11} \oplus Q_{12}$$

and

$$P_{11} \oplus Q_{11} \cong Q_1, P_{12} \oplus Q_{12} \cong Q_2.$$

We then get:

$$\frac{P_{11} \oplus P_{12}}{J(R)(P_{11} \oplus P_{12})} \cong \frac{P_1}{J(R)(P_1)},$$
$$\frac{Q_{11} \oplus Q_{12}}{J(R)(Q_{11} \oplus Q_{12}}) \cong \frac{P_2}{J(R)P_2}.$$

Thus,

$$P_1' \cong \frac{P_1}{J(R)P_1} \cong \frac{P_{11}}{J(R)P_{11}} \oplus \frac{P_{12}}{J(R)P_{12}},$$

$$P_2' \cong \frac{P_2}{J(R)P_2} \cong \frac{Q_{11} \oplus Q_{12}}{J(R)(Q_{11} \oplus Q_{12})},$$

$$Q_1' \cong \frac{Q_1}{J(R)Q_1} \cong \frac{P_{11}}{J(R)P_{11}} \oplus \frac{Q_{11}}{J(R)Q_{11}},$$

$$Q_2' \cong \frac{Q_2}{J(R)Q_2} \cong \frac{P_{12}}{J(R)P_{12}} \oplus \frac{Q_{12}}{J(R)Q_{12}}.$$

Then $\frac{R}{J(R)}$ is a refinement ring.

Conversely, assume that $\frac{R}{J(R)}$ is a refinement ring, and suppose that $P_1 \oplus P_2 \cong Q_1 \oplus Q_2$, for $P_1, P_2, Q_1, Q_2 \in FP(R)$. We have:

$$\frac{P_1 \oplus P_2}{J(R)(P_1 \oplus P_2)} \cong \frac{Q_1 \oplus Q_2}{J(R)(Q_1 \oplus Q_2)} \cong \frac{P_1}{J(R)P_1} \oplus \frac{P_2}{J(R)P_2} \cong \frac{Q_1}{J(R)Q_1} \oplus \frac{Q_2}{J(R)Q_2}.$$

Since $\frac{R}{J(R)}$ is a refinement ring, there are P_1', P_2', Q_1' and $Q_2' \in FP(\frac{R}{J(R)})$ such that

$$\frac{P_1}{J(R)P_1} \cong P_1' \oplus P_2', \frac{P_2}{J(R)P_2} \cong Q_1' \oplus Q_2',$$

$$\frac{Q_1}{J(R)Q_1} \cong P_1' \oplus Q_1', \frac{Q_2}{J(R)Q_2} \cong P_2' \oplus Q_2'.$$

As every finitely generated projective R-module lifts modulo J(R), there exist finitely generated projective R-modules P_{11}, P_{12}, Q_{11}, and Q_{12} such that $\frac{P_{11}}{J(R)P_{11}} \cong P_1'$, $\frac{P_{12}}{J(R)P_{12}} \cong P_2'$, $\frac{Q_{11}}{J(R)Q_{11}} \cong Q_1'$, and $\frac{Q_{12}}{J(R)Q_{22}} \cong Q_2'$, and then

$$\frac{P_1}{J(R)P_1} \cong \frac{P_{11}}{J(R)P_{11}} \oplus \frac{P_{12}}{J(R)P_{12}} \cong \frac{P_{11} \oplus P_{12}}{J(R)(P_{11} \oplus P_{12})},$$

$$\frac{P_2}{J(R)P_2} \cong \frac{Q_{11}}{J(R)Q_{11}} \oplus \frac{Q_{12}}{J(R)Q_{12}} \cong \frac{Q_{11} \oplus Q_{12}}{J(R)(Q_{11} \oplus Q_{12})},$$

$$\frac{Q_1}{J(R)Q_1} \cong \frac{P_{11}}{J(R)P_{11}} \oplus \frac{Q_{11}}{J(R)Q_{11}} \cong \frac{P_{11} \oplus Q_{11}}{J(R)(P_{11} \oplus Q_{11})},$$

$$\frac{Q_2}{J(R)Q_2} \cong \frac{P_{12}}{J(R)P_{12}} \oplus \frac{Q_{12}}{J(R)Q_{12}} \cong \frac{P_{12} \oplus Q_{12}}{J(R)(P_{12} \oplus Q_{12})}.$$

Again by [2, Theorem 5],

$$P_1 \cong P_{11} \oplus P_{12} \quad , \quad P_2 \cong Q_{11} \oplus Q_{12}$$
$$Q_1 \cong P_{11} \oplus Q_{11} \quad , \quad Q_2 \cong P_{12} \oplus Q_{12}.$$

□

Lemma 2.15 *If I is a two-sided ideal of a ring R such that every idempotent can be lifted modulo I and $\frac{R}{I}$ be a refinement ring, then every finitely generated projective R-module lifts modulo I.*

