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A note on Gorenstein projective complexes

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Abstract: As we know, a complex Q is projective if and only if Q is exact and $\mathrm{Z}_n(Q)$ is projective in R-Mod for each $n \in \mathbb{Z}$. In this article, we show that a complex G is Gorenstein projective with $\mathrm{Hom}_R(P, G)$ and $\mathrm{Hom}_R(G, P)$ exact for any Cartan–Eilenberg projective complex P if and only if G is exact and $\mathrm{Z}_n(G)$ is Gorenstein projective in R-Mod for each $n \in \mathbb{Z}$. Using the above result, a new equivalent characterization of some $\mathcal{A}$ complexes is obtained.

Key words: Gorenstein projective module, Cartan–Eilenberg projective complex, Gorenstein projective complex, $\mathcal{A}$ complex

1. Introduction and preliminaries

Throughout this paper, R denotes a ring with unity. A complex

$$\cdots \xrightarrow{\delta_2} C_1 \xrightarrow{\delta_1} C_0 \xrightarrow{\delta_0} C_{-1} \xrightarrow{\delta_{-1}} \cdots$$

of left R-modules will be denoted by (C, δ) or C. For a ring R, R-Mod denotes the category of left R-modules.

The extension of homological algebra from modules to complexes of modules started with the last chapter of [3] and was pursued in [4, 6, 7, 5, 8, 10, 11, 14, 17, 21, 25]. Establishing relationships between a complex X and the modules $X_n, n \in \mathbb{Z}$ is an important question. In [7], Enochs and García Rozas introduced and investigated the notion of Gorenstein projective and injective complexes. It was shown that a complex C is Gorenstein projective (respectively injective) if and only if C_m is Gorenstein projective (respectively injective) in R-Mod for each $m \in \mathbb{Z}$ over n-Gorenstein rings. Liu and Zhang proved that a complex C is Gorenstein injective if and only if C_m is Gorenstein injective in R-Mod for each $m \in \mathbb{Z}$ over left Noetherian rings [17], and a complex C is Gorenstein projective if and only if C_m is Gorenstein projective in R-Mod for each $m \in \mathbb{Z}$ over right coherent and left perfect rings [18]. This was further developed by Yang and Liu [25] and they proved that the same results hold over any associated ring.

Recently, Beligiannis developed a homological algebra in a triangulated category that parallels the homological algebra in an exact category in the sense of Quillen. In particular, he defined projective and injective objects in triangulated categories, and he called them ξ-projective objects and ξ-injective objects, respectively, where ξ denotes the proper class of triangles. However, in general it is not so easy to find a proper class ξ of triangles in a triangulated category having enough ξ-projective objects or ξ-injective objects [1].

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As we know, the homotopy category of complexes of R-modules is a triangulated category and so-called Cartan–Eilenberg projective complexes (or homotopically projective complexes) form the relative projective objects for a proper class of triangles in homotopy category; see [1], Section 12.4 and Section 12.5]. It is easy to see that C is Cartan–Eilenberg projective (or homotopically projective) in the homotopy category [1] if and only if C is Cartan–Eilenberg projective in the category of complexes [6].

Therefore, Cartan–Eilenberg complexes play an important role in the category of complexes and the homotopy category. In [6] and [20], the authors also considered Cartan–Eilenberg complexes. In the present article, we will study Gorenstein projective complexes using Cartan–Eilenberg complexes.

For the rest of the paper we will use the abbreviation C-E for Cartan–Eilenberg.

