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Weakly 2-absorbing submodules of modules

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Abstract: Let M be a module over a commutative ring R. A proper submodule N of M is called weakly 2-absorbing, if for $r, s \in R$ and $x \in M$ with $0 \neq rsx \in N$, either $rs \in (N : M)$ or $rx \in N$ or $sx \in N$. We study the behavior of $(N : M)$ and $\sqrt{(N : M)}$, when N is weakly 2-absorbing. The weakly 2-absorbing submodules when $R = R_1 \oplus R_2$ are characterized. Moreover we characterize the faithful modules whose proper submodules are all weakly 2-absorbing.

Key words: Prime submodule, 2-absorbing submodule, weakly 2-absorbing submodule, weakly prime submodule, weak prime submodule

1. Introduction

Throughout this paper all rings are commutative with identity and all modules are unitary. Furthermore, we consider R to be a commutative ring with identity and M an R-module, and $K[X, Y]$ denotes the ring of polynomials, where X and Y are independent indeterminates and K is a field.

The *colon ideal* of a submodule N of M is considered to be

$$(N : M) = \{r \in R | rM \subseteq N\}.$$

Moreover, $\sqrt{(N : M)}$ will be called the *radical ideal* of N.

Following [5], [resp. [4]] a proper ideal I of R is *weakly* 2*-absorbing*, [resp. 2*-absorbing*] if for $a, b, c \in R$ with $0 \neq abc \in I$, [resp. $abc \in I$] $ab \in I$ or $ac \in I$ or $bc \in I$.

Recall that a proper submodule N of M is called 2-absorbing, if for $r, s \in R$ and $x \in M$ with $rsx \in N$, $rs \in (N : M)$ or $rx \in N$ or $sx \in N$ (see [9, 10]).

According to [10], a proper submodule N of M is called weakly 2-absorbing, if for $r, s \in R$ and $x \in M$ with $0 \neq rsx \in N$, $rs \in (N : M)$ or $rx \in N$ or $sx \in N$.

A proper submodule N of M is called *prime*, when from $rx \in N$ for some $r \in R$ and $x \in M$, we can conclude either $x \in N$ or $rM \subseteq N$ (see for example [2, 7, 8]). If N is a prime submodule, then $P = (N : M)$ is a prime ideal of R.

Another generalization of prime ideals to modules was introduced in [6]. A proper submodule W of M is said to be *weakly prime*, if $rsx \in W$ for $r, s \in R$ and $x \in M$, implying that either $rx \in W$ or $sx \in W$.

Recall from [1] that a proper ideal I of a ring R is *a weakly prime ideal* if whenever $a, b \in R$ with

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$0 \neq ab \in I$, then either $a \in I$ or $b \in I$. For unifying with modules and preventing confusion, we name weakly prime ideals of [1] *weak prime ideals* in this paper. The following definition is a module version of this notion.

Definition 1 *A proper submodule N of M is said to be weak prime, if for $r \in R$ and $x \in M$ with $0 \neq rx \in N$ either $r \in (N : M)$ or $x \in N$.*

Note 1 *It is easy to see that:*

1. *Prime submodule $\Longrightarrow$ Weak prime $\Longrightarrow$ Weakly 2-absorbing.*
2. *Prime submodule $\Longrightarrow$ Weakly prime $\Longrightarrow$ 2-absorbing $\Longrightarrow$ Weakly 2-absorbing.*
3. *A submodule N is weakly prime if and only if N is 2-absorbing and $(N : M)$ is a prime ideal.*

See [9, Example 1], for examples of 2-absorbing submodules that are not weakly prime.

Example 1

1. *Let $R = K[X,Y]$, $M = R \oplus R$ and $N = \langle X\rangle \oplus \langle X,Y\rangle$. Then N is a 2-absorbing submodule of the R-module M, but it is not weak prime.*
2. *For the $\mathbb{Z}$-module $M = \mathbb{Z}_{12}$, the zero submodule is weakly 2-absorbing, but not 2-absorbing.*

PROOF. (1) One can easily see that N is a 2-absorbing submodule of M. However, N is not weak prime, because $0 \neq Y(0,1) \in N$, but $Y \notin \langle X\rangle = (N : M)$ and $(0,1) \notin N$.

(2) Evidently the zero submodule of any nonzero module is weakly 2-absorbing. Now consider $2.3.\overline{2} \in 0 = N$ to see that N is not 2-absorbing.

2. On a question from Badawi and Yousefian

The authors in [5] have asked the following question:

Question. Suppose that L is a weakly 2-absorbing ideal of a ring R and $0 \neq IJK \subseteq L$ for some ideals I, J, K of R. Does it imply that $IJ \subseteq L$ or $IK \subseteq L$ or $JK \subseteq L$?

This section is devoted to studying the above question and its generalization in modules.

Lemma 2.1 *Let N be a weakly 2-absorbing submodule of an R-module M and $a, b \in R$. If for some submodule K of M, $abK \subseteq N$ and $0 \neq 2abK$, then $ab \in (N : M)$ or $aK \subseteq N$ or $bK \subseteq N$.*

Proof Put $(N : M) = L$, and suppose $ab \notin L$. Then it is enough to prove that $K \subseteq (N :_M a) \cup (N :_M b)$. Let z be an arbitrary element of K. If $0 \neq abz$, then as N is weakly 2-absorbing and $ab \notin L$, either $az \in N$ or $bz \in N$ and so $z \in (N :_M a) \cup (N :_M b)$. Now let $0 = abz$. Since $0 \neq 2abK$, for some $x \in K$, we have $0 \neq 2abx$ and so $0 \neq abx \in N$. As N is weakly 2-absorbing and $ab \notin L$, either $ax \in N$ or $bx \in N$. Put $y = x + z$. Then $0 \neq aby \in N$ and since $ab \notin L$, either $ay \in N$ or $by \in N$. We consider three cases.

Case 1. $ax \in N$ and $bx \in N$. Note that $ay \in N$ or $by \in N$, and so either $az \in N$ or $bz \in N$.

Case 2. $ax \in N$ and $bx \notin N$. On the contrary let $az \notin N$. Then $ay \notin N$ and so $by \in N$. Therefore, $a(y + x) \notin N$ and $b(y + x) \notin N$. Now as N is weakly 2-absorbing and $ab \notin L$, then $0 = ab(y + x) = 2abx$, which is a contradiction. Thus $az \in N$.

Case 3. $ax \notin N$ and $bx \in N$. Then proof is similar to that of Case 2. □

Lemma 2.2 *Let J be an ideal of R and K, N two submodules of an R-module M, such that $aJK \subseteq N$, where $a \in R$. If N is weakly 2-absorbing and $0 \neq 4aJK$, then $aJ \subseteq (N:M)$ or $aK \subseteq N$ or $JK \subseteq N$.*

Proof Let $aJ \nsubseteq (N:M) = L$. Then $aj \notin L$ for some $j \in J$. First we claim that there exists $b \in J$ such that $0 \neq 4abK$, and $ab \notin L$.

Since $0 \neq 4aJK$, for some $j' \in J$, $0 \neq 4aj'K$. If $aj' \notin L$ or $0 \neq 4ajK$, then by putting $b = j'$ or $b = j$, we get the result. Therefore, let $aj' \in L$ and $4ajK = 0$. Hence $0 \neq 4a(j+j')K \subseteq N$ and $a(j+j') \notin L$. Consequently we find $b \in J$, such that $0 \neq 4abK$, and $ab \notin L$. Thus $0 \neq 2abK$ and by 2.1, $K \subseteq (N :_M a) \cup (N :_M b)$. If $aK \subseteq N$, there is nothing to prove. Therefore, assume that $aK \nsubseteq N$ and so $bK \subseteq N$.

Now we show that $J \subseteq (L:a) \cup (N:K)$. Let $c \in J$. If $0 \neq 2acK$, then by 2.1, $ac \in L$ or $aK \subseteq N$ or $cK \subseteq N$. However, as we assumed $aK \nsubseteq N$, $c \in (L:a) \cup (N:K)$.

Next assume $2acK = 0$. Then $0 \neq 2a(b+c)K \subseteq N$ and 2.1 implies that either $a(b+c) \in L$ or $aK \subseteq N$ or $(b+c)K \subseteq N$. Then as $aK \nsubseteq N$, $(b+c) \in (L:a) \cup (N:K)$. If $b+c \in (N:K)$, then $c \in (N:K)$, because $b \in (N:K)$. Therefore, let $(b+c) \in (L:a) \setminus (N:K)$.

