

## Li–Yorke chaos for invertible mappings on noncompact spaces

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**Abstract:** In this paper, we give two examples to show that an invertible mapping being Li–Yorke chaotic does not imply its inverse being Li–Yorke chaotic, in which one is an invertible bounded linear operator on an infinite dimensional Hilbert space and the other is a homeomorphism on the unit open disk. Moreover, we use the last example to prove that Li–Yorke chaos is not preserved under topological conjugacy.

**Key words:** Invertible dynamical systems, Li–Yorke chaos, noncompact spaces, topological conjugacy

### 1. Introduction

In this paper, we are interested in the invertible dynamical system  $(X, f)$ , where  $X$  is a metric space and  $f : X \rightarrow X$  is a homeomorphism. There is a natural problem in invertible dynamical systems as follows.

**Question 1.1** *Let  $(X, f)$  be an invertible dynamical system. If  $f$  has a dynamical property  $\mathfrak{P}$ , does its inverse  $f^{-1}$  also have property  $\mathfrak{P}$ ?*

It is not difficult to see that the answer is positive for many properties such as transitivity, mixing, and Devaney chaos. However, the conclusion for Li–Yorke chaos, which was defined by Li and Yorke in [2] in 1975, is not known.

**Definition 1.2** *Let  $(X, f)$  be a dynamical system.  $\{x, y\} \subseteq X$  is said to be a Li–Yorke chaotic pair if*

$$\limsup_{n \rightarrow +\infty} d(f^n(x), f^n(y)) > 0 \quad \text{and} \quad \liminf_{n \rightarrow +\infty} d(f^n(x), f^n(y)) = 0.$$

*Furthermore,  $f$  is called Li–Yorke chaotic if there exists an uncountable subset  $\Gamma \subseteq X$  such that each pair of two distinct points in  $\Gamma$  is a Li–Yorke chaotic pair.*

In the present article, we focus on invertible dynamical systems on noncompact metric spaces, and then we study Li–Yorke chaos for invertible mappings on noncompact spaces and give a negative answer to the above question for Li–Yorke chaos. We give two counterexamples on infinite dimensional space and finite dimensional space, respectively.

In Section 2, we will give an example to show that an invertible bounded linear operator on infinite dimensional Hilbert spaces being Li–Yorke chaotic does not imply its inverse being Li–Yorke chaotic. In fact,

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the example is distributional chaotic and was introduced in [1]. Distributional chaos that is defined in Schweizer and Smítal’s paper [5] is a kind of chaos stronger than Li–Yorke chaos. Notice that there exists an invertible bilateral forward weighted shift operator  $T$  introduced in [4], such that  $T$  is distributional chaotic but  $T^{-1}$  is not. However, the inverse of the operator  $T$  in [4] is Li–Yorke chaotic.

In Section 3, we will give a Li–Yorke chaotic homeomorphism  $f$  on the unit open disk such that its inverse  $f^{-1}$  is not Li–Yorke chaotic. Moreover, we obtain a homeomorphism  $g$  on two-dimensional Euclidean space being not Li–Yorke chaotic but topologically conjugate to the above  $f$ , which means that Li–Yorke chaos is not preserved under topological conjugacy.

**Definition 1.3** Let  $f : X \rightarrow X$  and  $g : Y \rightarrow Y$  be two continuous mappings.  $f$  is said to be topologically conjugate to  $g$  if there exists a homeomorphism  $h : X \rightarrow Y$  such that  $h \circ f = g \circ h$ . We also say that  $h$  is a topological conjugacy from  $f$  to  $g$ .

It is easy to see that many properties are preserved under topological conjugacy, such as density of periodic points, transitivity, mixing, and Devaney chaos. In [3], Lu et al. gave a homeomorphism on a discrete topology space such that it is Li–Yorke chaotic but its inverse is not.

**2. Invertible bounded linear operators on infinite dimensional Hilbert spaces**

**Theorem 2.1** There exists an invertible bounded linear operator  $T$  on an infinite dimensional Hilbert space  $\mathcal{H}$ , such that  $T$  is Li–Yorke chaotic but  $T^{-1}$  is not.