It is also an important question to establish relationships between a complex X and the modules $\mathrm{Z}_n(X), n \in \mathbb{Z}$. It is well known that a complex C is projective (respectively injective) if and only if C is exact and $\mathrm{Z}_n(C)$ is projective (respectively injective) in R-Mod for each $n \in \mathbb{Z}$; C is FP-injective if and only if C is exact and $\mathrm{Z}_n(C)$ is FP-injective in R-Mod for each $n \in \mathbb{Z}$ [22]. In [19], the authors proved that a complex C is FR-injective (respectively FR-flat) if and only if C is exact and $\mathrm{Z}_n(C)$ is FR-injective (respectively FR-flat) in R-Mod for each $n \in \mathbb{Z}$. It is natural to consider the relationships of Gorenstein projectivity of a complex C and Gorenstein projectivity of the modules $\mathrm{Z}_n(C), n \in \mathbb{Z}$. Our main results in this note can be stated as follows:

Theorem 1.1 *Let G be a complex. Then the following statements are equivalent*:

(1) *G is Gorenstein projective, and $\mathrm{Hom}_R(P,G)$ and $\mathrm{Hom}_R(G,P)$ are exact for all* C-E *projective complexes P*;

(2) *G is exact and $\mathrm{Z}_n(G)$ is Gorenstein projective in R-*Mod *for each $n \in \mathbb{Z}$.*

In [12], Gillespie introduced and investigated the notion of $\mathcal{A}$ complexes and $\mathcal{B}$ complexes whenever $(\mathcal{A},\mathcal{B})$ is a cotorsion pair on an abelian category $\mathcal{C}$. Applying Theorem 1.1, a new characterization of some $\mathcal{A}$ complexes is obtained.

Corollary 1.2 *Let R be a Gorenstein ring, $\mathcal{A}$ be the class of all Gorenstein projective left R-modules, and G a complex. Then the following statements are equivalent*:

(1) *G is an $\mathcal{A}$ complex.*

(2) *For every bounded above complex C with C_i of finite projective dimension in R-*Mod, $\mathrm{Ext}^1(G,C)=0$.

(3) *For every bounded complex C with C_i of finite projective dimension in R-*Mod, $\mathrm{Ext}^1(G,C)=0$.

(4) *For any module A of finite projective dimension and $n \in \mathbb{Z}$,* $\mathrm{Ext}^1(G,S^n(A))=0$.

(5) *G is Gorenstein projective with $\mathrm{Hom}_R(P,G)$ and $\mathrm{Hom}_R(G,P)$ exact for every* C-E *projective complex P.*

We will use superscripts to distinguish complexes, so if $\{C^i\}_{i\in I}$ is a family of complexes, C^i will be

$$\cdots \xrightarrow{\delta_2} C_1^i \xrightarrow{\delta_1} C_0^i \xrightarrow{\delta_0} C_{-1}^i \xrightarrow{\delta_{-1}} \cdots .$$

Given a left R-module M, we use the notation $D^m(M)$ to denote the complex

$$\cdots \longrightarrow 0 \longrightarrow M \xrightarrow{id} M \longrightarrow 0 \longrightarrow \cdots$$

with M in the mth and $(m-1)$th positions and set $\overline{M} = D^0(M)$. We also use the notation $S^m(M)$ to denote the complex with M in the mth place and 0 in the other places and set $\underline{M} = S^0(M)$.

Given a complex C and an integer l, the lth homology module of C is the module $\mathrm{H}_l(C) = \mathrm{Z}_l(C)/\mathrm{B}_l(C)$ where $\mathrm{Z}_l(C) = \mathrm{Ker}(\delta_l^C)$ and $\mathrm{B}_l(C) = \mathrm{Im}(\delta_{l+1}^C)$.

Let C and D be complexes of left R-modules. We will denote by $\mathrm{Hom}_R(C, D)$ the complex of abelian groups with $\mathrm{Hom}_R(C, D)_n = \prod_{t \in \mathbb{Z}} \mathrm{Hom}_R(C_t, D_{n+t})$ and such that if $f \in \mathrm{Hom}_R(C, D)_n$ then $(\delta_n(f))_m = \delta_{m+n}^D f_m - (-1)^n f_{m+1} \delta_m^C$. f is called a *chain map* of degree n if $\delta_n(f) = 0$. A chain map of degree 0 is called a *morphism*.

Given two complexes C and D, $\mathrm{Hom}(C, D)$ is the abelian group of morphisms from C to D and Ext^i for $i \geq 0$ will denote the groups we get from the right derived functor of Hom.

General background materials about complexes of R-modules can be found in [11].

We recall some notions and results needed in the paper.