Consider $2a(b+c+b)K = 4abK \neq 0$ and $2a(b+c+b)K \subseteq N$. Since $ab \notin L$ and $a(b+c) \in L$, $a(b+c+b) \notin L$. Thus, according to 2.1, $K \subseteq (N :_M a) \cup (N :_M b+c+b)$. However, since $b+c \notin (N:K)$ and $b \in (N:K)$, $b+c+b \notin (N:K)$, and so $K \subseteq (N :_M a)$, which is impossible. Therefore, $b+c \in (N:K)$ and since $b \in (N:K)$, $c \in (N:K)$. Consequently $J \subseteq (L:a) \cup (N:K)$ and hence as $aJ \nsubseteq L$, $JK \subseteq N$. □

Theorem 2.3 *Let I, J be ideals of R and N, K be submodules of an R-module M. If N is a weakly 2-absorbing submodule, $0 \neq IJK \subseteq N$, and $0 \neq 8\big(IJ + (I+J)(N:M)\big)\big(K+N\big)$, then $IJ \subseteq (N:M)$ or $IK \subseteq N$ or $JK \subseteq N$. In particular this holds if the group $(M,+)$ has no elements of order 2.*

Proof Note that $0 \neq 8\big(IJ + (I+J)(N:M)\big)\big(K+N\big) = 8IJK + 8IJN + 8I(N:M)K + 8J(N:M)K + 8I(N:M)N + 8J(N:M)N$. Therefore, one of the following different types is satisfied.

(i) $0 \neq 8IJK$. Then for some $a \in J$, we have $0 \neq 8aIK$. Therefore, $0 \neq 4aIK$ and by 2.2, either $aI \subseteq (N:M) = L$ or $aK \subseteq N$ or $IK \subseteq N$. If $IK \subseteq N$, then we have the result. Therefore, we suppose that $IK \nsubseteq N$ and so $a \in (L:I) \cup (N:K)$. Now we show that $J \subseteq (L:I) \cup (N:K)$. To see this let $c \in J$. If $0 \neq 4cIK$, then according to 2.2, since $IK \nsubseteq N$, $c \in (L:I) \cup (N:K)$.

Now let $4cIK = 0$. So $0 \neq 4(a+c)IK \subseteq N$. Thus, by 2.2, since $IK \nsubseteq N$, $a+c \in (L:I) \cup (N:K)$. We consider the following four cases.

Case 1. $a+c \in (L:I)$ and $a \in (L:I)$. Then $c \in (L:I)$.

Case 2. $a+c \in (N:K)$ and $a \in (N:K)$. Hence $c \in (N:K)$.

Case 3. $a \in (L:I) \setminus (N:K)$ and $a+c \in (N:K) \setminus (L:I)$. Therefore, $a+c+a \notin (L:I)$ and $a+c+a \notin (N:K)$ and so $a+c+a \notin (L:I) \cup (N:K)$. We consider $4(a+c+a)IK = 8aIK \neq 0$. Hence, by 2.2, as $IK \nsubseteq N$, $a+c+a \in (L:I) \cup (N:K)$, which is impossible. Hence as $a \in (L:I) \cup (N:K)$ and $a+c \in (L:I) \cup (N:K)$, one of the following holds.

(a) $a \in (N:K)$ and $a+c \in (N:K) \setminus (L:I)$. Thus $c \in (N:K)$.

(b) $a \in (L:I) \setminus (N:K)$ and $a+c \in (L:I)$. Hence $c \in (L:I)$.

Case 4. $a+c \in (L:I) \setminus (N:K)$ and $a \in (N:K) \setminus (L:I)$. Similar to Case 3, we get $c \in (L:I) \cup (N:K)$

Consequently $J \subseteq (L:I) \cup (N:K)$.

(ii) If $0 \neq 8IJN$ and $8IJK = 0$, then $0 \neq 8IJ(K+N) \subseteq N$, and then by part (i), $JI \subseteq (N:M)$ or $J(K+N) \subseteq N$ or $I(K+N) \subseteq N$ and so $JI \subseteq (N:M)$ or $JK \subseteq N$ or $IK \subseteq N$.

(iii) Let $0 \neq 8J(N:M)K$ and $8IJK = 0$. Then $8J(I+(N:M))K = 8J(N:M)K \neq 0$ and so according to part (i), either $J(I+(N:M)) \subseteq (N:M)$ or $JK \subseteq N$ or $(I+(N:M))K \subseteq N$ and so either $JI \subseteq (N:M)$ or $JK \subseteq N$ or $IK \subseteq N$. Similarly if $0 \neq 8I(N:M)K$, we get the result.

(iv) Let $0 \neq 8J(N:M)N$ and $8IJK = 8IJN = 8J(N:M)K = 8I(N:M)K = 0$. Then $8J(I+(N:M))(K+N) = 8J(N:M)N \neq 0$, and so part (i) implies that $J(I+(N:M)) \subseteq (N:M)$ or $J(K+N) \subseteq N$ or $(I+(N:M))(K+N) \subseteq N$. Hence $JI \subseteq (N:M)$ or $JK \subseteq N$ or $IK \subseteq N$. Clearly if $0 \neq 8I(N:M)N$, we have the result.

For the particular case suppose the group $(M,+)$ has no subgroups of order 2. Then we show that $0 \neq 8IJK$, and so by part (i), the result is given. If $0 = 8IJK$, then consider $0 \neq \ell \in IJK$. As $8\ell = 0$, so the group $(M,+)$ has a subgroup of order 2, 4, or 8, which implies that it has an element of order 2, a contradiction. □

The following result is the ring version of 2.1, 2.2, and 2.3. For the proof just consider $M = R$.

Corollary 2.4 *Let $a, b \in R$ and I, J, K be ideals of R and suppose that L is a weakly 2-absorbing ideal of R.*

(a) *If $0 \neq 2abI$ and $abI \subseteq L$ then $ab \in L$ or $aI \subseteq L$ or $bI \subseteq L$.*

(b) *If $0 \neq 4aIJ$ and $aIJ \subseteq L$, then either $aI \subseteq L$ or $aJ \subseteq L$ or $IJ \subseteq L$.*

(c) *If $0 \neq IJK \subseteq L$, then $IJ \subseteq L$ or $IK \subseteq L$ or $JK \subseteq L$, if $8\big(IJ(K+L) + IK(J+L) + JK(I+L) + IL(J+K) + JL(I+K) + KL(I+J) + L^2(I+J+K)\big) \neq 0$. In particular, this holds if the group $(R,+)$ has no elements of order 2.*

3. Weakly 2-absorbing submodules and their colon ideals

In this section we study when the quotient of a weakly 2-absorbing submodule is a weakly 2-absorbing ideal. We will also give a condition under which a weakly 2-absorbing submodule is 2-absorbing.

Lemma 3.1 *Let N be a weakly 2-absorbing submodule of an R-module M. If $a, b \in R$, $x \in M$ with $abx = 0$ and $ab \notin (N:M)$, $ax \notin N, bx \notin N$, then*

(i) *$abN = a(N:M)x = b(N:M)x = 0$.*

(ii) *$a(N:M)N = b(N:M)N = (N:M)^2x = 0$.*

Proof (i) If $abN \neq 0$, then for some $y \in N$, $0 \neq aby = ab(x+y) \in N$ and since N is weakly 2-absorbing, $ab \in (N:M)$ or $a(x+y) \in N$ or $b(x+y) \in N$. Hence $ab \in (N:M)$ or $ax \in N$ or $bx \in N$, which are impossible. Thus $abN = 0$ and the similar arguments prove that $a(N:M)x = b(N:M)x = 0$.

(ii) If on the contrary for some $t \in (N:M)$ and $y \in N$, $0 \neq aty$ then by part (i), $0 \neq aty = a(b+t)(x+y) \in N$. Then since N is weakly 2-absorbing, we get $a(b+t) \in (N:M)$ or $a(x+y) \in N$ or $(b+t)(x+y) \in N$. This implies that $ab \in (N:M)$ or $ax \in N$ or $bx \in N$, which are against our assumptions; consequently $a(N:M)N = 0$. Similarly $b(N:M)N = (N:M)^2x = 0$. □

Theorem 3.2 *The colon ideal of a weakly 2-absorbing submodule is a weakly 2-absorbing ideal if $Ann(M)$ is a weakly 2-absorbing ideal, particularly if M is faithful.*

Proof Let N be a weakly 2-absorbing submodule of M. First assume that M is a faithful R-module. Let $a,b,c \in R$ with $0 \neq abc \in (N:M)$ and $ab \notin (N:M)$, $ac \notin (N:M)$ and $bc \notin (N:M)$. As $Ann(M)=0$, for some $z \in M$, $0 \neq abcz \in N$. Thus since N is weakly 2-absorbing and $ab \notin (N:M)$, $acz \in N$ or $bcz \in N$. We claim that there exists $x \in M$ such that $0 \neq abcx \in N$ and one of the following holds.

(i) $acx \notin N$ and $bcx \in N, abx \in N$.

(ii) $bcx \notin N$ and $acx \in N, abx \in N$.

We consider the following two cases.

Case 1. $acz \in N$. Because of $ac \notin (N:M)$, there exists $z' \in M \setminus N$ such that $acz' \notin N$. Since $0 \neq abcz$, it is easy to see that either $0 \neq abc(2z+z')$ or $0 \neq abc(z+z')$. First we suppose that $0 \neq ab(c(2z+z')) \in N$. Therefore, as N is weakly 2-absorbing, $ab \in (N:M)$ or $ac(2z+z') \in N$ or $bc(2z+z') \in N$. However, by assumption, $ab \notin (N:M)$ and as $acz' \notin N$, $ac(2z+z') \notin N$ and so $bc(2z+z') \in N$. Hence as $0 \neq bc(a(2z+z')) \in N$ and $bc \notin (N:M)$, we have $ba(2z+z') \in N$. By the same way if $0 \neq ab(c(z+z')) \in N$, then $ac(z+z') \notin N$ and $bc(z+z') \in N$, $ba(z+z') \in N$. Consequently for some $x \in M$, we have $0 \neq abcx \in N$ and $acx \notin N$ and $bcx \in N$, $abx \in N$.

