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The q-analogue of the $\mathcal{E}_{2;1}$-transform and its applications

Ahmed SALEM[1], **Faruk UÇAR**[2,*]
[1]Department of Basic Science, Faculty of Information Systems and Computer Science, October 6 University, Sixth of October City, Giza, Egypt
[2]Department of Mathematics, Marmara University, Kadıköy, İstanbul, Turkey

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Abstract: In this paper, we introduce a new integral transform ${}_q\mathcal{E}_{2;1}$, which is the q-analogue of the $\mathcal{E}_{2;1}$-transform and can be regarded as a q-extension of the $\mathcal{E}_{2;1}$-transform. Some identities involving ${}_qL_2$-transfom, ${}_q\mathcal{L}_2$-transfom, and $\mathcal{P}_q$-transform are given. By making use of these identities and ${}_q\mathcal{E}_{2;1}$-transform, a new Parseval–Goldstein type theorem is obtained. Some examples are also given as an illustration of the main results presented here.

Key words: q-Exponential integral, ${}_qL_2$-transfom, ${}_q\mathcal{L}_2$-transfom, $\mathcal{P}_q$-transform, q-analogue of $\mathcal{E}_{2;1}$-transform

1. Introduction

The q-derivative D_qf of a function f is given as

$$(D_qf)(z) = \frac{f(z) - f(qz)}{(1-q)z}, \quad z \neq 0, \quad (D_qf)(0) = f'(0) \tag{1.1}$$

provided $f'(0)$ exists. If f is differentiable then $D_qf(z)$ tends to $f'(z)$ as $q \to 1^-$.

Recall that the q-derivative of two product functions states

$$D_q[f(z)g(z)] = g(z)D_qf(z) + f(zq)D_qg(z) = f(z)D_qg(z) + g(zq)D_qf(z). \tag{1.2}$$

The Jackson q-integral in a generic interval $[a, b]$ is given as

$$\int_a^b f(t)d_qt = \int_0^b f(t)d_qt - \int_0^a f(t)d_qt \tag{1.3}$$

where

$$\int_0^a f(x)d_qx = a(1-q)\sum_{n=0}^{\infty} q^n f(q^n a). \tag{1.4}$$

The improper integral is defined in the following way:

$$\int_0^{\frac{\infty}{A}} f(x)d_qx = (1-q)\sum_{n=-\infty}^{\infty} \frac{q^n}{A} f\left(\frac{q^n}{A}\right). \tag{1.5}$$

*Correspondence: fucar@marmara.edu.tr
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As a result of the last formula, one has the following reciprocity relations:

$$\int_0^{\infty} f(x)d_qx = \int_0^{\infty} \frac{1}{x^2} f\left(\frac{1}{x}\right) d_qx. \tag{1.6}$$

The q-integration by parts is given for suitable functions f and g as

$$\int_a^b f(t)D_qg(t)d_qt = f(b)g(b) - f(a)g(a) - \int_a^b g(tq)D_qf(t)d_qt. \tag{1.7}$$

The q-analogue of the integration theorem by change of variable is given when $u(z) = \alpha z^{\beta}$, $\alpha \in \mathbb{C}$, and $\beta > 0$ as follows:

$$\int_{u(a)}^{u(b)} f(u)d_qu = \int_a^b f(u(z))D_{q^{\frac{1}{\beta}}}u(z)d_{q^{\frac{1}{\beta}}}z. \tag{1.8}$$

The q-gamma function is defined as

$$\Gamma_q(a) = \int_0^{\frac{1}{1-q}} t^{a-1}E_q((1-q)qt)d_qt = \int_0^{\frac{\infty}{1-q}} t^{a-1}E_q((1-q)qt)d_qt, \qquad \Re(a) > 0. \tag{1.9}$$

