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$\mathcal{VW}$-Gorenstein categories

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Abstract: Let $\mathcal{A}$ be an abelian category, and $\mathcal{V},\mathcal{W}$ two additive full subcategories of $\mathcal{A}$. We introduce and study the $\mathcal{VW}$-Gorenstein subcategory of $\mathcal{A}$, which unifies many known notions, such as the Gorenstein category and the category consisting of G_C-projective (injective) modules, although they were defined in a different way. We also prove that the Bass class with respect to a semidualizing module is one kind of $\mathcal{VW}$-Gorenstein category. The connections between $\mathcal{VW}$-Gorenstein categories and Gorenstein categories are discussed. Some applications are given.

Key words: ($\mathcal{V}\mathcal{W}$-)Gorenstein categories, semidualizing modules, Gorenstein projective modules, Gorenstein injective modules, Bass class

1. Introduction

As generalizations of projective and injective modules, Enochs and Jenda introduced in [6] the notions of Gorenstein projective and injective modules. On the other hand, relative to a semidualizing module C, Holm and Jørgensen introduced in [12] the notions of C-Gorenstein (G_C- for short) projective and injective modules. In the special case that $C = R$, these recover the notions of Gorenstein projective and injective modules, respectively. They play an important role in relative homological algebra. Gorenstein projective and Gorenstein injective modules and some related generalized versions have been studied by many authors; see [1,3,5–17] and the literature listed in them.

Let $\mathcal{A}$ be an abelian category and $\mathcal{W}$ an additive full subcategory of $\mathcal{A}$. Sather-Wagstaff et al. introduced in [16] the Gorenstein category $\mathcal{G}(\mathcal{W})$, which unifies the following notions: the modules of G-dimension zero [1]; Gorenstein projective and Gorenstein injective modules [6]; V-Gorenstein projective and V-Gorenstein injective modules [8]. The authors proved that $\mathcal{G}(\mathcal{W})$ possesses many nice properties under the condition that $\mathcal{W}$ is self-orthogonal.

Note that the Gorenstein category $\mathcal{G}(\mathcal{W})$ does not recover the G_C-projective and G_C-injective modules. The complete resolutions used to define them are constructed by two different classes of modules. The main purpose of this paper is to define a new Gorenstein category that encompasses all of the aforementioned notions, and establish its homological properties in part by removing the restricted condition of orthogonality. This paper is organized as follows.

In Section 2, we give some notation and terminology needed in the later sections.

In Section 3, the notion of the $\mathcal{VW}$-Gorenstein category is introduced, which unifies the Gorenstein

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category, the category consisting of G_C-projective (injective) modules, and the Bass class with respect to a semidualizing module. This enables us to provide a unified approach for these notions. We then investigate the homological properties of the $\mathcal{VW}$-Gorenstein category, and show that many existing results are obtained as particular cases of them.

In Section 4, we first establish the stability of the $\mathcal{VW}$-Gorenstein category for any categories $\mathcal{V}$ and $\mathcal{W}$. As an application, more properties of the $\mathcal{VW}$-Gorenstein category are obtained. The rest of this section is devoted to discussing the connections between $\mathcal{VW}$-Gorenstein categories with other notions, in particular, with Gorenstein categories.

2. Preliminaries

Throughout this paper, $\mathcal{A}$ is an abelian category. We use the term 'subcategory' for an 'additive full subcategory' that is closed under isomorphisms, finite direct sums, and direct summands.

We fix subcategories $\mathcal{V}$, $\mathcal{W}$ and $\mathcal{X}$ of $\mathcal{A}$ such that $\mathcal{V}$, $\mathcal{W} \subseteq \mathcal{X}$. Write $\mathcal{V} \perp \mathcal{W}$ if $\mathrm{Ext}_{\mathcal{A}}^{i\geqslant 1}(V,W) = 0$ for each object V in $\mathcal{V}$ and each object W in $\mathcal{W}$. If $\mathcal{W} \perp \mathcal{W}$, $\mathcal{W}$ is called self-orthogonal. For an object A in $\mathcal{A}$, write $A \in {}^{\perp}\mathcal{X}$ ($A \in \mathcal{X}^{\perp}$) if $\mathrm{Ext}_{\mathcal{A}}^{i\geqslant 1}(A,X) = 0$ (resp. $\mathrm{Ext}_{\mathcal{A}}^{i\geqslant 1}(X,A) = 0$) for each object X in $\mathcal{X}$.

Definition 2.1 ([16]) *We say that $\mathcal{W}$ is a cogenerator for $\mathcal{X}$ if, for each object X in $\mathcal{X}$, there exists an exact sequence*

$$0 \to X \to W \to X' \to 0,$$

with $W \in \mathcal{W}$ and $X' \in \mathcal{X}$. The subcategory $\mathcal{W}$ is an injective cogenerator for $\mathcal{X}$ if $\mathcal{W}$ is a cogenerator for $\mathcal{X}$ and $\mathcal{X} \perp \mathcal{W}$.

Generator and projective generator are defined dually.

Definition 2.2 *Let A be an object in $\mathcal{A}$. An exact sequence (of finite or infinite length):*

$$\cdots \to X_1 \to X_0 \to A \to 0$$

in $\mathcal{A}$ with each $X_i \in \mathcal{X}$ is called an $\mathcal{X}$-resolution of A.

$\mathcal{X}$-coresolutions of A are defined dually.