As N is weakly 2-absorbing and $ab \notin (N:M)$, it suffices to show that there exists $x' \in M$, such that $0 \neq ab(cx') \in N$ and $acx' \notin N$, $bcx' \notin N$.

Since $ab \notin (N:M)$, for some $y' \in M$, $aby' \notin N$. Hence as $0 \neq acbx$, either $0 \neq acb(2x+y')$ or $0 \neq acb(x+y')$. First let $0 \neq ac(b(2x+y')) \in N$. Then since $abx \in N$ and $aby' \notin N$ we have $ab(2x+y') \notin N$ and hence as N is weakly 2-absorbing and $ac \notin (N:M)$, we have $cb(2x+y') \in N$. Then by considering $0 \neq bc(a(2x+y')) \in N$, since $bc \notin (N:M)$ and $ba(2x+y') \notin N$, we get $ca(2x+y') \in N$. Similarly in the case $0 \neq ac(b(x+y')) \in N$, we get $ab(x+y') \notin N$ and $cb(x+y') \in N$, $ca(x+y') \in N$.

Therefore, there exists $x'' \in M$ such that $0 \neq abcx''$ and $acx'' \in N, bcx'' \in N$ and $abx'' \notin N$. Thus as $0 \neq acx'' \in N$ and $ac \notin (N:M)$, either $ax'' \in N$ or $cx'' \in N$. However, since $abx'' \notin N$, $cx'' \in N$.

For some $y \in M$, we have $bcy \notin N$, because $bc \notin (N:M)$. Hence if $0 \neq ab(cy)$, then since N is weakly 2-absorbing, $acy \in N$ and $aby \in N$ and we consider $abc(x+y)$. If $0 = abc(x+y)$, then since $acx \notin N, acy \in N$ and $bcx \in N, bcy \notin N$, we have $bc(x+y) \notin N$ and $ac(x+y) \notin N$, and so by 3.1, since $ac \notin (N:M)$, we have $abN = 0$. Thus $abcx'' = 0$, which is a contradiction. Therefore, $0 \neq abc(x+y)$ and since $ab \notin (N:M)$ and $bc(x+y) \notin N, ac(x+y) \notin N$, we have the result.

Now let $ab(cy) = 0$. If $acy \notin N$, then since $ab \notin (N:M)$ and $bcy \notin N$, by 3.1, we have $abN = 0$ and so $abcx'' = 0$, which is impossible. Therefore, $acy \in N$. Then $bc(x+y) \notin N, ac(x+y) \notin N$ and since $abcy = 0$, $0 \neq abc(x+y)$. Consequently we find $x' \in M$, such that $0 \neq abcx' \in N$ and $acx' \notin N$ and $bcx' \notin N$.

Case 2. $bcz \in N$. The proof is given similar to that of Case 1.

Now if M is not a faithful R-module, then consider M as an $R' = R/Ann(M)$-module. It is easy to see that N is an R'-weakly 2-absorbing submodule of M and so by the above argument $(N:M)/Ann(M)$ is a weakly 2-absorbing ideal of R'. Now since $Ann(M)$ is a weakly 2-absorbing ideal, one can easily see that $(N:M)$ is a weakly 2-absorbing ideal of R. □

Now we show that the converse of 3.2 is not necessarily true.

Example 2 *It is easy to see that if* $(R, \mathfrak{M})$ *is a quasi-local ring with* $\mathfrak{M}^3 = 0$, *then every proper ideal of* R *is weakly* 2*-absorbing. Therefore, for the ring* $R = \frac{K[[X,Y,Z]]}{J}$, *where* $J = \langle X^3, Y^2, Z^2, XY, XZ\rangle$, *the ideal* $I = \frac{\langle X, Y^2, Z^2\rangle}{J}$ *is weakly* 2*-absorbing.*

Now consider the R*-module* $M = R \oplus R$ *and* $N = I \oplus R$. *Then* $(N : M) = I$ *is a weakly* 2*-absorbing ideal of* R, *but* N *is not a weakly* 2*-absorbing submodule of* M. *To see the proof note that* $(Y + J)(Z + J)(Y + Z + J, 1 + J) \in N$.

4. Weakly 2-absorbing submodules and their radical ideals

Let N be a 2-absorbing submodule of M. According to [9, Proposition 1(iii)] either $\sqrt{(N : M)}$ is a prime ideal of R, or $\sqrt{(N : M)} = P_1 \cap P_2$, where P_1, P_2 are the only distinct minimal prime ideals over $(N : M)$ and $P_1P_2 \subseteq (N : M)$. This is a motivation for studying $\sqrt{(N : M)}$ when N is a weakly 2-absorbing submodule in this section.

Let P be a prime ideal of R. The height of P denoted by $ht\ P$ is defined to be the supremum of the length of chains of $P_0 \subset P_1 \subset \cdots \subset P_n = P$ of prime ideals of R if the supermum exists, and ∞ otherwise.

The height of an ideal I denoted by $ht\ I$ is defined to be

$ht\ I = inf\{ht\ P \mid P$ is a minimal prime ideal containg $I\}$.

Proposition 4.1 *Let* I *be a weakly* 2*-absorbing ideal of* R *with* $\sqrt{I} = J$. *Then either* J *is a prime ideal of* R *or* $J = P_1 \cap P_2$, *where* P_1, P_2 *are the only distinct minimal prime ideals over* I *or* $I_P = 0$ *for every minimal prime ideal* P *over* I. *In the latter case* $ht\ I = 0$.

Proof Suppose that there are at least three minimal prime ideals P, Q, and L over I and $I_L \neq 0$. Consider $x \in P \setminus (L \cup Q)$ and $y \in Q \setminus (L \cup P)$. Since P, Q are minimal prime ideals over I, $\sqrt{I_P} = P_P$ and $\sqrt{I_Q} = Q_Q$ and so for some $s \in R \setminus P$ and $t \in R \setminus Q$, and $m, n > 0$ we have $sx^m \in I$ and $ty^n \in I$. Since $x \notin I$ and $y \notin I$, without loss of generality we can assume $sx^{m-1} \notin I$ and $ty^{n-1} \notin I$.

We claim that $sx \in I$ and $ty \in I$. If $0 \neq sx^m = sx^{m-1}x \in I$, then as I is weakly 2-absorbing and $x^m \notin I$, either $sx \in I$ or $sx^{m-1} \in I$. Hence $sx^{m-1} \notin I$ and we have $sx \in I$. Therefore, we can assume that $sx^m = 0$. Then as $sx^{m-1} \notin I$ and $x^m \notin I$, either $sx \in I$ or by 3.1, $x^m I = 0$ and so in this case $I_L = 0$, which is a contradiction and then $sx \in I$. Similarly $ty \in I$. Now we consider $(s + t)xy \in I$. If $(s + t)xy = 0$, then as $(s + t)x \notin I$ and $(s + t)y \notin I$, either $xy \in I$ or by 3.1, $xyI = 0$. If $xyI = 0$, then $I_L = 0$, which is impossible. Therefore, $xy \in I \subseteq L$, which is a contradiction.

Now let $I_P = 0$ for every minimal prime ideal P over I. To show that $ht\ I = 0$, let Q be a minimal prime ideal over I, and assume that Q' is a prime ideal with $Q' \subseteq Q$. If $I \subseteq Q'$, then evidently $Q' = Q$. Now let $x \in I \setminus Q'$. Since $I_Q = 0$, there exists $s \in R \setminus Q$ with $sx = 0$. Then $sx = 0 \in Q'$, which implies that $s \in Q' \subseteq Q$, a contradiction. □

To illustrate 4.1, in the following examples we introduce three different types of weakly 2-absorbing ideals.

Example 3

(i) *The zero ideal is a non-2-absorbing and weakly* 2*-absorbing ideal of* $\mathbb{Z}_8$, *and* $\sqrt{0} = 2\mathbb{Z}_8$ *is a prime ideal.*

(ii) *The zero ideal is a non-2-absorbing and weakly 2-absorbing ideal of* $\mathbb{Z}_{18}$, *and* $\sqrt{0} = 2\mathbb{Z}_{18} \cap 3\mathbb{Z}_{18}$, *which is the intersection of two distinct prime ideals.*

(iii) *If* P_1, P_2, *and* P_3 *are three incomparable prime ideals of a ring* R *with* $P_1P_2P_3 = 0$, *then* $I = P_1 \cap P_2 \cap P_3$ *is a weakly 2-absorbing ideal of* R *and* $\sqrt{I} = I$ *and* $I_{P_1} = I_{P_2} = I_{P_3} = 0$.