Furthermore, it has the representation

$$\Gamma_q(a) = K_q(a)\int_0^{\frac{\infty}{1-q}} t^{a-1}e_q(-(1-q)t)d_qt, \qquad \Re(a) > 0, \tag{1.10}$$

where

$$E_q(x) = \sum_{n=0}^{\infty} \frac{(-1)^n q^{\frac{n(n-1)}{2}} x^n}{(q;q)_n} = (x;q)_{\infty}, \tag{1.11}$$

$$e_q(x) = \sum_{n=0}^{\infty} \frac{x^n}{(q;q)_n} = \frac{1}{(x;q)_{\infty}}, \qquad |x| < 1, \tag{1.12}$$

$$K_q(a) = \frac{(-q;q)_{\infty}(-1;q)_{\infty}}{(-q^a;q)_{\infty}(-q^{1-a};q)_{\infty}}. \tag{1.13}$$

El-Shahed and Salem [3] introduced the definition of the complementary incomplete q-gamma function as

$$\Gamma_q(a,x) = \int_x^{\frac{1}{1-q}} t^{a-1}E_q((1-q)qt)d_qt = \int_x^{\frac{\infty}{1-q}} t^{a-1}E_q((1-q)qt)d_qt, \qquad \Re(a) > 0. \tag{1.14}$$

Salem [5] generalized the definition of complementary incomplete q-gamma function $\Gamma_q(a,x)$ in analytic continuation as an entire function for all complex a and x: $|\arg(z)| < \pi - \varepsilon$; $0 < \varepsilon < \pi$ and exploited this generalization to introduce the definition of the q-analogue of the exponential integral as

$$E_1(x;q) = \Gamma_q(0,x) = \int_x^{\frac{1}{1-q}} t^{-1}E_q((1-q)qt)d_qt = \int_x^{\frac{\infty}{1-q}} t^{-1}E_q((1-q)qt)d_qt. \tag{1.15}$$

In a slightly different form, we consider the definition of the q-analogue of the exponential integral as

$$E_1(x;q) = \int_x^{\infty} t^{-1}E_q(qt)d_qt. \tag{1.16}$$

Lemma 1.1 *Let x be a positive real. Then we have*

$$E_1(x;q) = E_q(x)\int_0^\infty \frac{e_q(-t)}{x+t}d_qt, \qquad x>0. \tag{1.17}$$

Proof. On q-differentiating (1.16) gives

$$D_qE_1(x;q) = -x^{-1}E_q(xq).$$

Since $\Gamma_q(1) = K_q(1) = 1$, then (1.10) enables us to rewrite the last formula as

$$\begin{aligned} D_qE_1(x;q) &= \frac{-1}{1-q}x^{-1}E_q(xq)\int_0^\infty e_q(-u)d_qu \\ &= \frac{-1}{1-q}E_q(xq)\int_0^\infty e_q(-xy)d_qy. \end{aligned}$$

On q-integrating from x to ∞ with changing the order of integration we obtain

$$E_1(x;q) = \frac{1}{1-q}\int_0^\infty\left(\int_x^\infty E_q(uq)e_q(-yu)d_qu\right)d_qy.$$

Using the q-derivative of the two product function rule (1.2) yields

$$D_q[E_q(u)e_q(-yu)] = \frac{-1}{1-q}E_q(uq)e_q(-yu)(1+y).$$

Substituting the previous formula into the above one we obtain

$$E_1(x;q) = E_q(x)\int_0^\infty \frac{e_q(-xy)}{1+y}d_qy.$$

Replacing xy by t gives the desired result.

2. The q-analogue of the $\mathcal{P}$-Widder transform

Recently Uçar and Albayrak [6] introduced q-analogues of the $\mathcal{L}_2$-transform in terms of the following q-integrals:

$${}_qL_2\left\{f(x);s\right\} = \frac{1}{1-q^2}\int_0^{s^{-1}} xE_{q^2}\left(q^2s^2x^2\right)f(x)\,d_qx, \qquad \Re(s)>0. \tag{2.1}$$

$${}_q\mathcal{L}_2\{f(x);s\} = \frac{1}{1-q^2}\int_0^\infty xe_{q^2}(-s^2x^2)f(x)d_qx, \qquad \Re(s)>0. \tag{2.2}$$

A q-analogue of $\mathcal{P}$-Widder potential transform is denoted $\mathcal{P}_q$ and defined by Albayrak et al. [1] as follows:

$$\mathcal{P}_q\{f(x);s\} = \frac{1}{1-q^2}\int_0^\infty \frac{xf(x)}{s^2+q^2x^2}d_qx. \tag{2.3}$$