A complex P is *projective* if the functor $\mathrm{Hom}(P, -)$ is exact. Equivalently, P is projective if and only if P is exact and $\mathrm{Z}_i(P)$ is projective in R-Mod for each $i \in \mathbb{Z}$.

Definition 1.3 ([[7], Definition 4.1]) *A complex C is called Gorenstein projective if there exists an exact sequence of projective complexes*

$$\cdots \to P^{-1} \to P^0 \to P^1 \to \cdots$$

with $C \cong \mathrm{Ker}(P^0 \to P^1)$ and it remains exact after applying $\mathrm{Hom}(-, P)$ for any projective complex P.

Definition 1.4 ([[9], Definition 3.1]) *A left R-module M is called Gorenstein projective if there exists an exact sequence of projective left R-modules*

$$\cdots \to P_{-1} \to P_0 \to P_1 \to \cdots$$

with $M \cong \mathrm{Ker}(P_0 \to P_1)$ and it remains exact after applying $\mathrm{Hom}(-, P)$ for any projective left R-module P.

Definition 1.5 ([[21], p.227]) *A complex P is said to be* C-E *projective if $P, \mathrm{Z}(P), \mathrm{B}(P)$, and $\mathrm{H}(P)$ are complexes consisting of projective modules.*

Definition 1.6 ([[6], Definition 5.3]) *A complex of complexes*

$$\cdots \to C^{-1} \to C^0 \to C^1 \to \cdots$$

is said to be C-E *exact if*

(1) $\cdots \to C^{-1} \to C^0 \to C^1 \to \cdots$,

(2) $\cdots \to \mathrm{Z}(C^{-1}) \to \mathrm{Z}(C^0) \to \mathrm{Z}(C^1) \to \cdots$,

(3) $\cdots \to \mathrm{B}(C^{-1}) \to \mathrm{B}(C^0) \to \mathrm{B}(C^1) \to \cdots$,

(4) $\cdots \to C^{-1}/\mathrm{Z}(C^{-1}) \to C^0/\mathrm{Z}(C^0) \to C^1/\mathrm{Z}(C^1) \to \cdots$,

(5) $\cdots \to C^{-1}/\mathrm{B}(C^{-1}) \to C^0/\mathrm{B}(C^0) \to C^1/\mathrm{B}(C^1) \to \cdots$,

(6) $\cdots \to \mathrm{H}(C^{-1}) \to \mathrm{H}(C^0) \to \mathrm{H}(C^1) \to \cdots$

are all exact.

By [[6], Proposition 6.3], we can compute derived functors of $\mathrm{Hom}(-,-)$ using C-E projective resolutions or C-E injective resolutions. For given C and D we will denote these derived functors applied to (C,D) as $\overline{\mathrm{Ext}}^n(C,D)$. It is obvious that $\overline{\mathrm{Ext}}^1(C,D) \subseteq \mathrm{Ext}^1(C,D)$.

2. Proof of Theorem 1.1

In this section, we give a proof of the main result, Theorem 1.1. First we establish several facts.

Lemma 2.1 *Let P be a complex. Then the following statements are equivalent*:

(1) *P is a* C-E *projective complex*;

(2) $\overline{\mathrm{Ext}}^i(P,D) = 0$ *for any complex D and all $i \geq 1$*;

(3) $\overline{\mathrm{Ext}}^1(P,D) = 0$ *for any complex D*;

(4) *For any* C-E *exact sequence of complexes $0 \to A \to B \xrightarrow{h} C \to 0$ and any morphism of complexes $f : P \to C$, there exists $g : P \to B$ such that $hg = f$, i.e. the following diagram*:

$$\begin{array}{ccccccccc} & & & & & & P & & \\ & & & & & \swarrow^{g} & \downarrow{\scriptstyle f} & & \\ 0 & \longrightarrow & A & \longrightarrow & B & \xrightarrow[h]{} & C & \longrightarrow & 0 \end{array}$$

is commutative.