Recall that a sequence $\mathbb{X}$ in $\mathcal{A}$ is called $\mathrm{Hom}_{\mathcal{A}}(\mathcal{X},-)$-exact if $\mathrm{Hom}_{\mathcal{A}}(X,\mathbb{X})$ is exact for each object X in $\mathcal{X}$. Dually, it is $\mathrm{Hom}_{\mathcal{A}}(-,\mathcal{X})$-exact if $\mathrm{Hom}_{\mathcal{A}}(\mathbb{X},X)$ is exact for each object X in $\mathcal{X}$.

Definition 2.3 [3] *An $\mathcal{X}$-resolution is said to be $\mathcal{X}$-proper (or simply proper) if it is $\mathrm{Hom}_{\mathcal{A}}(\mathcal{X},-)$-exact. We set*

$$res\widetilde{\mathcal{X}} = \text{the subcategory of objects of } \mathcal{A} \text{ admitting a proper } \mathcal{X}\text{-resolution.}$$

Proper coresolutions are defined dually, and we set

$$cores\widetilde{\mathcal{X}} = \text{the subcategory of objects of } \mathcal{A} \text{ admitting a proper } \mathcal{X}\text{-coresolution.}$$

Definition 2.4 [16] *The Gorenstein subcategory $\mathcal{G}(\mathcal{W})$ of $\mathcal{A}$ consists of all objects A isomorphic to $\mathrm{Im}(W_0 \to W^0)$ for some exact sequence in $\mathcal{W}$:*

$$\cdots \to W_1 \to W_0 \to W^0 \to W^1 \to \cdots$$

which is both $\mathrm{Hom}_{\mathcal{A}}(\mathcal{W},-)$-exact and $\mathrm{Hom}_{\mathcal{A}}(-,\mathcal{W})$-exact.

In this work, the rings R and S are associative with identity. We use Mod R (resp. Mod S^{op}) to denote the category of left R-modules (resp. right S-modules). We write $\mathcal{P}(R)$ (resp. $\mathcal{I}(R)$) for the subcategory of Mod R consisting of projective (resp. injective) R-modules. If $\mathcal{X}(R)$ is a subcategory of Mod R, then $\mathcal{X}^f(R)$ is the subcategory of finitely generated modules in $\mathcal{X}(R)$.

Definition 2.5 [13] *An (R,S)-bimodule $C = {}_RC_S$ is called semidualizing if it satisfies the following:*

($a1$) *${}_RC$ admits a (possibly unbounded) resolution by finitely generated projective left R-modules.*

($a2$) *C_S admits a (possibly unbounded) resolution by finitely generated projective right S-modules.*

($b1$) *The map $R \to \mathrm{Hom}_{S^{op}}(C,C)$ is an isomorphism.*

($b2$) *The map $S \to \mathrm{Hom}_R(C,C)$ is an isomorphism.*

($c1$) $\mathrm{Ext}_R^{i\geqslant 1}(C,C) = 0$.

($c2$) $\mathrm{Ext}_{S^{op}}^{i\geqslant 1}(C,C) = 0$.

Let ${}_RC_S$ be a semidualizing bimodule, and set

$\mathcal{P}_C(R)$ = the subcategory of left R-modules $C \otimes_S P$, where ${}_SP$ is projective,
$\mathcal{I}_C(S)$ =the subcategory of left S-modules $\mathrm{Hom}_R(C,I)$, where ${}_RI$ is injective.

Definition 2.6 [15] *A module $M \in$ Mod R is said to be G_C-projective if there exists an exact sequence*

$$\cdots \to P_1 \to P_0 \to C \otimes_S P^0 \to C \otimes_S P^1 \to \cdots$$

in Mod R with each P_i and P^i projective such that $M \cong \mathrm{Im}(P_0 \to C \otimes_S P^0)$ and that the sequence is $\mathrm{Hom}_R(-,\mathcal{P}_C(R))$-exact.

G_C-injective modules are defined dually.

In the case ${}_RC_S = {}_RR_R$, these are just the definitions of Gorenstein projective and Gorenstein injective modules. Denoted by $\mathcal{GP}_C(R)$ is the subcategory of G_C-projective left R-modules.

Over a noetherian ring, the next category was introduced by Avramov and Foxby [2] when C is dualizing, and by Christensen [4] for arbitrary C. In the non-noetherian setting, these definitions are from [13].

Definition 2.7 *The Bass class $\mathcal{B}_C(R)$ with respect to C is the subcategory of left R-modules N satisfying*

(1) $\mathrm{Ext}_R^{i\geqslant 1}(C,N) = 0 = \mathrm{Tor}^S_{i\geqslant 1}(C, \mathrm{Hom}_R(C,N))$ *and*

(2) *The natural evaluation map $C \otimes_S \mathrm{Hom}_R(C,N) \to N$ is an isomorphism.*

3. $\mathcal{V}\mathcal{W}$-Gorenstein categories

Definition 3.1 *An object A in $\mathcal{A}$ is said to be $\mathcal{VW}$-Gorenstein, if there exists an exact sequence*

$$\mathbb{X} = \cdots \to V_1 \to V_0 \to W^0 \to W^1 \to \cdots$$

in $\mathcal{X}$ with each V_i in $\mathcal{V}$ and W^i in $\mathcal{W}$, such that $A \cong \mathrm{Im}(V_0 \to W^0)$ and that $\mathbb{X}$ is $\mathrm{Hom}_{\mathcal{A}}(\mathcal{V},-)$-exact and $\mathrm{Hom}_{\mathcal{A}}(-,\mathcal{W})$-exact. The exact sequence $\mathbb{X}$ is called a complete $\mathcal{VW}$-resolution of A.

We will denote the category consisting of $\mathcal{VW}$-Gorenstein objects by $\mathcal{G}(\mathcal{VW})$.