They introduced a relation between the ${}_q\mathcal{L}_2$-transfom and $\mathcal{P}_q$-transform as

$${}_q\mathcal{L}_2\left\{{}_q\mathcal{L}_2\left\{f(x);u\right\};s\right\} = \frac{1}{[2]_q}\mathcal{P}_q\{f(x);s\} \tag{2.4}$$

but we see that the last formula is incorrect because they used in their proof the relation

$$e_{q^2}(-u^2s^2)e_{q^2}(-x^2u^2) = e_{q^2}(-(s^2+q^2x^2)u^2),$$

which is incorrect. Based on this remark, we redefine a q-analogue of $\mathcal{P}$-Widder potential transform as follows:

$$\mathcal{P}_q\{f(x);s\} = \frac{1}{1-q^2}\int_0^\infty \frac{xf(x)}{s^2+x^2}d_qx \tag{2.5}$$

and we introduce the relationship between our definition and ${}_q\mathcal{L}_2$, ${}_qL_2$-transforms in the following theorem:

Theorem 2.1 *The $\mathcal{P}_q$-Widder transform (2.5) can be regarded as iterated ${}_q\mathcal{L}_2$, ${}_qL_2$ transforms as*

$${}_qL_2\left\{{}_q\mathcal{L}_2\left\{f\left(x\right);u\right\};s\right\} = {}_q\mathcal{L}_2\left\{{}_qL_2\left\{f\left(x\right);u\right\};s\right\} = \frac{1}{[2]_q}\mathcal{P}_q\{f(x);s\} \tag{2.6}$$

provided that the q-integrals involved converge absolutely.

Proof. Indeed, to prove (2.6), we start by using definitions (2.1) and (2.2) of the ${}_qL_2$-transform and ${}_q\mathcal{L}_2$-transform to obtain

$${}_qL_2\left\{{}_q\mathcal{L}_2\left\{f\left(x\right);u\right\};s\right\} = \frac{1}{\left(1-q^2\right)^2}\int_0^\infty xf\left(x\right)\left(\int_0^{s^{-1}} uE_{q^2}\left(q^2s^2u^2\right)e_{q^2}\left(-u^2x^2\right)d_qu\right)d_qx.$$

Making use of (1.2), it is easy to see that

$$D_{q,u}\left[E_{q^2}\left(s^2u^2\right)e_{q^2}\left(-u^2x^2\right)\right] = -\frac{[2]_q\,u\left(s^2+x^2\right)}{1-q^2}E_{q^2}\left(s^2u^2\right)e_{q^2}\left(-u^2x^2\right).$$

Hence,

$${}_qL_2\left\{{}_q\mathcal{L}_2\left\{f\left(x\right);u\right\};s\right\} = -\frac{1}{[2]_q\left(1-q^2\right)}\int_0^\infty \frac{xf\left(x\right)}{s^2+x^2}\left(E_{q^2}\left(s^2u^2\right)e_{q^2}\left(-u^2x^2\right)\Big|_{u=0}^{s^{-1}}\right)d_qx,$$

which yields by (2.5) that

$${}_qL_2\left\{{}_q\mathcal{L}_2\left\{f\left(x\right);u\right\};s\right\} = \frac{1}{[2]_q\left(1-q^2\right)}\int_0^\infty \frac{xf\left(x\right)}{s^2+x^2}d_qx = \frac{1}{[2]_q}\mathcal{P}_q\left\{f\left(x\right);s\right\}. \tag{2.7}$$

Also, from (2.1) and (2.2), we find that

$${}_q\mathcal{L}_2\{{}_qL_2\{f(x);u\};s\} = \frac{1}{(1-q^2)^2}\int_0^\infty ue_{q^2}(-s^2u^2)\left(\int_0^{u^{-1}} xE_{q^2}(q^2u^2x^2)f(x)d_qx\right)d_qu.$$

Replacing xu by y into the internal q-integral followed by changing the order of q-integration, which is permissible by absolute convergence of the integrals involved, we get

$${}_q\mathcal{L}_2\{{}_qL_2\{f(x);u\};s\} = \frac{1}{(1-q^2)^2}\int_0^1 yE_{q^2}(q^2y^2)\left(\int_0^\infty u^{-1}e_{q^2}(-s^2u^2)f(y/u)d_qu\right)d_qy.$$