Proof Let Q be a finitely generated projective $\frac{R}{I}$-module. As $\frac{R}{I}$ is a refinement ring, then by Theorem 2.6, $Q \cong \bigoplus_{i=1}^{n} \frac{R}{I}\ \overline{e}_i \cong \bigoplus_{i=1}^{n} \frac{R}{I}(e_i + I)$. Since idempotents lift modulo I, there exist $f_i^2 = f_i \in R, i = 1, 2, \cdots, n$ such that $e_i - f_i \in I$, so $\bigoplus_{i=1}^{n} \frac{R}{I}(e_i + I) \cong \bigoplus_{i=1}^{n} \frac{R}{I}(f_i + I) \cong \frac{\bigoplus_{i=1}^{n} Rf_i}{\bigoplus_{i=1}^{n} I(Rf_i)} \cong \frac{P}{IP}$ for some finitely generated projective R-module P. □

Theorem 2.16 *Let I be a two-sided ideal of a ring R contained in $J(R)$ such that every idempotent of R lifts modulo I and $\frac{R}{I}$ be a refinement ring. Then R is a refinement ring.*

Proof As $I \subseteq J(R)$, we can easily verify that I is a superfluous ideal. Also, every idempotent lifts modulo $J(R)$. Therefore, the result follows from Theorem 2.14 and Lemma 2.15. □

Corollary 2.17 *Let I be a nil ideal of a ring R. If $\frac{R}{I}$ is refinement ring, then R is a refinement ring.*

Proof Since I is a nil ideal then we have by [14, Lemma 2] that finitely generated projective R-modules lift modulo I. Now assume that $I + K = R$ for some ideal K of R. Then $1 = x + y$ for some $x \in I, y \in K$. Since I is a nil ideal of R, for some $n \in \mathbb{N}$, we get $x^n = (1-y)^n = 0$. Now for some $y' \in K$ we have, $(1-y)^n = 1 - y' = 0$. That shows that $1 = y'$ and $y' \in K$, and then $K = R$. Therefore, I is superfluous. An application of Theorem 2.16 completes the proof. □

Corollary 2.18 *Let R be a ring such that $Soc^2(R)$ is a finitely generated ideal contained in $J(R)$. Then R is a refinement ring if and only if $\frac{R}{Soc(R)}$ is a refinement ring.*

Proof As $Soc^2(R)$ is a finitely generated ideal, we see by [14, Theorem 2] that every finitely generated projective R-module lifts modulo $Soc(R)$. The result follows from Theorem 2.16. □

Now we want to investigate the refinement property under ring homomorphism.

Proposition 2.19 *Let $f : R \longrightarrow S$ and $g : S \longrightarrow R$ be two ring homomorphisms such that $gf = 1_R$ and $Ker(g) \subseteq J(S)$. Then R is a refinement ring if and only if S is a refinement ring.*

Proof Let $A, B, C, D \in FP(S)$ and $A \oplus B \cong C \oplus D$. Then $A \otimes_g R \oplus B \otimes_g R \cong C \otimes_g R \oplus D \otimes_g R$. As R is a refinement ring, we have

$$A \otimes_g R \cong A' \oplus C', B \otimes_g R \cong B' \oplus D'$$

such that

$$C \otimes_g R \cong A' \oplus B', D \otimes_g R \cong C' \oplus D'.$$

Hence,

$$A \otimes_g R \cong A' \oplus C' \cong A' \otimes_R R \oplus C' \otimes_R R$$

$$\cong A' \otimes_{gf} R \oplus C' \otimes_{gf} R \cong (A' \otimes_f S \oplus C' \otimes_f S) \otimes_g R.$$

As g is surjective, we get $\frac{S}{Ker(g)} \cong R$. Hence,

$$A \otimes_g \frac{S}{Ker(g)} \cong (A' \otimes_f S \oplus C' \otimes_f S) \otimes_g \frac{S}{Ker(g)}.$$

It follows from $Ker(g) \subseteq J(S)$ that

$$A \cong A' \otimes_f S \oplus C' \otimes_f S.$$

Similarly, we get

$$B \cong B' \otimes_f S \oplus D' \otimes_f S,$$

$$C \cong A' \otimes_f S \oplus B' \otimes_f S,$$

$$D \cong C' \otimes_f S \oplus D' \otimes_f S.$$

This implies that S is a refinement ring.

Conversely, assume that for any $A, B, C, D \in FP(R)$, $A \oplus B \cong C \oplus D$. We get

$$A \otimes_f S \oplus B \otimes_f S \cong C \otimes_f S \oplus D \otimes_f S.$$

Since S is a refinement ring, we have

$$A \otimes_f S \cong A' \oplus C', B \otimes_f S \cong B' \oplus D'.$$

Also,

$$C \otimes_f S \cong A' \oplus B', D \otimes_f S \cong C' \oplus D'.$$

It follows from $A \otimes_f S \cong A' \oplus C'$ that

$$A \otimes_{gf} R \cong A \otimes_f S \otimes_g R \cong A' \otimes_g R \oplus C' \otimes_g R.$$