(iv) *If* $R = K[X, Y, Z]$ *and* $P_1 = \langle X, Y\rangle$, $P_2 = \langle X, Z\rangle$, *and* $P_3 = \langle Y, Z\rangle$, *then* $0 \neq I = \frac{P_1 \cap P_2 \cap P_3}{P_1P_2P_3}$ *is a weakly 2-absorbing ideal of the ring* $\frac{R}{P_1P_2P_3}$, $\sqrt{I} = I$ *and* $I_{\frac{P_1}{P_1P_2P_3}} = I_{\frac{P_2}{P_1P_2P_3}} = I_{\frac{P_3}{P_1P_2P_3}} = 0$.

PROOF. The proofs of (i) and (ii) are evident.

(iii) Let $0 \neq abc \in I$. If $a \in P_1 \cap P_2 \cap P_3$ or $a \notin P_1 \cup P_2 \cup P_3$, then there is nothing to prove. Therefore, we consider two cases.

Case 1. If a is in two of the P_i's, say P_1, P_2, then either $b \in P_3$ or $c \in P_3$ and so either $ab \in I$ or $ac \in I$.

Case 2. a is only in one of the P_i's. We can assume $a \in P_1 \setminus P_2 \cup P_3$. Hence $bc \in P_2 \cap P_3$ and since $P_1P_2P_3 = 0$ and $0 \neq abc$, either $b \in P_2 \cap P_3$ or $c \in P_2 \cap P_3$. Then similar to Case 1, we have the result.

It is easy to see that $\sqrt{I} = I$ and so I has three minimal prime ideals. Since $P_1P_2P_3 = 0$, for some $t \in P_2P_3 \setminus P_1$, we have $tI \subseteq tP_1 = 0$ and so $0 = I_{P_1}$. Similarly $I_{P_2} = I_{P_3} = 0$.

(iv) The proof is given by part (iii). □

The proof of the following result is given by 3.2 and 4.1.

Corollary 4.2 *Let* N *be a weakly 2-absorbing submodule of a faithful* R*-module* M. *Then either* $\sqrt{(N : M)}$ *is a prime ideal of* R *or* $\sqrt{(N : M)} = P_1 \cap P_2$, *where* P_1, P_2 *are the only distinct minimal prime ideals over* $(N : M)$ *or* $(N : M)_P = 0$ *for every prime ideal* P *containing* $(N : M)$. *In the latter case* $ht(N : M) = 0$.

Theorem 4.3 *Let* I *be a weakly 2-absorbing ideal of* R *and* P_1, P_2 *be two incomparable prime ideals, and suppose* $J = \sqrt{I} = P_1 \cap P_2$. *Then:*

If $0 \neq I_{P_1}$, $0 \neq I_{P_2}$, *then* $P_1P_2 \cup (P_1 + P_2)J \subseteq I$. *Furthermore, if* $J \neq I$, *then* $\{(I : r) \mid r \in J \setminus I\}$ *is a chain of prime ideals of* R.

Proof First we show that if $a \in P_1 \setminus P_2$, $b \in P_2 \setminus P_1$, then $ab \in I$ $(*)$.

As P_1, P_2 are minimal prime ideals over I, $\sqrt{I_{P_1}} = (P_1)_{P_1}$ and $\sqrt{I_{P_2}} = (P_2)_{P_2}$ and so for some $s \in R \setminus P_1$ and $t \in R \setminus P_2$, and $m, n > 0$, we have $sa^m \in I$ and $tb^n \in I$. Then by proof of 4.1, either $sa \in I$ or $a^mI = 0$ and $tb \in I$ or $b^nI = 0$. If $a^mI = 0$ or $b^nI = 0$, then $I_{P_2} = 0$ or $I_{P_1} = 0$; these two cases are impossible. Then $sa \in I$ and $tb \in I$. Now we consider $(s+t)ab \in I$. If $(s+t)ab = 0$, then as $(s+t)a \notin I$ and $(s+t)b \notin I$, either $ab \in I$ or by 3.1, $(s+t)aI = 0$. If $(s+t)aI = 0$, then $I_{P_2} = 0$, which is a contradiction. Therefore, $ab \in I$.

Suppose that $a', b' \in J$. Consider $t \in P_1 \setminus P_2$ and $s \in P_2 \setminus P_1$. Hence as $a' + t \in P_1 \setminus P_2$ and $b' + s \in P_2 \setminus P_1$, by $(*)$, $(a' + t)s$, $ts \in I$ and so $a's \in I$. Similarly $b't \in I$ and since $(a' + t)(s + b') \in I$, $a'b' \in I$. Thus $J^2 \subseteq I$.

For the proof of $P_1P_2 \subseteq I$, let $m \in P_1, n \in P_2$. By the last part we may assume $m \in J$ and $n \in P_2 \setminus P_1$. We consider $x \in P_1 \setminus P_2$ and by $(*)$, we get $nx \in I$, $n(m + x) \in I$ and so $mn \in I$ and completes the proof.

Put $I_r = (I : r)$ for each $r \in J \setminus I$. By the above paragraph, $rP_1 \subseteq I$, $rP_2 \subseteq I$ and so $P_1 \subseteq I_r$, $P_2 \subseteq I_r$. Now let $a''b'' \in I_r$. Then $a''b''r \in I$ and since I is weakly 2-absorbing, $a''b''r = 0$ or $a''b'' \in I$ or $a'' \in I_r$

or $b'' \in I_r$. Since $P_1 \subseteq I_r$ and $P_2 \subseteq I_r$, we can assume $a'' \notin P_1 \cup P_2$ and $b'' \notin P_1 \cup P_2$ and so $a''b'' \notin I$. If $a''b''r = 0$ and $a'' \notin I_r$, $b'' \notin I_r$, then by 3.1, $a''b''I = 0$ and so $I_{P_1} = 0$, which is a contradiction. Thus I_r is prime.

Now let $r', s' \in J \setminus I$ and $t' \in I_{r'} \setminus I_{s'}$. As $P_1, P_2 \subseteq I_{s'}$, $t' \notin P_1 \cup P_2$. To show that $I_{s'} \subseteq I_{r'}$, let $c \in I_{s'}$. We may assume that $c \notin P_1 \cup P_2$ and we conclude $t'c \notin P_1 \cup P_2$. Now consider $t'c(r'+s') \in I$. Since I is weakly 2-absorbing, $t'c(r'+s') = 0$ or $t'c \in I$ or $t'(r'+s') \in I$ or $c(r'+s') \in I$. However, since $t'c \notin P_1 \cup P_2$, $t'c \notin I$. Moreover, as $t' \in I_{r'} \setminus I_{s'}$, $t'(r'+s') \notin I$. Therefore, either $t'c(r'+s') = 0$ or $c(r'+s') \in I$. In the case $t'c(r'+s') = 0$, by 3.1, we have $t'cI = 0$ and so $I_{P_1} = 0$, which is a contradiction. Therefore, $c(r'+s') \in I$ and since $c \in I_{s'}$, we conclude $c \in I_{r'}$. □

Corollary 4.4 *Let I be a weakly 2-absorbing ideal of R and P_1, P_2 two incomparable prime ideals. If $\sqrt{I} = P_1 \cap P_2$ and $0 \neq I_{P_1}$, $0 \neq I_{P_2}$, then I is 2-absorbing.*

Proof Let $abc \in I$. As I is weakly 2-absorbing, we can assume that $abc = 0$. Put $J = \sqrt{I}$.

First assume that at least one of a or b or c is in J, for example $a \in J$. If $a \in I$, then we have the result. Therefore, let $a \in J \setminus I$. Thus, by 4.3, I_a is prime and so we have the result. Now let $a, b, c \notin J$. Hence as $abc \in I \subseteq J = P_1 \cap P_2$, we can assume $a \in P_1 \setminus P_2$ and $b \in P_2 \setminus P_1$. Therefore, according to 4.3, $ab \in I$. □

Proposition 4.5 *Let N be a weakly 2-absorbing submodule of an R-module M. Then the following statements hold:*

(i) *If there exists a submodule L of M such that $N \subsetneqq L$, then N is a weakly 2-absorbing submodule of L.*

(ii) *If for some submodule L and ideal I there exist positive integer numbers $m > n$ such that $I^m L \subseteq N \subsetneqq I^n L$, then N is a 2-absorbing submodule of $I^n L$ and $(\sqrt{(N : M)})^2 I^n L \subseteq N$.*

Proof (i) Let $a, b \in R, x \in L$ with $0 \neq abx \in N$. Hence as N is a weakly 2-absorbing submodule of M, $ab \in (N : M) \subseteq (N : L)$ or $ax \in N$ or $bx \in N$. Therefore, N is a weakly 2-absorbing submodule of L.

(ii) First suppose that $Ann(I^n L) = 0$. By part(i), N is a weakly 2-absorbing submodule of $I^n L$. Now we claim that N is 2-absorbing. Assume that $a, b \in R, x \in I^n L$, $abx \in N$ and $ab \notin (N : I^n L)$, $ax \notin N$ and $bx \notin N$. As N is weakly 2-absorbing, we may assume that $0 = abx$. Then, according to 3.1, $abN = 0$ and so $abI^m L = 0$ and then $abI^{m-n} = 0$, since $Ann(I^n L) = 0$. If $m - n \leq n$, then $abI^n L = 0$ and so $ab = 0 \in (N : I^n L)$. Now let $m - n > n$. Hence $abI^{m-2n} I^n L = 0$ and so $abI^{m-2n} = 0$. We repeat this until we get $ab = 0 \in (N : I^n L)$.