Again, replacing u by xy into the internal q-integral followed by changing the order of q-integration, we get

$$ {}_q\mathcal{L}_2\{ {}_qL_2\{f(x);u\};s\} = \frac{1}{(1-q^2)^2}\int_0^\infty x^{-1}f(x^{-1})\left(\int_0^1 yE_{q^2}(q^2y^2)e_{q^2}(-s^2x^2y^2)d_qy\right)d_qx. $$

As above, the previous double q-integral can be reduced as

$$ {}_q\mathcal{L}_2\{ {}_qL_2\{f(x);u\};s\} = \frac{1}{[2]_q(1-q^2)}\int_0^\infty \frac{x^{-1}f(x^{-1})}{1+s^2x^2}d_qx, $$

which can be read by using (1.6) as

$$ {}_q\mathcal{L}_2\{ {}_qL_2\{f(x);u\};s\} = \frac{1}{[2]_q(1-q^2)}\int_0^\infty \frac{xf(x)}{x^2+s^2}d_qx. \tag{2.8} $$

In view of (2.7) and (2.8), we get the desired result (2.6).

Now we give an example that will be used in the sequel.

Example 2.2 *Let α be a real number. Then we have*

$$ \mathcal{P}_q\{x^\alpha;t\} = \frac{1}{[2]_q}\frac{1}{(1-q^2)}\frac{\Gamma_{q^2}(\alpha/2+1)\Gamma_{q^2}(-\alpha/2)}{K_{q^2}(\alpha/2+1)}t^\alpha, \qquad -2<\alpha<0. \tag{2.9} $$

Proof. We put $f(x)=x^\alpha$ $(-2<\alpha<0)$ in (2.1) and (2.2). Utilizing the known results due to Albayrak et al. [1], namely

$$ {}_q\mathcal{L}_2\{x^\alpha;u\} = \frac{1}{[2]_q}\frac{(1-q^2)^{\alpha/2}}{u^{\alpha+2}}\frac{\Gamma_{q^2}(\alpha/2+1)}{K_{q^2}(\alpha/2+1)}, \qquad \Re(\alpha/2+1)>0, \tag{2.10} $$

$$ {}_qL_2\{x^\alpha;u\} = \frac{1}{[2]_q}\frac{(1-q^2)^{\alpha/2}}{u^{\alpha+2}}\Gamma_{q^2}(\alpha/2+1), \qquad \Re(\alpha/2+1)>0, \tag{2.11} $$

we have

$$ \begin{aligned} \mathcal{P}_q\{x^\alpha;t\} &= [2]_q\left({}_qL_2\left\{{}_q\mathcal{L}_2\{x^\alpha;u\};t\right\}\right) \\ &= [2]_q\left({}_qL_2\left\{\frac{1}{[2]_q}\frac{(1-q^2)^{\alpha/2}}{u^{\alpha+2}}\frac{\Gamma_{q^2}(\alpha/2+1)}{K_{q^2}(\alpha/2+1)};t\right\}\right) \\ &= [2]_q\frac{(1-q^2)^{\alpha/2}}{[2]_q}\frac{\Gamma_{q^2}(\alpha/2+1)}{K_{q^2}(\alpha/2+1)}\ {}_qL_2\left\{u^{-\alpha-2};t\right\} \\ &= [2]_q\frac{(1-q^2)^{\alpha/2}}{[2]_q}\frac{\Gamma_{q^2}(\alpha/2+1)}{K_{q^2}(\alpha/2+1)}\frac{1}{[2]_q}\frac{(1-q^2)^{-\alpha/2-1}}{t^{-\alpha}}\Gamma_{q^2}(-\alpha/2) \\ &= \frac{1}{[2]_q}\frac{1}{(1-q^2)}\frac{\Gamma_{q^2}(\alpha/2+1)\Gamma_{q^2}(-\alpha/2)}{K_{q^2}(\alpha/2+1)}t^\alpha. \end{aligned} $$

Example 2.3 *We have the identity*

$$ \mathcal{P}_q\{e_{q^2}(-a^2x^2);s\} = \frac{1}{[2]_q(1-q^2)}e_{q^2}(a^2s^2)E_1(a^2s^2;q^2) \tag{2.12} $$

that holds true.