Proof It is easy by the definition. □

Lemma 2.2 ([[6], Proposition 10.1]) (1) *For a complex C the equality $\overline{\mathrm{Ext}}^1(C,-) = \mathrm{Ext}^1(C,-)$ holds if and only if C is exact*;

(2) *For a complex C the equality $\overline{\mathrm{Ext}}^1(-,C) = \mathrm{Ext}^1(-,C)$ holds if and only if C is exact.*

To prove Theorem 1.1, we give the following definition.

Definition 2.3 ([[6], Dual version of Definition 8.6]) *A complex G is said to be* C-E *Gorenstein projective if there is a* C-E *exact sequence of* C-E *projective complexes*

$$\mathbb{P} : \cdots \to P^{-2} \to P^{-1} \to P^0 \to P^1 \to P^2 \to \cdots$$

such that $G \cong \mathrm{Ker}(P^0 \to P^1)$ and the functor $\mathrm{Hom}(-,Q)$ leaves $\mathbb{P}$ exact whenever Q is C-E *projective.*

The following lemmas describe C-E Gorenstein projective complexes. The first statement can be obtained by the similar argument of [6], Theorem 8.5], and the second follows from the definition of C-E Gorenstein projective complexes and Lemma 2.2.

Lemma 2.4 *Let G be a* C-E *Gorenstein projective complex. Then:*

(1) $\mathrm{Z}_n(G)$, $G_n/\mathrm{Z}_n(G), G_n/\mathrm{B}_n(G)$ *and* $\mathrm{H}_n(G)$ *are Gorenstein projective in R-*Mod *for all $n \in \mathbb{Z}$*;

(2) $\overline{\mathrm{Ext}}^n(G,P) = 0$ *for any* C-E *projective complex P and each $n \geq 1$. Moreover, $\mathrm{Ext}^1(G,P) = 0$ whenever G is exact.*

Lemma 2.5 *Let G be a complex with G_n a Gorenstein projective module for each $n \in \mathbb{Z}$. Then $\mathrm{Hom}_R(G,P)$ is exact for any* C-E *projective complex P if and only if $\mathrm{Ext}^1(G,P)=0$ for any* C-E *projective complex P.*

Proof It follows from [[12], Lemma 2.1]. □

Lemma 2.6 ([[24], Lemma 2.4]) *Let $0 \to M \xrightarrow{f} P \to N \to 0$ be a short exact sequence of modules. If M and N are Gorenstein projective and P is projective, then $\mathrm{Coker}(\alpha)$ is Gorenstein projective for any homomorphism $f' : M \to P'$ with P' projective, where $\alpha = (f, f') : M \to P \oplus P'$ is defined by $\alpha(x) = f(x) + f'(x)$ for any $x \in M$.*

Lemma 2.7 ([[12], Lemma 3.1]) *For any left R-module M and any complex of left R-modules X, we have the following natural isomorphisms*:

(1) $\mathrm{Hom}(D^n(M), X) \cong \mathrm{Hom}_R(M, X_n)$;

(2) $\mathrm{Hom}(S^n(M), X) \cong \mathrm{Hom}_R(M, \mathrm{Z}_n(X))$;

(3) $\mathrm{Hom}(X, D^n(M)) \cong \mathrm{Hom}_R(X_{n-1}, M)$;

(4) $\mathrm{Hom}(X, S^n(M)) \cong \mathrm{Hom}_R(X_n/\mathrm{B}_n(X), M)$.

We need the following lemma, which provides a specific C-E exact sequence of complexes.

Lemma 2.8 *Let $0 \to A \to B \to C \to 0$ be a short exact sequence of complexes with A exact. Then $0 \to A \to B \to C \to 0$ is* C-E *exact.*

Proof Since $0 \to A \to B \to C \to 0$ is exact, we have the exact sequence

$$0 \to \mathrm{Hom}(\mathrm{S}^n(R), A) \to \mathrm{Hom}(\mathrm{S}^n(R), B) \to \mathrm{Hom}(\mathrm{S}^n(R), C) \to \mathrm{Ext}^1(\mathrm{S}^n(R), A).$$