Next we let $Ann(I^n L) \neq 0$. We consider $I^n L$ a $\frac{R}{Ann(I^n L)}$-module. Clearly N is a weakly 2-absorbing $\frac{R}{Ann(I^n L)}$-submodule of $I^n L$. By the above argument, N is a 2-absorbing $\frac{R}{Ann(I^n L)}$-submodule of $I^n L$. It is easy to see N is a 2-absorbing R-submodule of $I^n L$. Then, by [9, Proposition 2.2], $(\sqrt{(N : I^n L)})^2 I^n L \subseteq N$ and since $(\sqrt{(N : M)})^2 I^n L \subseteq (\sqrt{(N : I^n L)})^2 I^n L$, we have the result. □

Corollary 4.6 *Let I be a finitely generated weakly 2-absorbing ideal of R. Then $(\sqrt{I})^3 \subseteq I$. Furthermore, either $8(\sqrt{I})^3 = 0$ or $(\sqrt{I})^2 \subseteq I$.*

Proof There exists a positive integer number m such that $(\sqrt{I})^m \subseteq I \subseteq \sqrt{I}$. If $I = \sqrt{I}$, then evidently we have the result. Then let $I \neq \sqrt{I}$. Thus, according to 4.5**(ii)**, $(\sqrt{I})^3 \subseteq I$. Now if $0 \neq 8(\sqrt{I})^3$, then by 2.3, $(\sqrt{I})^2 \subseteq I$. □

5. Weakly 2-absorbing submodules in direct sum of modules

Throughout this section R_1 and R_2 are two commutative rings with identity, N_1 is a submodule of an R_1-module M_1, and N_2 is a submodule of an R_2-module M_2, the ring $R = R_1 \oplus R_2$, $M = M_1 \oplus M_2$, and $N = N_1 \oplus N_2$. We will characterize the weakly 2-absorbing submodules of the R-module M, and some applications of this study are given in the next section.

Lemma 5.1 *Let K^* be a proper submodule of an R^*-module M^* and $I^*M^* \neq 0$, where I^* is an ideal of R^*. Then there exist $r \in I^*$ and $x \in (M^* \setminus K^*)$ with $rx \neq 0$.*

Proof If $I^*x = 0$ for each $x \in (M^* \setminus K^*)$, then $(M^* \setminus K^*) \subseteq (0 :_{M^*} I^*)$. Therefore, $M^* = K^* \cup (M^* \setminus K^*) \subseteq K^* \cup (0 :_{M^*} I^*)$, and since $M^* \not\subseteq K^*$, $M^* \subseteq (0 :_{M^*} I^*)$, that is $I^*M^* = 0$, which is a contradiction. □

Lemma 5.2 *[10, Theorem 2.5] Let N be a weakly 2-absorbing submodule of an R-module M, which is not 2-absorbing. Then $(N : M)^2 N = 0$, and particularly $(N : M)^3 \subseteq Ann(M)$.*

The weakly 2-absorbing submodules of the form $N_1 \oplus M_2$ are characterized in part (a) of the following result.

Lemma 5.3 *Let $0 \neq M_1$ and $0 \neq M_2$.*

(a) *The following are equivalent:*

(i) *$N_1 \oplus M_2$ is a weakly 2-absorbing submodule of the R-module M;*

(ii) *$N_1 \oplus M_2$ is a 2-absorbing submodule of the R-module M;*

(iii) *N_1 is a 2-absorbing submodule of M_1.*

(b) *If $N = N_1 \oplus N_2$ is a weakly 2-absorbing submodule of M, $N_1 \neq M_1$, and $N_2 \neq M_2$, then N_1 is a weak prime submodule of M_1; moreover, if $0 \neq N_2$, then N_1 is a weakly prime submodule of M_1.*

(c) *If N_1 is a prime submodule of M_1 and N_2 is a prime submodule of M_2, then $N = N_1 \oplus N_2$ is a 2-absorbing submodule of M.*

(d) *If $N = N_1 \oplus N_2$ is a weakly 2-absorbing submodule of M and $N_1 \neq M_1$, $N_2 \neq M_2$, and $(N_2 : M_2)M_2 \neq 0$, then N_1 is a prime submodule of M_1.*

Proof (a)(i)$\Rightarrow$(ii) If $K = N_1 \oplus M_2$ is not 2-absorbing, then by 5.2, $(0,0) = (K : M)^2 K = \big((N_1 : M_1) \oplus (M_2 : M_2)\big)^2 (N_1 \oplus M_2) = \big((N_1 : M_1)^2 N_1\big) \oplus M_2$ and so $M_2 = 0$, which is a contradiction.

(ii)$\Rightarrow$(iii) The proof is clear.

(iii)$\Rightarrow$(i) It is straightforward.

(b) Let $0 \neq rx \in N_1$, where $r \in R$ and $x \in M_1$. Consider $z \in M_2 \backslash N_2$ Then $(0,0) \neq (1,0)(r,1)(x,z) \in N$ and as N is weakly 2-absorbing, $(1,0)(r,1) \in (N : M)$ or $(r,1)(x,z) \in N$ or $(1,0)(x,z) \in N$. Note that $z \in M_2 \setminus N_2$, $(r,1)(x,z) \notin N$; thus $(1,0)(r,1) \in (N : M) = (N_1 : M_1) \oplus (N_2 : M_2)$ or $(1,0)(x,z) \in N$. Therefore, $r \in (N_1 : M)$ or $x \in N_1$. This shows that N_1 is a weak prime submodule of M_1.

Now let $0 \neq N_2$. Consider $a_1, b_1 \in R_1$ and $y_1 \in M_1$ with $a_1 b_1 y_1 \in N_1$, and let $0 \neq y_2 \in N_2$. Then $(0,0) \neq (a_1,1)(b_1,1)(y_1,y_2) \in N$, and so $(a_1,1)(b_1,1) \in (N : M)$ or $(a_1,1)(y_1,y_2) \in N$ or $(b_1,1)(y_1,y_2) \in N$. If $(a_1,1)(b_1,1) \in (N : M)$, then $1 \in (N_2 : M_2)$, which is impossible. If $(a_1,1)(y_1,y_2) \in N$ or $(b_1,1)(y_1,y_2) \in N$, then $a_1 y_1 \in N_1$ or $b_1 y_1 \in N_1$ as required.

(c) Suppose that $(a,c),(b,d) \in R$ and $(m,n) \in M$ with $(a,c)(b,d)(m,n) \in N = N_1 \oplus N_2$. Then $abm \in N_1$. Therefore, $a \in (N_1 : M_1)$ or $b \in (N_1 : M_1)$ or $m \in N_1$. Moreover, since $cdn \in N_2$, $c \in (N_2 : M_2)$ or $d \in (N_2 : M_2)$ or $n \in N_2$. In any of these cases we get $(a,c)(b,d) \in (N : M)$ or $(a,c)(m,n) \in N$ or $(b,d)(m,n) \in N$, which completes the proof.

(d) Let $rx \in N_1$, where $r \in R$ and $x \in M_1$. We show that $r \in (N_1 : M)$ or $x \in N_1$.

Apply 5.1 for $I^* = (N_2 : M_2)$, $K^* = N_2$, and $M^* = M_2$ to see that there exist $s \in (N_2 : M_2)$ and $z \in (M_2 \setminus N_2)$ with $sz \neq 0$.

Note that $(0,0) \neq (1,s)(r,1)(x,z) \in N$ and since N is weakly 2-absorbing, $(1,s)(r,1) \in (N : M)$ or $(r,1)(x,z) \in N$ or $(1,s)(x,z) \in N$. As $z \in M_2 \setminus N_2$, $(r,1)(x,z) \notin N$; hence $(1,s)(r,1) \in (N : M) = (N_1 : M_1) \oplus (N_2 : M_2)$ or $(1,s)(x,z) \in N$. This implies that $r \in (N_1 : M)$ or $x \in N_1$. □

The weakly 2-absorbing submodules of the form $N_1 \oplus 0$ are characterized in the following.

Theorem 5.4 *Let $N_1 \neq M_1$ and $0 \neq M_2$. The submodule $N_1 \oplus 0$ is a weakly 2-absorbing submodule of M if and only if one of the following holds:*

(i) *N_1 is a weak prime submodule of M_1 and 0 is a prime submodule of M_2 and $0 \neq (N_1 : M_1)M_1$.*

(ii) *N_1 is a weak prime submodule of M_1 and 0 is a weakly prime submodule of M_2 and $0 = (N_1 : M_1)M_1$.*

(iii) *$N_1 = 0$.*

Moreover if (i) *holds, then $N_1 \oplus 0$ is 2-absorbing if and only if N_1 is a prime submodule of M_1.*

Proof $(\Longrightarrow)$ Let $N_1 \oplus 0$ be a weakly 2-absorbing submodule of M and $0 \neq N_1$. Then by 5.3**(b)**, N_1 is weak prime.