As $gf = 1$, we see that

$$A \cong A' \otimes_g R \oplus C' \otimes_g R.$$

Likewise, we get

$$B \cong B' \otimes_g R \oplus D' \otimes_g R,$$

$$C \cong A' \otimes_g R \oplus B' \otimes_g R,$$

$$D \cong C' \otimes_g R \oplus D' \otimes_g R.$$

This completes the proof. □

Corollary 2.20 *For a ring R, the following statements are equivalent.*

(1) *R is a refinement ring.*

(2) *$R[[x_1, x_2, ..., x_n]]$ is a refinement ring.*

Proof Let $f : R \to R[[x_1, x_2, ..., x_n]]$ be defined by $f(r) = r$, and let $g : R[[x_1, x_2, ..., x_n]] \to R$ be defined by $g(\phi(x_1, x_2, ..., x_n) = \phi(0, 0, ..., 0)$. Then $gf = 1_R$ and $Ker(g) \subseteq J(R[[x_1, x_2, ..., x_n]])$. Then the result follows from Proposition 2.19. □

Most of the properties that we know, such as, for example, the exchange property, are Morita invariant and closed under a finite direct product. In the next theorem we prove that these facts also hold for refinement rings.

Theorem 2.21 *Let R, R_1, and R_2 be rings and let P be a finitely generated R-progenerator. Then the following statements hold.*

(1) *$R_1 \times R_2$ is a refinement ring if and only if R_1 and R_2 are refinement rings.*

(2) *If R is a refinement ring, then every ring S that is Morita equivalent to R is a refinement ring. In particular, for any full idempotent $e \in R$ and any $n \in \mathbb{N}$, eRe and $M_n(R)$ are refinement rings.*

(3) *The ring $End_R(P)$ is a refinement ring.*

(4) *If the Krull–Schmidt Theorem holds for finitely generated projective R-modules, then R is a refinement ring.*

Proof

(1). Let R_1 and R_2 be refinement rings. Since $V(R_1 \times R_2) \cong V(R_1) \times V(R_2)$ and the multiplication of two refinement monoids is refinement, then $R_1 \times R_2$ is refinement. Conversely, assume that $R_1 \times R_2$ is a refinement ring. Then $V(R_1 \times R_2)$ is a refinement monoid that implies that $V(R_1)$ and $V(R_2)$ are refinement monoids. Then R_1 and R_2 are refinement rings.

(2). Since the V functor is invariant by Morita equivalence, the result is clear.

(3). As P is a finitely generated projective generator, $P \cong e(R^n)$ for some $n \in \mathbb{N}$ and $e \in M_n(R)$, a full idempotent, so $End(P)_R \cong eM_n(R)e$, and now the assertion follows from (2).

(4). If $P_1 \oplus Q_1 \cong P_2 \oplus Q_2$ for $P_1, Q_1, P_2, Q_2 \in V(R)$, then $P_i \cong \sigma_i(Q_i)$, $(1 \leq i \leq 2)$ for some permutation. Thus, R is a refinement ring. □

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References

[1] Ara P, Pardo E. Refinement monoids with weak comparability and applications to regular rings and $\mathbb{C}^\star$-algebras. P Am Math Soc 1996; 124: 1–6.

[2] Beck I. Projective and free modules. Math Z 1972; 129: 231–234.

[3] Bergman GM. Coproducts and some universal ring constructions. T Am Math Soc 1974; 200: 33–88.

[4] Brookfield G. Cancellation in primely generated refinement monoids. Algebra Universalis 2001; 46: 343–371.

[5] Chen H. Rings Related to Stable Range Conditions. Series in Algebra, 11. Hackensack, NJ, USA: World Scientific Publishing, 2011.

[6] Cohn PM. Free Rings and Their Relations. 2nd ed. London, UK; London Mathematical Society, 1985.

[7] Crawley P, Jonsson B. Refinement for infinite decompositions of algebraic systems. Pacific J Math 1964; 14: 797–855.

[8] Dobbertin H. On Vaught's criterion for isomorphisms of countable Boolean algebra. Algebra Universalis 1982; 15: 95–144.

[9] Huang CH. Refinement rings,exchange property and comparability. Bull Korean Math Soc 2011; 48: 455–468.

[10] Lam TY. Lectures on Modules and Rings. New York, NY, USA: Springer Verlag, 1999.

[11] Moreira C. A refinement monoid whose maximal antisymmetric quotient is not a refinement monoid. Semigroup Forum 2002; 65: 249–263.

[12] Nicholson WK. Lifting idempotents and exchange rings. T Am Math Soc 1977; 229: 1–10.

[13] Tungsuo WU. Finitely generated projective modules over exchange rings. Manuscripta Math 1995; 86: 149–157 .

[14] Tungsuo WU. Global lifting of projective modules modulo socles. Chinese Sci Bull 1996; 41: 1–3.

[15] Warfield RB. A Krull-Schmidt theorem for infinite sums of modules. P Am Math Soc 1969; 22: 460–465.

[16] Warfield RB. Exchange rings and decompositions of modules. Math An 1972; 199: 31–36.