On the other hand, $\mathrm{Ext}^1(\mathrm{S}^n(R), A) \cong \overline{\mathrm{Ext}}^1(\mathrm{S}^n(R), A)$ by Lemma 2.2. Note that $\overline{\mathrm{Ext}}^1(\mathrm{S}^n(R), A){=}0$ by Lemma 2.1 since $\mathrm{S}^n(R)$ is C-E projective, and so the sequence

$$0 \to \mathrm{Hom}(\mathrm{S}^n(R), A) \to \mathrm{Hom}(\mathrm{S}^n(R), B) \to \mathrm{Hom}(\mathrm{S}^n(R), C) \to 0$$

is exact. Then $0 \to \mathrm{Z}_n(A) \to \mathrm{Z}_n(B) \to \mathrm{Z}_n(C) \to 0$ is exact by Lemma 2.7. Therefore, $0 \to A \to B \to C \to 0$ is C-E exact by [[6], Lemma 5.2]. □

The following observation is useful; its proof is routine.

Lemma 2.9 *Let C be a complex. If $\mathrm{Hom}_R(P,C)$ is exact for all* C-E *projective complexes P, then C is exact.*

Proof of Theorem 1.1

We will divide the proof of Theorem 1.1 into three steps.

Step 1. An exact complex G is C-E Gorenstein projective if and only if G_n is Gorenstein projective in R-Mod for each $n \in \mathbb{Z}$ and $\mathrm{Hom}_R(G,P)$ is exact for all C-E projective complexes P.

" $\Rightarrow$ ". Note that $0 \to \mathrm{H}_n(G) \to G_n/\mathrm{B}_n(G) \to G_n/\mathrm{Z}_n(G) \to 0$ and $0 \to \mathrm{B}_n(G) \to G_n \to G_n/\mathrm{B}_n(G) \to 0$ are exact and $\mathrm{B}_n(G) \cong G_{n+1}/\mathrm{Z}_{n+1}(G)$. Then G_n are Gorenstein projective in R-Mod for all $n \in \mathbb{Z}$ by Lemma

2.4 and [[15], Theorem 2.5]. By Lemma 2.4 and Lemma 2.5, $\mathrm{Hom}_R(G,P)$ is exact for all C-E projective complexes P.

" $\Leftarrow$ ". Note that it can be obtained by [[24], Theorem 2.6]. For completeness we include a proof.

Since G_n is Gorenstein projective, there exists an exact sequence of modules

$$0 \to G_n \stackrel{f_n}{\to} X_n \to H_n \to 0$$

with X_n projective and H_n Gorenstein projective for each $n \in \mathbb{Z}$. Put

$$P^0 =: \cdots \to P^0_{n+1} \stackrel{\delta^{P^0}_{n+1}}{\to} P^0_n \stackrel{\delta^{P^0}_{n}}{\to} P^0_{n-1} \stackrel{\delta^{P^0}_{n-1}}{\to} P^0_{n-2} \to \cdots$$

with $P^0_n = X_n \oplus X_{n-1}$ and $\delta^{P^0}_n : P^0_n \to P^0_{n-1}$ defined via $\delta^{P^0}_n(x,y) = (y,0)$ for all $n \in \mathbb{Z}$ and any $(x,y) \in X_n \oplus X_{n-1}$. Clearly, P^0 is exact, and P^0_n and $\mathrm{Z}_n(P^0)$ are projective in R-Mod for all $n \in \mathbb{Z}$. Then P^0 is C-E projective. Now we have a morphism $\alpha = (\alpha_n)_{n\in\mathbb{Z}} : G \to P^0$ of complexes as follows:

$$\begin{array}{ccccccccc}
\cdots & \longrightarrow & G_{n+1} & \xrightarrow{\delta^G_{n+1}} & G_n & \xrightarrow{\delta^G_n} & G_{n-1} & \longrightarrow & \cdots \\
 & & \downarrow{\scriptstyle (f_{n+1}, f_n\delta^G_{n+1})} & & \downarrow{\scriptstyle (f_n, f_{n-1}\delta^G_n)} & & \downarrow{\scriptstyle (f_{n-1}, f_{n-2}\delta^G_{n-1})} & & \\
\cdots & \longrightarrow & X_{n+1}\oplus X_n & \xrightarrow{\delta^{P^0}_{n+1}} & X_n \oplus X_{n-1} & \xrightarrow{\delta^{P^0}_n} & X_{n-1}\oplus X_{n-2} & \longrightarrow & \cdots .
\end{array}$$