If $0 \neq (N_1 : M_1)M_1$, then by 5.3**(d)**, the zero submodule of M_2 is prime. Otherwise since $0 \neq N_1$, then by 5.3**(b)**, the zero submodule of M_2 is weakly prime.

$(\Longleftarrow)$ Assume that $(0,0) \neq (a,b)(c,d)(x,y) \in N_1 \oplus 0$, where $(a,b),(c,d) \in R$, $(x,y) \in M$. Then $0 \neq acx \in N_1$ and $bdy = 0$. Since N_1 is weak prime, $a \in (N_1 : M_1)$ or $c \in (N_1 : M_1)$ or $x \in N_1$. First suppose that (i) is satisfied.

As 0 is a prime submodule of M_2, we have $b \in (0 : M_2)$ or $d \in (0 : M_2)$ or $y = 0$.

Now it is easy to see that in any of the above cases $(a,b)(c,d) \in (N_1 \oplus 0 : M)$ or $(a,b)(x,y) \in N_1 \oplus 0$ or $(c,d)(x,y) \in N_1 \oplus 0$. Consequently $N_1 \oplus 0$ is weakly 2-absorbing.

Now assume that (ii) holds. If $a \in (N_1 : M_1)$ or $c \in (N_1 : M_1)$, then $acx \in (N_1 : M_1)M_1 = 0$, and so $acx = 0$, which is impossible. Thus $x \in N_1$. Since $bdy = 0$ and 0 is weakly prime, $by = 0$ or $dy = 0$. Therefore, either $(a,b)(x,y) \in N_1 \oplus 0$ or $(c,d)(x,y) \in N_1 \oplus 0$.

To prove the second part of this theorem, assume that (i) holds. Then N_1 is a weak prime submodule of M_1 and 0 is a prime submodule of M_2.

If N_1 is not a prime submodule, then for some $t \in R_1 \setminus (N_1 : M_1)$, and $z \in M_1 \setminus N_1$, we have $tz \in N_1$. Now choose $0 \neq u \in M_2$. Then $(0,0) = (1,0)(t,1)(z,u) \in N_1 \oplus 0$ and $(1,0)(t,1) \notin (N_1 \oplus 0 : M)$ and $(t,1)(z,u) \notin N_1 \oplus 0$; also $(1,0)(z,u) \notin N_1 \oplus 0$. Therefore, $N_1 \oplus 0$ is not 2-absorbing.

Conversely if N_1 is a prime submodule of M_1, then as 0 is prime, by 5.3**(c)**, $N_1 \oplus 0$ is 2-absorbing. □

Example 4 *It is easy to see that if $(R_1, \mathfrak{M})$ is a quasi-local ring with $\mathfrak{M}^2 = 0$, then every proper ideal of R_1 is weak prime. Particularly if $R_1 = \frac{K[X,Y]}{\langle X^2, XY, Y^2\rangle}$, where K is a field, then $I_1 = \frac{\langle X, Y^2\rangle}{\langle X^2, XY, Y^2\rangle}$ is a weak prime ideal of R_1, but it is not prime. Therefore, by 5.4 the ideal $I_1 \oplus 0$ is a weakly 2-absorbing ideal of the ring $R_1 \oplus K$, but it is not a 2-absorbing ideal.*

Theorem 5.5 *Let $0 \neq N_1 \neq M_1$ and $0 \neq N_2 \neq M_2$. Then N is a weakly 2-absorbing submodule of M if and only if for each $i = 1, 2$ one of the following holds:*

(1) *$0 \neq (N_i : M_i)M_i$ and N_{3-i} is a prime submodule of M_{3-i}.*

(2) *$0 = (N_i : M_i)M_i$ and N_{3-i} is a weak prime and a weakly prime submodule of M_{3-i}.*

Proof ($\Longrightarrow$) Suppose that N is a weakly 2-absorbing submodule of M. According to 5.3**(b)**, N_{3-i} is a weak prime and a weakly prime submodule of M_{3-i} for each $i = 1, 2$.

Now if $0 \neq (N_i : M_i)M_i$, then by 5.3**(d)**, N_{3-i} is a prime submodule of M_{3-i}.

($\Longleftarrow$) First suppose that (1) holds for $i = 1, 2$. Then by 5.3**(c)**, N is a weakly 2-absorbing submodule of M.

Let $(0,0) \neq (r_1, r_2)(r_1', r_2')(m_1, m_2) \in N = N_1 \oplus N_2$, where $(r_1, r_2), (r_1', r_2') \in R$ and $(m_1, m_2) \in M$. Then $r_i r_i' m_i \in N_i$ for $i = 1, 2$.

Now assume that (2) holds for $i = 1, 2$. Without loss of generality we can suppose that $0 \neq r_1 r_1' m_1$. Since N_1 is weak prime, $r_1 \in (N_1 : M_1)$ or $r_1' \in (N_1 : M_1)$ or $m_1 \in N_1$. If $r_1 \in (N_1 : M_1)$ or $r_1' \in (N_1 : M_1)$, then $r_1 r_1' m_1 \in (N_1 : M_1)M_1 = 0$, which is impossible; hence $m_1 \in N_1$. Also note that $r_2 r_2' m_2 \in N_2$ and N_2 is weakly prime; then $r_2 m_2 \in N_2$ or $r_2' m_2 \in N_2$. Therefore, either $(r_1, r_2)(m_1, m_2) \in N$ or $(r_1', r_2')(m_1, m_2) \in N$, as required.

Now let (1) hold for $i = 1$ and (2) hold for $i = 2$. Note that $r_2 r_2' m_2 \in N_2$ and N_2 is prime, then $r_2 \in (N_2 : M_2)$ or $r_2' \in (N_2 : M_2)$ or $m_2 \in N_2$. We have one of the following two cases:

Case 1. $0 \neq r_1 r_1' m_1$. As N_1 is weak prime, $r_1 \in (N_1 : M_1)$ or $r_1' \in (N_1 : M_1)$ or $m_1 \in N_1$. Now it is easy to see that in any of the above cases $(r_1, r_2)(m_1, m_2) \in N$ or $(r_1', r_2')(m_1, m_2) \in N$ or $(r_1, r_2)(r_1', r_2') \in (N : M)$, as required.

Case 2. $0 \neq r_2 r_2' m_2$. If $r_2 \in (N_2 : M_2)$ or $r_2' \in (N_2 : M_2)$, then $r_2 r_2' m_2 \in (N_2 : M_2)M_2 = 0$, which is impossible; thus $m_2 \in N_2$. As $r_1 r_1' m_1 \in N_1$ and N_1 is weakly prime, either $r_1 m_1 \in N_1$ or $r_1' m_1 \in N_1$, and so either $(r_1, r_2)(m_1, m_2) \in N$ or $(r_1', r_2')(m_1, m_2) \in N$. □

6. Modules whose proper submodules are all weakly 2-absorbing

A well-known result states that if every proper ideal of a commutative ring with identity R is a prime ideal, then R is a field. As a generalization, in [3, Proposition 2.1] it is proved that if every proper submodule of a nontorsion R-module module M is a prime submodule of M, then R is a field. In this section we study the modules whose proper submodules are all weakly 2-absorbing.

Theorem 6.1 *Let M be a nonzero R-module such that every proper submodule of M is weakly 2-absorbing. Then R has at most three maximal ideals containing $Ann(M)$.*

Proof Let N be a nonzero finitely generated submodule of M. We prove that R has at most three maximal ideals containing $Ann(N)$. By 4.5, every proper submodule of N is a weakly 2-absorbing submodule of N. Let $\mathfrak{M}_1, \mathfrak{M}_2, \mathfrak{M}_3$, and $\mathfrak{M}_4$ be distinct maximal ideals of R containing $Ann(N)$. Put $J = \mathfrak{M}_1 \cap \mathfrak{M}_2 \cap \mathfrak{M}_3$ and $N' = JN$.

Evidently for each i, $\mathfrak{M}_i N \neq N$; otherwise by Nakayama's lemma there exists $t \in \mathfrak{M}_i$ with $(t-1) \in Ann(N) \subseteq \mathfrak{M}_i$, which is impossible. Now since $\mathfrak{M}_i \subseteq (\mathfrak{M}_i N : N)$, we get $\mathfrak{M}_i = (\mathfrak{M}_i N : N)$. Therefore, $J \subseteq (N' : N) \subseteq \cap_{i=1}^{3}(\mathfrak{M}_i N : N) = J$, and so $\sqrt{(N' : N)} = \sqrt{J} = J = \mathfrak{M}_1 \cap \mathfrak{M}_2 \cap \mathfrak{M}_3$. By [9, Section 2, Proposition 1(iii)], the radical ideal of a 2-absorbing submodule is the intersection of at most 2 prime ideals; therefore, N' is not a 2-absorbing submodule of N. Hence by 5.2, $J^3 = (N' : N)^3 \subseteq Ann(N) \subseteq \mathfrak{M}_4$, which implies that $\mathfrak{M}_j = \mathfrak{M}_4$ for some $1 \leq j \leq 3$, a contradiction. Thus R has at most three maximal ideals $\mathfrak{M}_1$, $\mathfrak{M}_2$, $\mathfrak{M}_3$ containing $Ann(N)$.