Example 3.2 (1) *If $\mathcal{V} = \mathcal{P}^f(R)$ and $\mathcal{W} = \mathcal{P}_C^f(R)$, where R is a left and right Noetherian ring, then by [17, Lemma 4.3], $\mathcal{G}(\mathcal{VW})$ coincides with the category consisting of left R-modules with G_C-dimension zero defined in* [10].

(2) *If $\mathcal{V} = \mathcal{P}(R)$ and $\mathcal{W} = \mathcal{P}_C(R)$, then $\mathcal{G}(\mathcal{VW})$ is the category of G_C-projective left R-modules. If $\mathcal{V} = \mathcal{I}_C(S)$ and $\mathcal{W} = \mathcal{I}(S)$, then $\mathcal{G}(\mathcal{VW})$ is the category of G_C-injective left S-modules* [15].

(3) *Let $\mathcal{V} = \mathcal{W}$ be a subcategory of $\mathcal{A}$; then $\mathcal{G}(\mathcal{VW})$ is the Gorenstein category $\mathcal{G}(\mathcal{W})$ defined in* [16]. *In addition, if $\mathcal{W}$ is a self-orthogonal subcategory of Mod R, then it coincides with the notion defined in* [9].

The following result shows that the Bass class $\mathcal{B}_C(R)$ is another example of $\mathcal{VW}$-Gorenstein categories.

Proposition 3.3 *If $\mathcal{V} = \mathcal{P}_C(R)$ and $\mathcal{W} = \mathcal{I}(R)$, then $\mathcal{G}(\mathcal{VW})$ coincides with the subcategory $\mathcal{B}_C(R)$ of Mod R.*

Proof From [13, Theorem 6.1], it suffices to prove that a sequence $\mathbb{Y}$ of R-modules is $\mathrm{Hom}_R(\mathcal{P}_C(R), -)$-exact if and only if it is $\mathrm{Hom}_R(C, -)$-exact. Indeed, if $\mathrm{Hom}_R(C \otimes_S P, \mathbb{Y})$ is exact for any projective S-module P, in particular, for $P = S$, $\mathrm{Hom}_R(C, \mathbb{Y})$ is exact. Conversely, the assertion follows from the adjoint isomorphism

$$\mathrm{Hom}_R(C \otimes_S P, \mathbb{Y}) \cong \mathrm{Hom}_S(P, \mathrm{Hom}_R(C, \mathbb{Y}))$$

for any projective left S-module P. □

Lemma 3.4 *Let*

$$\mathbb{X} = \cdots \to V_1 \to V_0 \to W^0 \to W^1 \to \cdots$$

be a complete $\mathcal{VW}$-resolution of A. Set $A_{i-1} = \mathrm{Im}(V_i \to V_{i-1})$ and $A^{i-1} = \mathrm{Im}\,(W^{i-1} \to W^i)$ for any $i \geq 1$.

(1) *If $\mathcal{V} \perp \mathcal{W}$, then A, $A_i \in {}^{\perp}\mathcal{W}$ and A, $A^i \in \mathcal{V}^{\perp}$ for any $i \geq 0$. If further,*

(2) *$\mathcal{V} \perp \mathcal{V}$, then $A_i \in \mathcal{V}^{\perp}$ for any $i \geq 0$.*

(3) *$\mathcal{W} \perp \mathcal{W}$, then $A^i \in {}^{\perp}\mathcal{W}$ for any $i \geq 0$.*

Proof Consider the short exact sequences: $0 \to A_0 \to V_0 \to A \to 0$, and $0 \to A_{i+1} \to V_{i+1} \to A_i \to 0$ for any $i \geq 0$.

(1) Because $\mathcal{V} \perp \mathcal{W}$, for any $W \in \mathcal{W}$, by applying $\mathrm{Hom}_{\mathcal{A}}(-, W)$ to the first exact sequence, one has the following long exact sequence

$$\begin{aligned} 0 \to \mathrm{Hom}_{\mathcal{A}}(A, W) \to \mathrm{Hom}_{\mathcal{A}}(V_0, W) \to \mathrm{Hom}_{\mathcal{A}}(A_0, W) \\ \to \mathrm{Ext}^1_{\mathcal{A}}(A, W) \to 0 \to \mathrm{Ext}^1_{\mathcal{A}}(A_0, W) \\ \to \mathrm{Ext}^2_{\mathcal{A}}(A, W) \to 0 \to \cdots. \end{aligned}$$

Thus $\mathrm{Ext}^1_{\mathcal{A}}(A, W) = 0$ since $\mathrm{Hom}_{\mathcal{A}}(\mathbb{X}, \mathcal{W})$ is exact, and $\mathrm{Ext}^{j+1}_{\mathcal{A}}(A, W) \cong \mathrm{Ext}^j_{\mathcal{A}}(A_0, W)$ for any $j \geq 1$. By a similar argument to the second exact sequence, we have $\mathrm{Ext}^{j \geqslant 1}_{\mathcal{A}}(A, W) = 0$ and $\mathrm{Ext}^{j \geqslant 1}_{\mathcal{A}}(A_i, W) = 0$ for any $i \geq 0$ by dimension shifting. Therefore, A and all $A_i \in {}^{\perp}\mathcal{W}$ for any $i \geq 0$. Dually, one has A and each $A^i \in \mathcal{V}^{\perp}$ for any $i \geq 0$.

(2) Since $\mathrm{Hom}_{\mathcal{A}}(\mathcal{V},\mathbb{X})$ is exact and $A \in \mathcal{V}^{\perp}$ by (1), the assumption $\mathcal{V}\perp\mathcal{V}$ implies that $A_i \in \mathcal{V}^{\perp}$ for any $i \geq 0$.