It is clear that α is injective and so we have a short exact sequence of complexes

$$0 \to G \stackrel{\alpha}{\to} P^0 \to K^1 \to 0, \qquad (*)$$

where $K^1 = \mathrm{Coker}(\alpha)$. Note that G and P^0 are exact. Then the sequence $(*)$ is C-E exact by Lemma 2.8. By Lemma 2.6, K^1_n is Gorenstein projective for each $n \in \mathbb{Z}$, and so we get that the sequence of abelian groups

$$0 \to \mathrm{Hom}_R(K^1,P) \to \mathrm{Hom}_R(P^0,P) \to \mathrm{Hom}_R(G,P) \to 0$$

is exact for any C-E projective complex P. Since $\mathrm{Ext}^1(P^0,P) = 0$ by Lemma 2.1 and Lemma 2.2, then $\mathrm{Hom}_R(P^0,P)$ is exact by Lemma 2.5. Therefore, $\mathrm{Hom}_R(K^1,P)$ is exact since $\mathrm{Hom}_R(G,P)$ is so, and hence $\mathrm{Ext}^1(K^1,P) = 0$. This yields exactness of the sequence

$$0 \to \mathrm{Hom}(K^1,P) \to \mathrm{Hom}(P^0,P) \to \mathrm{Hom}(G,P) \to 0.$$

Continuously using the methods above, we get a C-E exact sequence of complexes

$$0 \to G \to P^0 \to P^1 \to \cdots$$

with each P^i C-E projective for $i \geq 0$ and which remains exact after applying $\mathrm{Hom}(-,P)$ for any C-E projective complex P.

Take the C-E projective resolution of G

$$\cdots \to P_1 \to P_0 \to G \to 0. \qquad (\star)$$

Then each $(K_i)_j$ is a Gorenstein projective module by [[15], Theorem 2.5] for $j \in \mathbb{Z}$ and $i \geq 0$, where $K_0 = \mathrm{Ker}(P_0 \to G)$ and $K_i = \mathrm{Ker}(P_i \to P_{i-1})$ for $i \geq 1$. It is easy to see that the sequence $(\star)$ remains exact after applying $\mathrm{Hom}(-, P)$ for any C-E projective complex P, and so G is C-E Gorenstein projective.

Step 2. Using [[24], Proposition 2.2], we get that an exact complex G is C-E Gorenstein projective if and only if $\mathrm{Z}_n(G)$ is Gorenstein projective in R-Mod for each $n \in \mathbb{Z}$.

Step 3. By step 1 and step 2, we infer that an exact complex G is Gorenstein projective and $\mathrm{Hom}_R(G, P)$ is exact for all C-E projective complexes P if and only if $\mathrm{Z}_n(G)$ is Gorenstein projective in R-Mod for each $n \in \mathbb{Z}$. Thus, for any exact complex G, $\mathrm{Z}_n(G)$ is Gorenstein projective in R-Mod for each $n \in \mathbb{Z}$ if and only if G_n is Gorenstein projective in R-Mod for each $n \in \mathbb{Z}$ and $\mathrm{Hom}_R(G, P)$ is exact for all C-E projective complexes P by [[25], Theorem 2.2]. Therefore, we get that G is Gorenstein projective with $\mathrm{Hom}_R(P, G)$ and $\mathrm{Hom}_R(G, P)$ are exact for all C-E projective complexes P if and only if G is exact and $\mathrm{Z}_n(G)$ is Gorenstein projective in R-Mod for each $n \in \mathbb{Z}$ by Lemma 2.9. This finishes the proof of Theorem 1.1. □

3. Some remarks on Theorem 1.1

Now we illustrate the main result in the paper by examples and corollaries.