**Proof** First of all, let us review the construction of a distributional chaotic operator defined in [1]. Given any  $\epsilon > 0$ , let  $C_i$  be a sequence of positive numbers increasing to  $+\infty$ . For each  $i \in \mathbb{N}$ , set  $\epsilon_i = 4^{-i}\epsilon$ . Then we can select  $L_i$  to satisfy  $(1 + \epsilon_i)^{L_i} \geq \sqrt{2}C_i$ . Moreover, choose  $m_i$  such that  $\frac{L_i}{m_i} < \frac{1}{i}$  and put  $n_i = 2m_i$ .

Let  $\mathcal{H}$  be a separable complex Hilbert space with an orthogonal decomposition  $\mathcal{H} = \bigoplus_{i=1}^{\infty} H_i$ , where  $H_i$  is an  $n_i$ -dimensional subspace. For each  $i \in \mathbb{N}$ , define  $T_i : H_i \rightarrow H_i$  by

$$T_i = \begin{bmatrix} 1 - \epsilon_i & 2\epsilon_i & & & \\ & \ddots & \ddots & & \\ & & \ddots & & \\ & & & \ddots & 2\epsilon_i \\ & & & & 1 - \epsilon_i \end{bmatrix}_{(n_i \times n_i)} .$$

Then we obtain an invertible bounded linear operator

$$T = \bigoplus_{i=1}^{\infty} T_i : \mathcal{H} \rightarrow \mathcal{H}.$$

Following from Theorem 9 in [1],  $T$  is distributional chaotic (Li–Yorke chaotic). Now consider the inverse of  $T$ . Notice that  $T^{-1} = \bigoplus_{i=1}^{\infty} T_i^{-1}$  and

$$T_i^{-1} = \begin{bmatrix} \frac{1}{1-\epsilon_i} & * & \cdots & * \\ & \ddots & \ddots & \vdots \\ & & \ddots & * \\ & & & \frac{1}{1-\epsilon_i} \end{bmatrix}_{(n_i \times n_i)} .$$

For any  $0 \neq x = \bigoplus_{i=1}^{\infty} u_i \in \mathcal{H}$ , where  $u_i \in H_i$ , there exists some certain  $u_i \neq 0$ . Since  $T_i^{-1}$  is an upper-triangular matrix with all eigenvalues more than 1, then

$$\lim_{n \rightarrow +\infty} \|T^{-n}(x)\| \geq \lim_{n \rightarrow +\infty} \|T_i^{-n}(u_i)\| = +\infty.$$

Therefore,  $T^{-1}$  can not be Li–Yorke chaotic. □

### 3. Invertible nonlinear mappings on finite dimensional spaces

Denote  $\mathbb{R}$  by the set of all real numbers and by  $\mathbb{Q}$  the set of all rational numbers. Notice that  $(\mathbb{R}, +)$  is an abelian group and  $\mathbb{Q}$  is a subgroup of  $\mathbb{R}$ . Let us consider a simple conclusion of the quotient group  $\mathbb{R}/\mathbb{Q}$  first.

**Lemma 3.1**  $\mathbb{R}/\mathbb{Q}$  is an uncountable infinite set.

**Proof** For each  $x \in \mathbb{R}$ , denote  $[x]$  the equivalence class of  $x$  in  $\mathbb{R}/\mathbb{Q}$ . Then  $[x] = \{x + q; q \in \mathbb{Q}\}$  is a countable finite subset of  $\mathbb{R}$ . Since  $\mathbb{R}$  is uncountable,  $\mathbb{R}/\mathbb{Q}$  is an uncountable infinite set. □

**Lemma 3.2** There exists an uncountable infinite subset  $B$  of the open interval  $(0, 1)$  such that, for any distinct  $x, y \in B$ ,  $\ln \frac{x}{1-x} - \ln \frac{y}{1-y}$  is an irrational number.

**Proof** Define a homeomorphism  $\phi : (0, 1) \rightarrow \mathbb{R}$ ,

$$\phi(r) = \ln \frac{r}{1-r}, \quad \text{for any } r \in (0, 1).$$

For each element  $[x]$  in  $\mathbb{R}/\mathbb{Q}$ , select one number  $x \in [x]$  and consequently denote  $A$  the collection of such numbers. Then  $A \subseteq \mathbb{R}$  is an uncountable set by Lemma 3.1, in which distinct numbers belong to distinct equivalence classes.