(3) By a similar argument to (2). □

From the definition of $\mathcal{G}(\mathcal{VW})$, it is clear that $\mathcal{G}(\mathcal{VW}) \subseteq \mathrm{res}\widetilde{\mathcal{V}} \cap \mathrm{cores}\widetilde{\mathcal{W}}$.

Proposition 3.5 *If $\mathcal{V} \perp \mathcal{W}$, then $\mathcal{G}(\mathcal{VW}) = \mathcal{V}^{\perp} \cap {}^{\perp}\mathcal{W} \cap res\widetilde{\mathcal{V}} \cap cores\widetilde{\mathcal{W}}$.*

Proof Assume that $A \in \mathcal{G}(\mathcal{VW})$. From Lemma 3.4 (1), we get that $A \in \mathcal{V}^{\perp} \cap {}^{\perp}\mathcal{W}$, and hence $\mathcal{G}(\mathcal{VW}) \subseteq \mathcal{V}^{\perp} \cap {}^{\perp}\mathcal{W} \cap \mathrm{res}\widetilde{\mathcal{V}} \cap \mathrm{cores}\widetilde{\mathcal{W}}$. The reverse containment is obvious. □

The next result investigates when the kernels and cokernels in a complete $\mathcal{VW}$-resolution are also in $\mathcal{G}(\mathcal{VW})$.

Proposition 3.6 (1) *If $\mathcal{V}$ is a projective generator for $\mathcal{X}$ and $\mathcal{W} \perp \mathcal{W}$, then every cokernel in the right part of a complete $\mathcal{VW}$-resolution is in $\mathcal{G}(\mathcal{VW})$, and hence $\mathcal{W}$ is an injective cogenerator for $\mathcal{G}(\mathcal{VW})$.*

(2) *If $\mathcal{W}$ is an injective cogenerator for $\mathcal{X}$ and $\mathcal{V} \perp \mathcal{V}$, then every kernel in the left part of a complete $\mathcal{VW}$-resolution is in $\mathcal{G}(\mathcal{VW})$, and hence $\mathcal{V}$ is a projective generator for $\mathcal{G}(\mathcal{VW})$.*

Proof We prove only part (1); the proof of part (2) is dual. Let $\mathbb{X}$ be a complete $\mathcal{VW}$-resolution of A and write $A^0 = \mathrm{Coker}(V_0 \to W^0)$, $A^i = \mathrm{Coker}(W^{i-1} \to W^i)$ for any $i \geq 1$. Observe first that each $A^i \in \mathrm{cores}\widetilde{\mathcal{W}} \cap \mathcal{V}^{\perp}$ by Lemma 3.4 (1) for any $i \geq 0$. Consider the short exact sequence $0 \to A \to W^0 \to A^0 \to 0$. Because $\mathcal{V}$ is a projective generator for $\mathcal{X}$ and $W^0 \in \mathcal{X}$, $W^0 \in \mathrm{res}\widetilde{\mathcal{V}}$. From [14, Theorem 3.6 (5)], $A^0 \in \mathrm{res}\widetilde{\mathcal{V}}$. Since $\mathcal{W} \perp \mathcal{W}$, $A^0 \in {}^{\perp}\mathcal{W}$ by Lemma 3.4 (3). Thus $A^0 \in \mathcal{G}(\mathcal{VW})$ by Proposition 3.5. Using a similar argument, we get that each A^i is also in $\mathcal{G}(\mathcal{VW})$ for any $i \geq 1$. □

In the rest of this section, we will study two out of the three properties of $\mathcal{VW}$-Gorenstein categories within a short exact sequence.

Theorem 3.7 *Let $0 \to A' \to A \to A'' \to 0$ be an exact sequence in $\mathcal{A}$ with A' and A'' in $\mathcal{G}(\mathcal{VW})$. If it is $\mathrm{Hom}_{\mathcal{A}}(\mathcal{V},-)$-exact and $\mathrm{Hom}_{\mathcal{A}}(-,\mathcal{W})$-exact, then A is in $\mathcal{G}(\mathcal{VW})$.*

Proof Let $\mathbb{X}'$ and $\mathbb{X}''$ be complete $\mathcal{VW}$-resolutions for A' and A'', respectively. One has a degreewise split exact sequence of complexes $0 \to \mathbb{X}' \to \mathbb{X} \to \mathbb{X}'' \to 0$ from [16, Lemma 1.9]. Since the complexes $\mathbb{X}'$ and $\mathbb{X}''$ are both $\mathrm{Hom}_{\mathcal{A}}(\mathcal{V},-)$-exact and $\mathrm{Hom}_{\mathcal{A}}(-,\mathcal{W})$-exact, so is $\mathbb{X}$. Therefore, $\mathbb{X}$ is a complete $\mathcal{VW}$-resolution of A, and so A is in $\mathcal{G}(\mathcal{VW})$. □

Corollary 3.8 *If $\mathcal{V} \perp \mathcal{W}$, then $\mathcal{G}(\mathcal{VW})$ is closed under extensions.*

Proof Let $0 \to A' \to A \to A'' \to 0$ be an exact sequence in $\mathcal{A}$ with A' and A'' in $\mathcal{G}(\mathcal{VW})$. Since $\mathcal{V} \perp \mathcal{W}$, Proposition 3.5 implies that the exact sequence above is $\mathrm{Hom}_{\mathcal{A}}(\mathcal{V},-)$-exact and $\mathrm{Hom}_{\mathcal{A}}(-,\mathcal{W})$-exact. The conclusion follows from Theorem 3.7. □

By Proposition 3.6 and Corollary 3.8, we get the following result, which may be of independent interest.