Remark 3.1 *The complexes that satisfy one of conditions of Theorem 1.1 are said to be strongly Gorenstein projective complexes.*

Note that projective complexes are strongly Gorenstein projective complexes that are Gorenstein projective. However, the converse is not true in general. The following examples (Example 2.2 and Example 3.3) which show that strongly Gorenstein projective complexes that are described by Theorem 1.1 lie strictly between projective complexes and Gorenstein projective complexes.

Example 3.2 ([[2], Example 2.5]) *Consider the quasi-Frobenius local ring* $R = k[X]/(X^2)$ *where* k *is a field. Then*

$$P =: \cdots \to R \xrightarrow{x} R \xrightarrow{x} R \to \cdots$$

is a complete projective resolution in R-Mod *and* $\mathrm{Z}_i(P)$ *is not projective in* R-Mod*. Thus,* P *is a strongly Gorenstein projective complex and* P *is not a projective complex.*

Example 3.3 *The complex* A *in [[24], Example 2.3] (also see [[16], Example 2.4]) is Gorenstein projective but it is not strongly Gorenstein projective.*

Remark 3.4 *It is well known that a complex* P *is projective if and only if* P *is exact and* $\mathrm{Z}_n(P)$ *is projective modules. However, the Gorenstein version of this above result need not be true by Theorem 1.1, Remark 3.1, and Example 3.3.*

It is obvious that any C-E Gorenstein projective complex is Gorenstein projective from Lemma 2.4. The following result establishes a new relationship between C-E Gorenstein projective and Gorenstein projective complexes, which is implied by the proof of Theorem 1.1.

Corollary 3.5 *Let* G *be a right bounded exact complex. Then* G *is* C-E *Gorenstein projective if and only if* G *is Gorenstein projective.*

In [12], $\mathcal{A}$ complexes and $\mathcal{B}$ complexes are introduced and investigated whenever $(\mathcal{A},\mathcal{B})$ is a cotorsion pair on an abelian category $\mathcal{C}$.

Definition 3.6 ([[12], Definition 3.3]) *Let $(\mathcal{A},\mathcal{B})$ be a cotorsion pair in R-Mod and X a complex.*

(1) *X is called an $\mathcal{A}$ complex if it is exact and $\mathrm{Z}_n(X)\in\mathcal{A}$ for all $n\in\mathbb{Z}$.*

(2) *X is called a $\mathcal{B}$ complex if it is exact and $\mathrm{Z}_n(X)\in\mathcal{B}$ for all $n\in\mathbb{Z}$.*

As we know, $(\mathcal{A},\mathcal{A}^{\perp})$ is a complete cotorsion pair whenever R is a Gorenstein ring and $\mathcal{A}$ is the class of all Gorenstein projective left R-modules [13].

Using Theorem 1.1, Remark 3.1, and [[23], Proposition 2.2], we get the following result.

Corollary 3.7 *Let R be a Gorenstein ring, $\mathcal{A}$ be the class of all Gorenstein projective left R-modules, and G be a complex. Then the following statements are equivalent:*

(1) *G is an $\mathcal{A}$ complex.*

(2) *For every bounded above complex C with C_i of finite projective dimension in R-Mod, $\mathrm{Ext}^1(G,C)=0$.*

(3) *For every bounded complex C with C_i of finite projective dimension in R-Mod, $\mathrm{Ext}^1(G,C)=0$.*

(4) *For any module A of finite projective dimension and $n\in\mathbb{Z}$, $\mathrm{Ext}^1(G,S^n(A))=0$.*

(5) *G is Gorenstein projective with $\mathrm{Hom}_R(P,G)$ and $\mathrm{Hom}_R(G,P)$ exact for every C-E projective complex P.*

(6) *G is strongly Gorenstein injective.*

Remark 3.8 *On one hand, the result above gives some characterizations of $\mathcal{A}$ complexes whenever $\mathcal{A}$ is the class of all Gorenstein projective left R-modules; on the other hand, it establishes a relationship between Gorenstein projective complexes and strongly Gorenstein projective complexes.*

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