Proof. From (2.5), we get

$$\mathcal{P}_q\{e_{q^2}(-a^2x^2);s\} = \frac{1}{1-q^2}\int_0^\infty \frac{xe_{q^2}(-a^2x^2)}{s^2+x^2}d_qx.$$

Using the q-analogue of the integration theorem by a change of variable (1.8) with changing x by $y^{\frac{1}{2}}$ yields

$$\mathcal{P}_q\{e_{q^2}(-a^2x^2);s\} = \frac{1}{[2]_q(1-q^2)}\int_0^\infty \frac{e_{q^2}(-a^2y)}{s^2+y}d_{q^2}y.$$

Again, putting $a^2y=t$ leads to

$$\begin{aligned}\mathcal{P}_q\{e_{q^2}(-a^2x^2);s\} &= \frac{1}{[2]_q(1-q^2)}\int_0^\infty \frac{e_{q^2}(-t)}{a^2s^2+t}d_{q^2}t \\ &= \frac{1}{[2]_q(1-q^2)}e_{q^2}(a^2s^2)E_1(a^2s^2;q^2).\end{aligned}$$

In view of Lemma 1.1 and noting that $E_q(x)e_q(x)=1$, we get the desired result.

Proposition 2.4 *The $\mathcal{P}_q$-Widder potential transform and the $\mathbb{S}_q$-Stieltjes transform are related by the identity*

$$\mathcal{P}_q\{f(x);s\} = \frac{1}{[2]_q}\mathbb{S}_{q^2}\{f(\sqrt{x});s^2\} \tag{2.13}$$

where the $\mathbb{S}_q$-Stieltjes transform is defined by Küren and Vulaş [4] as

$$\mathbb{S}_q\{f(x);s\} = \frac{1}{1-q}\int_0^\infty \frac{f(x)}{x+s}d_qx.$$

Proof. The proof comes immediately by changing the variable x in (2.5) by $u^{\frac{1}{2}}$ and using the rule (1.8).

3. The main results

Brown et al. [2] introduced the $\mathcal{E}_{2;1}$-transform as

$$\mathcal{E}_{2;1}\{f(x);s\} = \int_0^\infty x\exp(x^2s^2)E_1(x^2s^2)f(x)dx \tag{3.1}$$

where $E_1(x)$ is the exponential integral defined as

$$E_1(x) = \int_x^\infty t^{-1}e^{-t}dt, \qquad x>0.$$

A function f is q-integrable on $[0,\infty)$ if the series $\sum_{n\in\mathbb{Z}} q^n f(q^n)$ converges absolutely. We write $L_q^1(\mathbb{R}_{q,+})$ for the set of all functions that are absolutely q-integrable on $[0,\infty)$, where $\mathbb{R}_{q,+}$ is the set

$$\mathbb{R}_{q,+} = \{q^n : n\in\mathbb{Z}\},$$

that is

$$L_q^1(\mathbb{R}_q) := \left\{ f : \sum_{n\in\mathbb{Z}} q^n |f(q^n)| < \infty \right\}$$
$$:= \left\{ f : \frac{1}{1-q} \int_0^\infty |f(x)|\, d_q x < \infty \right\}.$$

Now we introduce the following q-integral transform, which may be regarded as a q-extension of the $\mathcal{E}_{2;1}$-transform (3.1).

Definition 3.1 *A q-analogue of $\mathcal{E}_{2;1}$-transform will be denoted ${}_q\mathcal{E}_{2;1}$ and defined by*

$$ {}_q\mathcal{E}_{2;1}\{f(x); s\} = \frac{1}{(1-q^2)^2} \int_0^\infty x f(x) e_{q^2}(s^2x^2) E_1\left(s^2x^2; q^2\right) d_q x \tag{3.2}$$

where $E_1(x;q)$ is the q-exponential integral defined as in (1.16). In view of (1.5), (3.2) can be expressed as

$$ {}_q\mathcal{E}_{2;1}\{f(x); s\} = \frac{1-q}{(1-q^2)^2} \sum_{n\in\mathbb{Z}} q^n e_{q^2}\left(s^2 q^{2n}\right) E_1\left(s^2 q^{2n}; q^2\right) f(q^n). \tag{3.3}$$