Proposition 3.9 (1) *Suppose that $\mathcal{V}$ is a projective generator for $\mathcal{X}$ and $\mathcal{W} \perp \mathcal{W}$. If $A \in \mathcal{A}$ has a $\mathcal{G}(\mathcal{VW})$-coresolution that is $\mathrm{Hom}_{\mathcal{A}}(-,\mathcal{W})$-exact, then A has a proper $\mathcal{W}$-coresolution.*

(2) *Suppose that $\mathcal{W}$ is an injective cogenerator for $\mathcal{X}$ and $\mathcal{V} \perp \mathcal{V}$. If an object B in $\mathcal{A}$ has a $\mathcal{G}(\mathcal{VW})$-resolution that is* $\mathrm{Hom}_{\mathcal{A}}(\mathcal{V}, -)$*-exact, then B has a proper $\mathcal{V}$-resolution.*

Proof (1) Let

$$0 \to A \to G^0 \to G^1 \to \cdots$$

be a $\mathcal{G}(\mathcal{VW})$-coresolution of A, and $A^i = \mathrm{Im}(G^i \to G^{i+1})$ for any $i \geq 0$. Consider the exact sequence $0 \to A \to G^0 \to A^0 \to 0$. Because $\mathcal{W}$ is an injective cogenerator for $\mathcal{G}(\mathcal{VW})$ by Proposition 3.6 (1), and $\mathcal{G}(\mathcal{VW})$ is closed under extensions by Corollary 3.8, there exist exact sequences $0 \to A \to W^0 \to B^0 \to 0$ and $0 \to A^0 \to B^0 \to G \to 0$ with W^0 in $\mathcal{W}$ and G in $\mathcal{G}(\mathcal{VW})$ by [14, Theorem 5.3 (1)] for the case $n = 1$. Since both A^0 and G are in ${}^{\perp}\mathcal{W}$, the exactness of the second sequence above yields $B^0 \in {}^{\perp}\mathcal{W}$, and hence the first one is $\mathrm{Hom}_{\mathcal{A}}(-, \mathcal{W})$-exact. Consider the following pushout diagram:

$$\begin{array}{ccccccccc} & & 0 & & 0 & & & & \\ & & \downarrow & & \downarrow & & & & \\ 0 & \to & A^0 & \to & B^0 & \to & G & \to & 0 \\ & & \downarrow & & \downarrow & & \| & & \\ 0 & \to & G^1 & \to & G' & \to & G & \to & 0 \\ & & \downarrow & & \downarrow & & & & \\ & & A^1 & = & A^1 & & & & \\ & & \downarrow & & \downarrow & & & & \\ & & 0 & & 0 & & & & \end{array}$$

Because both G^1 and G are in $\mathcal{G}(\mathcal{VW})$, so is G' by Corollary 3.8 again. Since $A^i \in {}^{\perp}\mathcal{W}$ for any $i \geq 1$ by Lemma 3.4 (3), one gets a $\mathcal{G}(\mathcal{VW})$-coresolution of B^0

$$0 \to B^0 \to G' \to G^2 \to G^3 \to \cdots,$$

which is also $\mathrm{Hom}_{\mathcal{A}}(-, \mathcal{W})$-exact. Repeating the proceeding process, we obtain the exact sequence

$$0 \to A \to W^0 \to W^1 \to \cdots$$

in $\mathcal{A}$, with each W^i in $\mathcal{W}$ and $\mathrm{Hom}_{\mathcal{A}}(-, \mathcal{W})$ leaves it exact, as desired.

(2) The proof is dual to that of (1). □

Theorem 3.10 *Suppose*

$$0 \to A' \to A \to A'' \to 0 \qquad (*)$$

is an exact sequence in $\mathcal{A}$, and it is $\mathrm{Hom}_{\mathcal{A}}(\mathcal{V}, -)$*-exact and* $\mathrm{Hom}_{\mathcal{A}}(-, \mathcal{W})$*-exact.*

(1) *If A', A are objects in $\mathcal{G}(\mathcal{VW})$, then so is A''.*

(2) *If A, A'' are objects in $\mathcal{G}(\mathcal{VW})$, then so is A'.*

Proof (1) Let

$$\cdots \to V_1' \to V_0' \to (W^0)' \to (W^1)' \to \cdots$$

and

$$\cdots \to V_1 \to V_0 \to W^0 \to W^1 \to \cdots$$

be complete $\mathcal{VW}$-resolutions for A' and A, respectively. Since the exact sequence $(*)$ is $\mathrm{Hom}_{\mathcal{A}}(\mathcal{V},-)$-exact and $\mathrm{Hom}_{\mathcal{A}}(-,\mathcal{W})$-exact, [14, Theorem 3.6 (5)] gives rise to a proper $\mathcal{V}$-resolution of A'', which is also $\mathrm{Hom}_{\mathcal{A}}(-,\mathcal{W})$-exact by [14, Theorem 3.6 (2)]. On the other hand, from [14, Theorem 3.4 (1)], there exist two exact sequences:

$$0 \to A'' \to W \to W^1 \oplus (W^2)' \to \cdots \to W^i \oplus (W^{i+1})' \to \cdots$$

and

$$0 \to (W^0)' \to W^0 \oplus (W^1)' \to W \to 0 \qquad (**)$$

with the first exact sequence both $\mathrm{Hom}_{\mathcal{A}}(\mathcal{V},-)$-exact by [14, Theorem 3.4 (2)] and $\mathrm{Hom}_{\mathcal{A}}(-,\mathcal{W})$-exact by [14, Theorem 3.4 (3)]. We claim that W is an object in $\mathcal{W}$. In fact, since the exact sequence $(*)$ is $\mathrm{Hom}_{\mathcal{A}}(-,\mathcal{W})$-exact, so is the exact sequence $(**)$ by [14, Theorem 3.4 (4)]. Thus the sequence $0 \to \mathrm{Hom}_{\mathcal{A}}(W,(W^0)') \to \mathrm{Hom}_{\mathcal{A}}(W^0 \oplus (W^1)',(W^0)') \to \mathrm{Hom}_{\mathcal{A}}((W^0)',(W^0)') \to 0$ is exact, which implies that the sequence $(**)$ is split. Hence, $W \in \mathcal{W}$ since $\mathcal{W}$ is closed under direct summands. Therefore, A'' is in $\mathcal{G}(\mathcal{VW})$.