Now if N^* is another nonzero finitely generated submodule of M, then by the same argument $Ann(N^*)$ is contained in at most three maximal ideals, say $\mathfrak{M}_1^*$, $\mathfrak{M}_2^*$, $\mathfrak{M}_3^*$. Thus $Ann(N + N^*)$ is contained in $\mathfrak{M}_1$, $\mathfrak{M}_2$, $\mathfrak{M}_3$, $\mathfrak{M}_1^*$, $\mathfrak{M}_2^*$, $\mathfrak{M}_3^*$, and since $N + N^*$ is finitely generated, $\{\mathfrak{M}_1, \mathfrak{M}_2, \mathfrak{M}_3\} = \{\mathfrak{M}_1^*, \mathfrak{M}_2^*, \mathfrak{M}_3^*\}$.

Hence R has at most three fixed maximal ideals $\mathfrak{M}_1$, $\mathfrak{M}_2$, $\mathfrak{M}_3$ such that for each nonzero finitely generated submodule L of M, we have $Ann(L) \subseteq U = \mathfrak{M}_1 \cup \mathfrak{M}_2 \cup \mathfrak{M}_3$.

Now we prove that $J^3 M = 0$, where $J = \mathfrak{M}_1 \cap \mathfrak{M}_2 \cap \mathfrak{M}_3$. $(*)$

On the contrary let $a, b, c \in J$ and $x \in M$ such that $abcx \neq 0$. If $Rabcx = M$, then $M = Rabcx \subseteq Rcx$ and so $Rabcx = Rcx$. Then there exists $s \in R$ with $(1 - sab)cx = 0$, and since $0 \neq cx$, $(1 - sab) \in Ann(cx) \subseteq U$, which is impossible. Thus $Rabcx \neq M$.

Note that $0 \neq abcx \in Rabcx$ and since $Rabcx$ is weakly 2-absorbing, $acx \in Rabcx$ or $bcx \in Rabcx$ or $ab \in (Rabcx : M)$.

If $acx \in Rabcx$, then for some $r \in R$, $acx = rabcx$ and so $(1 - rb)acx = 0$ and note that $0 \neq acx$; thus $(1 - rb) \in Ann(acx) \subseteq U$, which is a contradiction. Consequently $acx \notin Rabcx$ and similarly $bcx \notin Rabcx$. Furthermore, if $ab \in (Rabcx : M)$, then for some $t \in R$, $abx = tabcx$ and so $(1 - tc)abx = 0$ and we get $(1 - tc) \in Ann(abx) \subseteq U$, which is impossible. Whence $J^3 \subseteq Ann(M)$.

Now if $Ann(M)$ is contained in a maximal ideal $\mathfrak{M}^*$, then $(\mathfrak{M}_1 \cap \mathfrak{M}_2 \cap \mathfrak{M}_3)^3 = J^3 \subseteq Ann(M) \subseteq \mathfrak{M}^*$. This implies that $\mathfrak{M}_j = \mathfrak{M}^*$ for some $1 \leq j \leq 3$, which completes the proof. □

Recall that $J(R)$ is the intersection of all maximal ideals of R.

Corollary 6.2 *Let M be a nonzero R-module such that every proper submodule of M is weakly 2-absorbing. Then $(J(R))^3 M = 0$.*

Proof According to $(*)$ in the proof of 6.1, $J^3 M = 0$, and evidently $J(R) \subseteq J$. □

Theorem 6.3 *Let* $(R_1, \mathfrak{M}_1), (R_2, \mathfrak{M}_2)$ *be quasi-local rings and* $R = R_1 \oplus R_2$*. Then the following are equivalent:*

(i) *There exists a faithful* R*-module* M *such that every proper submodule of* M *is weakly* 2*-absorbing;*

(ii) $\mathfrak{M}_1^2 = 0$, $\mathfrak{M}_2^2 = 0$; *furthermore,* R_1 *or* R_2 *is a field.*

Moreover:

(a) *If* R_2 *is not a field and* (i) *holds, then* $(1,0)M \cong R_1$.

(b) *If* R_1 *is not a field and* (i) *holds, then* $(0,1)M \cong R_2$.

(c) *If* R_1 *and* R_2 *are fields, then every proper submodule of any arbitrary* R*-module is weakly* 2*-absorbing.*

Proof (i) $\Longrightarrow$ (ii) Put $M_1 = (1,0)M$ and $M_2 = (0,1)M$. Since M is faithful, $M_1, M_2 \neq 0$. One can easily see that M_1 is a faithful R_1-module with the multiplication $r_1\big((1,0)m\big) = (r_1, 0)m$ for each $r_1 \in R_1$ and $m \in M$. Similarly M_2 is a faithful R_2-module and $M \cong M_1 \oplus M_2$ as R-modules.

To show that $\mathfrak{M}_1^2 = 0$, let $a, b \in \mathfrak{M}_1$ with $0 \neq ab$. As M_1 is faithful, $0 \neq abM_1$ and so for some $x \in M_1$, $0 \neq abx$.

Note that $0 \neq M_2$ and so $R_1abx \oplus 0$ is a proper submodule of M; thus it is weakly 2-absorbing. Now by 5.3**(b)**, R_1abx is a weak prime submodule of M_1, and as $0 \neq abx \in R_1abx$, we have $a \in (R_1abx : M_1)$ or $bx \in R_1abx$. Hence $ax \in R_1abx$ or $bx \in R_1abx$.

Therefore, either $ax = rabx$ for some $r \in R_1$, or $bx = sabx$ for some $s \in R_1$. As $(1 - rb)$ and $(1 - sa)$ are unit, either $ax = 0$ or $bx = 0$, which is a contradiction. Then we conclude that $\mathfrak{M}_1^2 = 0$. With the same argument we get $\mathfrak{M}_2^2 = 0$.

If R_1 is not a field, then $\mathfrak{M}_1 \neq 0$ and as M_1 is faithful, $\mathfrak{M}_1 M_1 \neq 0$. Then $0 \neq m_1 x_1$ for some $m_1 \in \mathfrak{M}_1, x_1 \in M_1$. Now we show that $\mathfrak{M}_2 M_2 = 0$. Let $x_2 \in M_2$ and $m_2 \in \mathfrak{M}_2$. Since $\mathfrak{M}_2^2 = 0$, we have $m_2^2 = 0$.

If $\mathfrak{M}_1 M_1 = M_1$, then as $0 = \mathfrak{M}_1^2$, we get $0 = \mathfrak{M}_1^2 M = \mathfrak{M}_1 M_1 = M_1$, which is impossible; thus $\mathfrak{M}_1 M_1 \neq M_1$.

Put $N = \mathfrak{M}_1 M_1 \oplus 0$. Note that $(0,0) \neq (1, m_2)(1, m_2)(m_1 x_1, x_2) \in N$. As N is weakly 2-absorbing, either $(1, m_2)(1, m_2) \in (N : M)$ or $(1, m_2)(m_1 x_1, x_2) \in N$, and as $\mathfrak{M}_1 M_1 \neq M_1$, $(1, m_2)(1, m_2) \notin (N : M)$ and then $(1, m_2)(m_1 x_1, x_2) \in N$, and so $0 = m_2 x_2$. Thus $\mathfrak{M}_2 M_2 = 0$, that is $\mathfrak{M}_2 \subseteq Ann(M_2) = 0$. Hence R_2 is a field.

(ii) $\Longrightarrow$ (i) Put $M = R$. Then the proof is given by [5, Theorem 3.4].

(a) Now if R_2 is not a field and (i) holds, then we show that $M_1 \cong R_1$.

If for some $y_1 \in R_1$, $M_1 = Ry_1$, then as $0 = Ann(M_1) = Ann(y_1)$, we get $M_1 = Ry_1 \cong \frac{R}{Ann(y_1)} \cong R_1$. Now assume that $M_1 \neq R_1 y_1$ for each $0 \neq y_1 \in M_1$. Since R_2 is not a field and M_2 is faithful, $0 \neq \mathfrak{M}_2 M_2$ and so for some $t_2 \in \mathfrak{M}_2$ and $y_2 \in M_2$, $0 \neq t_2 y_2$. As $\mathfrak{M}_2^2 = 0$, $t_2^2 = 0$ and so $(0,0) \neq (1, t_2)(1, t_2)(y_1, y_2) \in R_1 y_1 \oplus 0$. Note that $R_1 y_1 \neq M_1$ and so $(1, t_2)(1, t_2) \notin (R_1 y_1 \oplus 0 : M)$ and since $R_1 y_1 \oplus 0$ is weakly 2-absorbing, $(1, t_2)(y_1, y_2) \in R_1 y_1 \oplus 0$, which is impossible because $t_2 y_2 \neq 0$. Consequently $M_1 \cong R_1$.

(b) The proof is similar to that of (a).