Theorem 3.1 *If $f \in L_q^1(\mathbb{R}_{q,+})$, then the q-integral defined by (3.2) is convergent.*

Proof. We have

$$\left|{}_q\mathcal{E}_{2;1}\{f(x); s\}\right| \le \left|\frac{(1-q)^2}{(1-q^2)^2}\right| \sum_{n\in\mathbb{Z}} \left| e_{q^2}\left(s^2(1-q)^2\right) E_1\left(s^2(1-q)^2; q^2\right)\right| |q^n f(q^n)|$$
$$= \frac{(1-q)^2}{(1-q^2)^2} e_{q^2}\left(s^2(1-q)^2\right) E_1\left(s^2(1-q)^2; q^2\right) \sum_{n\in\mathbb{Z}} |q^n f(q^n)|.$$

If $f \in L_q^1(\mathbb{R}_{q,+})$, then $\sum_{n\in\mathbb{Z}} q^n |f(q^n)| < \infty$. This completes the proof.

Theorem 3.2 *Let x be real and s be complex. Then we have that*

$$ {}_q\mathcal{L}_2\{\{{}_qL_2\{{}_q\mathcal{L}_2\{f(x); u\}; t\}; s\} = \frac{1}{[2]_q}\, {}_q\mathcal{L}_2\{\mathcal{P}_q\{f(x); t\}; s\} \tag{3.4}$$
$$= \frac{1}{[2]_q}\, {}_q\mathcal{P}_2\{\mathcal{L}_q\{f(x); t\}; s\}$$

and

$$ {}_q\mathcal{L}_2\{{}_qL_2\{{}_q\mathcal{L}_2\{f(x); u\}; t\}; s\} = \frac{1}{[2]_q^2}\, {}_q\mathcal{E}_{2;1}\{f(x); s\} \tag{3.5}$$

hold true, provided that the integrals involved converge absolutely.

Proof. The proof of (3.4) comes immediately from (2.6). In order to prove (3.5), inserting (2.2) and (2.5) into (3.4), we get

$$ {}_q\mathcal{L}_2\{\{{}_qL_2\left\{{}_q\mathcal{L}_2\left\{f\left(x\right);u\right\};t\right\};s\} = \frac{1}{[2]_q(1-q^2)^2}\int_0^\infty te_{q^2}(-s^2t^2)\left(\int_0^\infty \frac{xf(x)}{t^2+x^2}d_qx\right)d_qt. $$

Changing the order of integration, which is permissible by absolute convergence of the integrals involved, and then using definition (2.5) once again, it follows that

$$ {}_q\mathcal{L}_2\{\{{}_qL_2\left\{{}_q\mathcal{L}_2\left\{f\left(x\right);u\right\};t\right\};s\} = \frac{q^2}{[2]_q(1-q^2)}\int_0^\infty xf(x)\mathcal{P}_q\{e_{q^2}(-s^2t^2);x\}d_qx. $$

From Theorem 2.1, we find that

$$ {}_q\mathcal{L}_2\{\{{}_qL_2\left\{{}_q\mathcal{L}_2\left\{f\left(x\right);u\right\};t\right\};s\} = \frac{1}{[2]_q^2(1-q^2)^2}\int_0^\infty xf(x)e_{q^2}(s^2x^2)E_1\left(s^2x^2;q^2\right)d_qx, $$

which is the desired result according to Definition 3.1.

Example 3.3 *Let α be real. Then we have*

$$ {}_q\mathcal{E}_{2;1}\left\{x^\alpha;s\right\} = \frac{1}{[2]_q}\frac{\Gamma_{q^2}(\alpha/2+1)\Gamma_{q^2}(-\alpha/2)\Gamma_{q^2}(\alpha/2+1)}{K_{q^2}^2(\alpha/2+1)}\frac{(1-q^2)^{\alpha/2-1}}{u^{\alpha+2}} \tag{3.6} $$

where $-2<\alpha<0$.