(2) The proof is similar to that of (1). □

The following result is an immediate consequence of Theorem 3.10.

Corollary 3.11 *Let $0 \to A' \to A \to A'' \to 0$ be an exact sequence in $\mathcal{A}$ with A in $\mathcal{G}(\mathcal{VW})$ and $\mathcal{V} \perp \mathcal{W}$.*

(1) *If $\mathrm{Ext}^1_{\mathcal{A}}(A'',\mathcal{W}) = 0$ and $A' \in \mathcal{G}(\mathcal{VW})$, then so is A''.*

(2) *If $\mathrm{Ext}^1_{\mathcal{A}}(\mathcal{V},A') = 0$ and $A'' \in \mathcal{G}(\mathcal{VW})$, then so is A'.*

Recall that a class of modules is called resolving (coresolving) if it is closed under extensions and kernels of surjections (cokernels of injections), and it contains all projective (injective) modules. From Corollary 3.8 and 3.11, it is not hard to get the following result.

Corollary 3.12 (1) ([17, Theorem 2.8]) *The class of G_C-projective (resp. injective) modules is resolving (resp. coresolving).*

(2) ([13, Theorem 6.2]) *The Bass class $\mathcal{B}_C(R)$ is coresolving.*

4. The connections with Gorenstein categories

In this section, we mainly discuss the relations between $\mathcal{VW}$-Gorenstein categories and Gorenstein categories. First of all, we investigate the stability of the $\mathcal{VW}$-Gorenstein category under the procedure used to define these entities, which recovers the one defined in [16].

We denote $\mathcal{G}^2(\mathcal{VW}) = \{A \in \mathcal{A} \mid$ there exists an exact sequence $\cdots \to G_1 \to G_0 \to G^0 \to G^1 \to \cdots$ in $\mathcal{A}$ with each G_i and G^i in $\mathcal{G}(\mathcal{VW})$, such that $A \cong \mathrm{Im}(G_0 \to G^0)$ and $\mathrm{Hom}_{\mathcal{A}}(\mathcal{V},-)$, $\mathrm{Hom}_{\mathcal{A}}(-,\mathcal{W})$ leave it exact.$\}$.

Lemma 4.1 *Let*

$$0 \to Y \to Y^0 \to Y^1 \to \cdots \to Y^n \qquad (4.1)$$

and

$$0 \to Y^j \to W^j_0 \to W^j_1 \to \cdots \to W^j_{n-j} \qquad (4.2(j))$$

be exact sequences in $\mathcal{A}$ for any $0 \le j \le n$.

If the exact sequence (4.1) *is* $\mathrm{Hom}_{\mathcal{A}}(-,\mathcal{W})$*-exact and* (4.2($j$)) *is a proper* $\mathcal{W}$*-coresolution of* Y^j *for any* $0 \le j \le n$*. Then*

$$0 \to Y \to W_0^0 \to W_1^0 \bigoplus W_0^1 \to \cdots \to \bigoplus_{i=0}^{n-1} W_{(n-1)-i}^i \to \bigoplus_{i=0}^{n} W_{n-i}^i \tag{4.3}$$

is a proper $\mathcal{W}$*-coresolution of* Y*; furthermore, if* (4.1) *and all* (4.2(j)) *are* $\mathrm{Hom}_{\mathcal{A}}(\mathcal{V},-)$*-exact, then so is* (4.3)*.*

Proof From [14, Theorem 3.8 (2) and (5)], using induction on n, it is not difficult to get the assertion. □

Theorem 4.2 $\mathcal{G}^2(\mathcal{VW}) = \mathcal{G}(\mathcal{VW})$*.*

Proof It is easy to see that $\mathcal{G}(\mathcal{VW}) \subseteq \mathcal{G}^2(\mathcal{VW})$ from the definition of $\mathcal{G}^2(\mathcal{VW})$.

Conversely, let A be an object in $\mathcal{G}^2(\mathcal{VW})$ and

$$\cdots \to G_1 \to G_0 \to G^0 \to G^1 \to \cdots$$

a complete resolution of A with $A \cong \mathrm{Im}(G_0 \to G^0)$. Then for any $j \ge 0$, by definition, there exist exact sequences:

$$\cdots \to V_j^i \to \cdots \to V_j^1 \to V_j^0 \to G_j \to 0$$

and

$$0 \to G^j \to W_0^j \to W_1^j \to \cdots \to W_i^j \to \cdots$$

in $\mathcal{A}$ with each V_j^i in $\mathcal{V}$ and W_i^j in $\mathcal{W}$, which are both $\mathrm{Hom}_{\mathcal{A}}(\mathcal{V},-)$-exact and $\mathrm{Hom}_{\mathcal{A}}(-,\mathcal{W})$-exact. By Lemma 4.1 and its dual version, we get exact sequences:

$$\cdots \to \bigoplus_{i=0}^{n} V_i^{n-i} \to \cdots \to V_0^1 \bigoplus V_1^0 \to V_0^0 \to A \to 0$$

and

$$0 \to A \to W_0^0 \to W_1^0 \bigoplus W_0^1 \to \cdots \to \bigoplus_{i=0}^{n} W_{n-i}^i \to \cdots,$$

which are both $\mathrm{Hom}_{\mathcal{A}}(\mathcal{V},-)$-exact and $\mathrm{Hom}_{\mathcal{A}}(-,\mathcal{W})$-exact. Gluing these two exact sequences, one has a complete $\mathcal{VW}$-resolution of A with $A \cong \mathrm{Im}(V_0^0 \to W_0^0)$, and hence A is in $\mathcal{G}(\mathcal{VW})$. Thus $\mathcal{G}^2(\mathcal{VW}) = \mathcal{G}(\mathcal{VW})$ as desired. □

As an immediate consequence of Theorem 4.2, we get the following two corollaries. The first one is [15, Theorem 2.9]. However, the methods of proof differ.

Corollary 4.3 $\mathcal{G}^2\mathcal{P}_C(R) = \mathcal{G}\mathcal{P}_C(R)$*.*

Proof Set $\mathcal{V} = \mathcal{P}(R)$ and $\mathcal{W} = \mathcal{P}_C(R)$ in Theorem 4.2; then $\mathcal{G}(\mathcal{VW})$ is the category $\mathcal{G}\mathcal{P}_C(R)$. The assertion follows from Theorem 4.2. □

The second corollary generalizes [14, Theorem 4.1].

Corollary 4.4 $\mathcal{G}^2(\mathcal{W}) = \mathcal{G}(\mathcal{W})$*.*

Proof By setting $\mathcal{V}=\mathcal{W}$ in Theorem 4.2, we obtain the assertion. □

As another application of Theorem 4.2, we have the following result showing that the $\mathcal{VW}$-Gorenstein category possesses a kind of symmetry similar to the Gorenstein category.

Proposition 4.5 *If $\mathcal{V},\mathcal{W}\subseteq\mathcal{G}(\mathcal{VW})$, then an object $A\in\mathcal{A}$ is in $\mathcal{G}(\mathcal{VW})$ if and only if there is an exact sequence*

$$\cdots\to U_1\to U_0\to U^0\to U^1\to\cdots$$

in $\mathcal{A}$ with each U_i and $U^i\in\mathcal{U}=\mathcal{V}\cup\mathcal{W}$, such that $A\cong\operatorname{Im}(U_0\to U^0)$ and $\operatorname{Hom}_{\mathcal{A}}(\mathcal{V},-)$, $\operatorname{Hom}_{\mathcal{A}}(-,\mathcal{W})$ leave it exact. In this case, $\mathcal{V}\cup\mathcal{W}$ is a generator–cogenerator for $\mathcal{G}(\mathcal{VW})$.

Proof The necessity is obvious by the definition of $\mathcal{G}(\mathcal{VW})$.

Assume that there is an exact sequence $\cdots\to U_1\to U_0\to U^0\to U^1\to\cdots$ in $\mathcal{A}$ with each U_i and $U^i\in\mathcal{U}=\mathcal{V}\cup\mathcal{W}$; thus each U_i and U^i are in $\mathcal{G}(\mathcal{VW})$ by the hypothesis. This implies that $A\in\mathcal{G}(\mathcal{VW})$ from Theorem 4.2. □

The next result is an immediate consequence of Proposition 4.5, which generalizes both [9, Remark 2.3 (1)] and [17, Proposition 2.9]. The reader is invited to compare the following result to Proposition 3.6.

Corollary 4.6 *If $\mathcal{V},\mathcal{W}\subseteq\mathcal{G}(\mathcal{VW})$, then every kernel and cokernel in a complete $\mathcal{VW}$-resolution are in $\mathcal{G}(\mathcal{VW})$.*

Based on the results above, we obtain the first relation between $\mathcal{VW}$-Gorenstein categories and Gorenstein categories.

Theorem 4.7 *If $\mathcal{V},\mathcal{W}\subseteq\mathcal{G}(\mathcal{VW})$, then $\mathcal{G}(\mathcal{V}\cup\mathcal{W})\subseteq\mathcal{G}(\mathcal{VW})$.*

Proof Suppose that $A\in\mathcal{G}(\mathcal{V}\cup\mathcal{W})$; there exists a complete resolution of A

$$\cdots\to U_1\to U_0\to U^0\to U^1\to\cdots$$

in $\mathcal{A}$, with each U_i and $U^i\in\mathcal{V}\cup\mathcal{W}$. Since the sequence above is $\operatorname{Hom}_{\mathcal{A}}(\mathcal{V}\cup\mathcal{W},-)$-exact and $\operatorname{Hom}_{\mathcal{A}}(-,\mathcal{V}\cup\mathcal{W})$-exact, in particular, it is $\operatorname{Hom}_{\mathcal{A}}(\mathcal{V},-)$-exact and $\operatorname{Hom}_{\mathcal{A}}(-,\mathcal{W})$-exact, Proposition 4.5 yields that $A\in\mathcal{G}(\mathcal{VW})$. □

As another application of Lemma 4.1, we get the following homological property of $\mathcal{VW}$-Gorenstein categories for any $\mathcal{V}$ and $\mathcal{W}$, which generalizes [14, Theorem 4.6].