(c) Let R_1 and R_2 be two fields and M be an arbitrary R-module. Then $M \cong M_1 \oplus M_2$, where M_i is an R_i-module for each $i = 1, 2$. Furthermore, every proper submodule of M is of the form $N = N_1 \oplus N_2$, where N_i is a submodule of M_i for each $i = 1, 2$ and at least one of N_1 or N_2 is a proper submodule.

Note that every proper subspace of a vector space is prime and so for each $i = 1, 2$ either $N_i = M_i$ or N_i is a prime submodule of M_i. Hence, by 5.3**(a)** and 5.3**(c)**, the submodule N is a weakly 2-absorbing submodule of M. □

Proposition 6.4 *Let $R = R_1 \oplus R_2 \oplus R_3$, where R_1, R_2, and R_3 are three rings. If M is a faithful R-module such that every proper submodule of M is weakly 2-absorbing, then R_1, R_2, R_3 are fields and $M \cong R$.*

Proof Put $M_1 = (1, 0, 0)M$, $M_2 = (0, 1, 0)M$, and $M_3 = (0, 0, 1)M$. Then it is easy to see that M_i is an R_i-module for each $i = 1, 2, 3$, and also $M \cong M_1 \oplus M_2 \oplus M_3$ as R-modules. Since M is faithful, the R_i-module M_i is faithful, for each $i = 1, 2, 3$.

Let $\mathfrak{M}_i$ be a maximal ideal of R_i for each $i = 1, 2, 3$. Evidently $\mathfrak{M}_1 \oplus R_2 \oplus R_3$ and $R_1 \oplus \mathfrak{M}_2 \oplus R_3$ and $R_1 \oplus R_2 \oplus \mathfrak{M}_3$ are the the maximal ideals of R and by 6.1, R has at most three maximal ideals; therefore, $(R_1, \mathfrak{M}_1)$ and $(R_2, \mathfrak{M}_2)$ and $(R_3, \mathfrak{M}_3)$ are quasi-local rings, and $J(R) = \mathfrak{M}_1 \oplus \mathfrak{M}_2 \oplus \mathfrak{M}_3$.

According to 6.2, $(J(R))^3M = 0$ and since M is faithful, $(J(R))^3 = 0$; hence $\mathfrak{M}_i^3 = 0$ for each $i = 1, 2, 3$. If $\mathfrak{M}_iM_i = M_i$, then $0 = \mathfrak{M}_i^3M_i = M_i$, which is a contradiction. Hence $\mathfrak{M}_iM_i \neq M_i$ for each $i = 1, 2, 3$.

If on the contrary $0 \neq \mathfrak{M}_1$, then $0 \neq \mathfrak{M}_1M_1$, because M_1 is faithful. Now apply 5.1, for $I^* = \mathfrak{M}_1$, $K^* = \mathfrak{M}_1M_1$, and $M^* = M_1$ to see that there exist $x_1 \in (M_1 \setminus \mathfrak{M}_1M_1)$ and $a_1 \in \mathfrak{M}_1$ with $a_1x_1 \neq 0$.

For $N = \mathfrak{M}_1M_1 \oplus 0 \oplus 0$ and $0 \neq x_2 \in M_2$, $(0, 0, 0) \neq (a_1, 1, 1)(1, 0, 1)(x_1, x_2, 0) \in N$, and N is a weakly 2-absorbing submodule of M and $(a_1, 1, 1)(x_1, x_2, 0) = (a_1x_1, x_2, 0) \notin N$, $(1, 0, 1)(x_1, x_2, 0) = (x_1, 0, 0) \notin N$, and so $(a_1, 0, 1) = (a_1, 1, 1)(1, 0, 1) \in (N : M)$. Hence $M_3 = (0, 0, 1)M = (a_1, 0, 1)(0, 0, 1)M \subseteq N$, and this implies that $M_3 = 0$, which is impossible. Therefore, $0 = \mathfrak{M}_1$, that is R_1 is a field. Similarly R_2 and R_3 are fields.

Now we prove that $M \cong R$. If $M_1 \ncong R_1$, then since M_1 is a nonzero vector space over the field R_1, there exists a nontrivial submodule (subspace) K_1 of M_1. Consider $(0, 0, 0) \neq (1, 0, 1)(1, 1, 0)(x_1, x_2, x_3) \in K_1 \oplus 0 \oplus 0 = K$, where $0 \neq x_1 \in K_1$ and $0 \neq x_2 \in M_2$ and $0 \neq x_3 \in M_3$.

Note that $(1, 0, 1)(x_1, x_2, x_3) = (x_1, 0, x_3) \notin K$ and $(1, 1, 0)(x_1, x_2, x_3) = (x_1, x_2, 0) \notin K$, and $(1, 0, 1)(1, 1, 0) = (1, 0, 0) \notin (K : M)$. Thus the proper submodule K is not a weakly 2-absorbing submodule of M, which is a contradiction. Therefore, $M_1 \cong R_1$ and similarly $M_2 \cong R_2$ and $M_3 \cong R_3$. Thus $M \cong R$. □

Theorem 6.5 *There exists a nonzero faithful R-module M such that every proper submodule of M is weakly 2-absorbing if and only if one of the following statements holds:*

(i) *$(R, \mathfrak{M})$ is a quasi-local ring with $\mathfrak{M}^3 = 0$.*

(ii) *$R \cong R_1 \oplus R_2$, where $(R_1, \mathfrak{M})$ is a quasi-local ring with $\mathfrak{M}^2 = 0$ and R_2 is a field.*

(iii) *$R \cong R_1 \oplus R_2 \oplus R_3$, where R_1, R_2, R_3 are fields.*

Proof First suppose that there exists a nonzero faithful R-module M such that every proper submodule of M is weakly 2-absorbing. By 6.2, $(J(R))^3 = 0$.

By 6.1, R has at most three maximal ideals. We consider the following three cases.

Case 1. The ring R has only one maximal ideal, say $\mathfrak{M}$. Then in this case $\mathfrak{M}^3 = ((J(R))^3) = 0$.

Case 2. The ring R has two maximal ideals $\mathfrak{M}_1, \mathfrak{M}_2$. Note that $\mathfrak{M}_1^3 \cap \mathfrak{M}_2^3 = (J(R))^3 = 0$. Therefore, $R \cong \frac{R}{\mathfrak{M}_1^3} \oplus \frac{R}{\mathfrak{M}_2^3}$ and clearly $(R_1, \overline{\mathfrak{M}_1})$ and $(R_2, \overline{\mathfrak{M}_2})$ are quasi-local rings, where $R_1 = \frac{R}{\mathfrak{M}_1^3}$, $R_2 = \frac{R}{\mathfrak{M}_2^3}$, $\overline{\mathfrak{M}_1} = \frac{\mathfrak{M}_1}{\mathfrak{M}_1^3}$, $\overline{\mathfrak{M}_2} = \frac{\mathfrak{M}_2}{\mathfrak{M}_2^3}$. By 6.3((i) $\Longrightarrow$ (ii)), $\overline{\mathfrak{M}_1}^2 = 0$ and $\overline{\mathfrak{M}_2}^2 = 0$ and R_1 or R_2 is a field.

Case 3. The ring R has three maximal ideals $\mathfrak{M}_1, \mathfrak{M}_2, \mathfrak{M}_3$. Again since $(J(R))^3 = \mathfrak{M}_1^3 \cap \mathfrak{M}_2^3 \cap \mathfrak{M}_2^3 = 0$, clearly $R \cong \frac{R}{\mathfrak{M}_1^3} \oplus \frac{R}{\mathfrak{M}_2^3} \oplus \frac{R}{\mathfrak{M}_3^3}$. Therefore, by 6.4, $\frac{R}{\mathfrak{M}_1^3}, \frac{R}{\mathfrak{M}_2^3}, \frac{R}{\mathfrak{M}_3^3}$ are fields.

For proving the converse of this theorem, put $M = R$, and apply [5, Theorem 3.7]. □

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References

[1] Anderson DD, Smith E. Weakly prime ideals. Houston J Math 2003; 29: 831–840.

[2] Azizi A. Prime submodules and flat modules. Acta Math Sinica 2007; 23: 147–152.

[3] Azizi A. Weak multiplication modules. Czech Math J 2003; 53: 529–534.

[4] Badawi A. On 2-absorbing ideals of commutative rings. Bull Austral Math Soc 2007; 75: 417–429.

[5] Badawi A, Yousefian Darani A. On weakly 2-absorbing ideals of commutative rings. Houston J Math 2013; 39: 441–452.

[6] Behboodi M, Koohy H. Weakly prime submodules. Vietnam J Math 2004; 32: 185–195.

[7] Lu CP. Spectra of modules. Comm Algebra 1995; 23: 3741–3752.

[8] McCasland RL, Moore ME. Prime submodules. Comm Algebra 1992; 20: 1803–1817.

[9] Moradi S, Azizi A. 2-absorbing and n-weakly prime submodules. Miskolc Math Notes 2012; 13: 75--86.

[10] Yousefian Darani A, Soheilnia F. 2-absorbing and weakly 2-absorbing submodules. Thai J Math 2011; 9: 577–584.