Proof. We put $f\left(x\right)=x^\alpha$ $(-2<\alpha<0)$ in Theorem 3.2, and hence we get

$$ {}_q\mathcal{L}_2\{\{{}_qL_2\left\{{}_q\mathcal{L}_2\left\{x^\alpha;u\right\};t\right\};s\} = \frac{1}{[2]_q^2}\,{}_q\mathcal{E}_{2;1}\left\{x^\alpha;s\right\}. \tag{3.7} $$

At first, we calculate the left-hand side of equation (3.7). From (2.6), we get

$$ {}_q\mathcal{L}_2\{\{{}_qL_2\left\{{}_q\mathcal{L}_2\left\{x^\alpha;u\right\};t\right\};s\} = \frac{1}{[2]_q}\,{}_q\mathcal{L}_2\left\{\mathcal{P}_q\{x^\alpha;u\};s\right\}. \tag{3.8} $$

From (2.9) and (2.10), identity (3.8) gives rise to

$$ \begin{aligned} {}_q\mathcal{L}_2\{\{{}_qL_2\left\{{}_q\mathcal{L}_2\left\{x^\alpha;u\right\};t\right\};s\} &= \frac{1}{[2]_q^2}\frac{1}{(1-q^2)}\frac{\Gamma_{q^2}(\alpha/2+1)\Gamma_{q^2}(-\alpha/2)}{K_{q^2}(\alpha/2+1)}t^\alpha\ {}_q\mathcal{L}_2\{t^\alpha;s\} \\ &= \frac{(1-q^2)^{\alpha/2-1}}{[2]_q^3}\frac{\Gamma_{q^2}(\alpha/2+1)\Gamma_{q^2}(-\alpha/2)}{K_{q^2}^2(\alpha/2+1)}\frac{\Gamma_{q^2}(\alpha/2+1)}{u^{\alpha+2}}. \end{aligned} \tag{3.9} $$

Therefore, in view of (3.7) and (3.9), we obtain the complete proof of the example.

Remark. Letting $q\to 1^-$ and making use of the limits formulas

$$ \lim_{q\to 1^-}\Gamma_q(t)=\Gamma(t), \quad \lim_{q\to 1^-}K_q(A;t)=1, $$

we observe identity (3.6), the q-extension of the known result

$$\mathcal{E}_{2;1}\left\{x^{\alpha-1};s\right\}=\frac{\pi}{2}\sec\left(\frac{\alpha\pi}{2}\right)\Gamma\left(\frac{\alpha+1}{2}\right)s^{-\alpha-1}$$

due to Brown et al. [2, Example 2, 2.8].

Theorem 3.4 *The Parseval–Goldstein type relations: The identities*

$$\int_0^\infty t\ {}_q\mathcal{L}_2\{f(x);t\}\mathcal{P}_q\{g(u);t\}d_qt=\frac{1}{[2]_q}\int_0^\infty xf(x)\mathcal{E}_{2;1}\{g(u);x\}d_qx \tag{3.10}$$

and

$$\int_0^\infty t\ {}_q\mathcal{L}_2\{f(x);t\}\mathcal{P}_q\{g(u);t\}d_qt=\frac{1}{[2]_q}\int_0^\infty ug(u)\mathcal{E}_{2;1}\{f(x);u\}d_qu \tag{3.11}$$

hold true, provided that the q-integrals involved converge absolutely.

Proof. Inserting (2.2) into the left-hand side of (3.10) followed by changing the order of q-integrations, we get

$$\int_0^\infty t\ {}_q\mathcal{L}_2\{f(x);t\}\mathcal{P}_q\{g(u);t\}d_qt=\frac{1}{1-q^2}\int_0^\infty xf(x)\left(\int_0^\infty te_{q^2}(-x^2t^2)\mathcal{P}_q\{g(u);t\}d_qt\right)d_qx.$$

Once again, from (2.2), we get

$$\int_0^\infty t\ {}_q\mathcal{L}_2\{f(x);t\}\mathcal{P}_q\{g(u);t\}d_qt=\int_0^\infty xf(x)\ {}_q\mathcal{L}_2\{\mathcal{P}_q\{g(u);t\};x\}d_qx.$$

In view of (3.4), (3.5), and the last formula, we obtain the identity (3.10). The proof of (3.11) is similar.