Proposition 4.8 *$\mathcal{G}(\mathcal{VW})$ is closed under direct summands.*

Proof Assume that $A=B\oplus C$, and A is an object in $\mathcal{G}(\mathcal{VW})$. Splicing together the split short exact sequences $0\to C\xrightarrow{\binom{0}{id}}A\xrightarrow{(id,0)}B\to 0$ and $0\to B\xrightarrow{\binom{0}{id}}A\xrightarrow{(id,0)}C\to 0$, one obtains the following exact sequence

$$0\to B\to A\to A\to\cdots.$$

Since $A\in\mathcal{G}(\mathcal{VW})$, B has a proper $\mathcal{W}$-coresolution that is also $\operatorname{Hom}_{\mathcal{A}}(\mathcal{V},-)$-exact by Lemma 4.1. Dually, from the exact sequence $\cdots\to A\to A\to B\to 0$, one has a proper $\mathcal{V}$-resolution of B that is also $\operatorname{Hom}_{\mathcal{A}}(-,\mathcal{W})$-exact. Combining these two exact sequences, we conclude that B is in $\mathcal{G}(\mathcal{VW})$. □

The next relation extends [16, Theorem 4.9] to any category $\mathcal{X}$, and gives a sufficient condition when $\mathcal{V},\mathcal{W}\subseteq\mathcal{G}(\mathcal{VW})$.

Theorem 4.9 *If $\mathcal{V}$ is a projective generator and $\mathcal{W}$ an injective cogenerator for $\mathcal{X}$, respectively, then $\mathcal{G}(\mathcal{X}) \subseteq \mathcal{G}(\mathcal{V}\mathcal{W})$. In this case, we have that $\mathcal{V}, \mathcal{W} \subseteq \mathcal{G}(\mathcal{V}\mathcal{W})$.*

Proof Let A be an object in $\mathcal{G}(\mathcal{X})$ and

$$\cdots \to X_1 \to X_0 \to X^0 \to X^1 \to \cdots \tag{4.4}$$

a complete $\mathcal{X}$-resolution of A with $A \cong \mathrm{Im}(X_0 \to X^0)$. Since $\mathcal{W}$ is an injective cogenerator for $\mathcal{X}$, it is clear that $\mathcal{X} \subseteq \mathrm{cores}\widetilde{\mathcal{W}}$, and hence each X^i has a proper $\mathcal{W}$-coresolution

$$0 \to X^i \to W_0^i \to W_1^i \to \cdots$$

with all images in $\mathcal{X}$. Therefore, it is also $\mathrm{Hom}_{\mathcal{A}}(\mathcal{V}, -)$-exact since $\mathcal{V} \perp \mathcal{X}$. Because (4.4) is $\mathrm{Hom}_{\mathcal{A}}(\mathcal{X}, -)$-exact and $\mathrm{Hom}_{\mathcal{A}}(-, \mathcal{X})$-exact, in particular, it is $\mathrm{Hom}_{\mathcal{A}}(\mathcal{V}, -)$-exact and $\mathrm{Hom}_{\mathcal{A}}(-, \mathcal{W})$-exact; it follows from Lemma 4.1 that A has a proper $\mathcal{W}$-coresolution, which is $\mathrm{Hom}_{\mathcal{A}}(\mathcal{V}, -)$-exact. By a dual argument, one has a proper $\mathcal{V}$-resolution of A, which is $\mathrm{Hom}_{\mathcal{A}}(-, \mathcal{W})$-exact. Therefore, $A \in \mathcal{G}(\mathcal{V}\mathcal{W})$.

Since $\mathcal{X} \subseteq \mathcal{G}(\mathcal{X})$ from [16, Remark 4.2], we obtain that $\mathcal{V}, \mathcal{W} \subseteq \mathcal{X} \subseteq \mathcal{G}(\mathcal{V}\mathcal{W})$. □

The $\mathcal{V}\mathcal{W}$-Gorenstein category has nice properties under the condition $\mathcal{V} \perp \mathcal{W}$. In this case, we have the following result.

Proposition 4.10 *Assume that $\mathcal{V}'$ is another subcategory of $\mathcal{X}$ with $\mathcal{V} \perp \mathcal{V}'$ and $\mathcal{W} \perp \mathcal{V}'$. If $\mathcal{V} \perp \mathcal{W}$ and $\mathcal{W} \perp \mathcal{W}$, then $\mathcal{G}(\mathcal{W}) \cap \mathcal{G}(\mathcal{V}\mathcal{V}') = \mathcal{G}(\mathcal{V}\mathcal{W}) \cap \mathcal{G}(\mathcal{W}\mathcal{V}')$.*

Proof The assumption of orthogonality of subcategories implies that the conclusion holds true by Proposition 3.5. □

By Proposition 4.10, it is easy to have the following relation.

Corollary 4.11 *Suppose that $\mathcal{V}$ is a projective generator and $\mathcal{V}'$ is an injective cogenerator for $\mathcal{A}$, respectively. If $\mathcal{W} \perp \mathcal{W}$, then $\mathcal{G}(\mathcal{W}) = \mathcal{G}(\mathcal{V}\mathcal{W}) \cap \mathcal{G}(\mathcal{W}\mathcal{V}')$.*

Corollary 4.12 ([16, Proposition 5.2]) $\mathcal{G}(\mathcal{P}_C(R)) = \mathcal{G}\mathcal{P}_C(R) \cap \mathcal{B}_C(R)$.

Proof In Corollary 4.11, set $\mathcal{V} = \mathcal{P}(R)$, $\mathcal{V}' = \mathcal{I}(R)$, and $\mathcal{W} = \mathcal{P}_C(R)$. Since $\mathcal{P}_C(R) \perp \mathcal{P}_C(R)$, the assertion follows from Proposition 3.3 and Corollary 4.11. □

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