Theorem 3.5 *Let ν be a complex number. Then we have*

$$\int_0^\infty y^\nu\ {}_q\mathcal{L}_2\{f(x);y\}d_qy=\frac{1}{[2]_q(1-q^2)^{\frac{\nu+1}{2}}}\frac{\Gamma_{q^2}(\frac{\nu+1}{2})}{K_{q^2}(\frac{\nu+1}{2})}\int_0^\infty\frac{f(x)}{x^\nu}d_qx, \tag{3.12}$$

holding true for $\Re(\nu)>-1$,

$$\int_0^\infty\frac{\mathcal{P}_q\{f(x);y\}}{y^\nu}d_qy=\frac{1}{[2]_q(1-q^2)}\frac{\Gamma_{q^2}(\frac{1+\nu}{2})\Gamma_{q^2}(\frac{1-\nu}{2})}{K_{q^2}(\frac{1-\nu}{2})}\int_0^\infty\frac{f(x)}{x^\nu}d_qx, \tag{3.13}$$

holding true for $-1<\Re(\nu)<1$, and

$$\int_0^\infty\frac{\mathcal{P}_q\{f(x);y\}}{y^\nu}d_qy=\frac{\Gamma_{q^2}(\frac{1-\nu}{2})}{(1-q^2)^{\frac{1+\nu}{2}}}\int_0^\infty y^\nu\ {}_q\mathcal{L}_2\{f(x);y\}d_qy, \tag{3.14}$$

holding true for $\Re(\nu)<1$, provided that the q-integrals involved converge absolutely.

Proof. Insert the definition of ${}_q\mathcal{L}_2$-transform (2.2) into the left-hand side of identity (3.12) followed by changing the order of integration to obtain

$$\int_0^\infty y^\nu \, {}_q\mathcal{L}_2\{f(x);y\}d_qy = \frac{1}{1-q^2}\int_0^\infty xf(x)\left(\int_0^\infty y^\nu e_{q^2}(-x^2y^2)d_qy\right)d_qx.$$

Replace xy by $u^{\frac{1}{2}}\sqrt{1-q^2}$ in the internal q-integral with using the rule (1.8) and the definition of the q-gamma function (1.10) to obtain the first identity (3.12). The proof of identity (3.13) is similar by inserting the definition of the q-Widder transform (2.5) instead of the definition of ${}_q\mathcal{L}_2$-transform. To prove identity (3.14), inserting relation (2.6) into the left-hand side of identity (3.14) gives

$$\int_0^\infty \frac{\mathcal{P}_q\{f(x);y\}}{y^\nu}d_qy = [2]_q\int_0^\infty \frac{{}_qL_2\{\,{}_q\mathcal{L}_2\{f(x);u\};y\}}{y^\nu}d_qy.$$

From the definition of ${}_qL_2$-transform (2.1), we get

$$\int_0^\infty \frac{\mathcal{P}_q\{f(x);y\}}{y^\nu}d_qy = \frac{1}{1-q}\int_0^\infty y^{-\nu}\left(\int_0^{y^{-1}} uE_{q^2}(q^2u^2y^2)\,{}_q\mathcal{L}_2\{f(x);u\}d_qu\right)d_qy.$$

Replace uy by t in the internal q-integral followed by changing the order of integration to obtain

$$\int_0^\infty \frac{\mathcal{P}_q\{f(x);y\}}{y^\nu}d_qy = \frac{1}{1-q}\int_0^1 tE_{q^2}(q^2t^2)\left(\int_0^\infty y^{-\nu-2}\,{}_q\mathcal{L}_2\{f(x);ty^{-1}\}d_qy\right)d_qt.$$

Again, replacing y by tu in the internal q-integral gives

$$\int_0^\infty \frac{\mathcal{P}_q\{f(x);y\}}{y^\nu}d_qy = \frac{1}{1-q}\int_0^1 t^{-\nu}E_{q^2}(q^2t^2)d_qt\int_0^\infty u^{-\nu-2}\,{}_q\mathcal{L}_2\{f(x);u^{-1}\}d_qu.$$

To compute the first q-integral, put $t = u^{\frac{1}{2}}\sqrt{1-q^2}$ with using rule (1.8) and the definition of the q-gamma function (1.9). Using rule (1.6) ends the proof of the theorem.

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