Let  $B = \{r \in (0, 1); \phi(r) \in A\}$ . Then  $B$  is uncountable, and for any two distinct numbers  $x, y \in B$ ,

$$\ln \frac{x}{1-x} - \ln \frac{y}{1-y} \notin \mathbb{Q}.$$

□

**Theorem 3.3** There exist a homeomorphism  $f$  on the unit open disk  $\mathbb{D} \triangleq \{z \in \mathbb{C}; |z| < 1\}$  and a homeomorphism  $g$  on  $\mathbb{R}^2$ , such that  $f$  and  $g$  are topologically conjugate,  $f$  is Li–Yorke chaotic but  $f^{-1}$  is not, and  $g$  and  $g^{-1}$  are not Li–Yorke chaotic.

**Proof**

Define a mapping  $f : \mathbb{D} \rightarrow \mathbb{D}$  by

$$f(0) = 0 \quad \text{and} \quad f(z) = \frac{ez}{e^{|z|} - |z| + 1} e^{2\pi i \ln \frac{|z|}{1-|z|}}, \quad \text{for all } 0 \neq z \in \mathbb{D}.$$

Then  $f$  is a homeomorphism on  $\mathbb{D}$  and its inverse is

$$f^{-1}(0) = 0 \quad \text{and} \quad f^{-1}(w) = \frac{e^{-1}w}{e^{-1}|w| - |w| + 1} e^{-2\pi i \ln \frac{|w|}{1-|w|}}, \quad \text{for all } 0 \neq w \in \mathbb{D}.$$

Define a mapping  $g : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  by

$$g(0) = 0 \text{ and } g(z) = ze^{2\pi i \ln |z|}, \text{ for all } 0 \neq z \in \mathbb{R}^2.$$

Then  $g$  is a homeomorphism on  $\mathbb{R}^2$  and its inverse is

$$g^{-1}(0) = 0 \text{ and } g^{-1}(w) = e^{-1}we^{-2\pi i \ln |w|}, \text{ for all } 0 \neq w \in \mathbb{R}^2.$$

Define a mapping  $h : \mathbb{D} \rightarrow \mathbb{R}^2$  by

$$h(z) = \frac{z}{1 - |z|}, \text{ for all } z \in \mathbb{D}.$$

Then  $h$  is a homeomorphism and its inverse is

$$h^{-1}(w) = \frac{w}{1 + |w|}, \text{ for all } w \in \mathbb{R}^2.$$

One can see

$$h \circ f(0) = 0 = g \circ h(0) \text{ and } h \circ f(z) = \frac{ez}{1 - |z|} e^{2\pi i \ln \frac{|z|}{1 - |z|}} = g \circ h(z), \text{ for all } 0 \neq z \in \mathbb{D}.$$

Consequently,  $h$  is a topological conjugacy from  $f$  to  $g$ .

It is easy to see that for any  $k \in \mathbb{Z}$

$$g^k(0) = 0 \text{ and } g^k(z) = e^k z e^{2\pi i k \ln |z|}, \text{ for all } 0 \neq z \in \mathbb{R}^2.$$

Then,

$$\lim_{n \rightarrow +\infty} g^n(0) = \lim_{n \rightarrow +\infty} g^{-n}(0) = 0,$$

and for any  $0 \neq z \in \mathbb{R}^2$ ,

$$\lim_{n \rightarrow +\infty} |g^n(z)| = +\infty,$$

and

$$\lim_{n \rightarrow +\infty} |g^{-n}(z)| = 0.$$

Therefore, both  $g$  and  $g^{-1}$  are not Li-Yorke chaotic.

Notice that for any  $n \in \mathbb{N}$

$$f^{-n}(0) = 0 \text{ and } f^{-n}(z) = \frac{e^{-n}z}{e^{-n}|z| - |z| + 1} e^{2\pi i (-n) \ln \frac{|z|}{1 - |z|}}, \text{ for all } 0 \neq z \in \mathbb{D}.$$

Then

$$\lim_{n \rightarrow +\infty} |f^{-n}(z)| = 0, \text{ for all } 0 \neq z \in \mathbb{D}.$$

Therefore,  $f^{-1}$  is not Li-Yorke chaotic.

To complete this proof, it suffices to show that  $f$  is Li-Yorke chaotic now. Choose an uncountable infinite subset  $B$  of the open interval  $(0, 1)$  defined in Lemma 3.2. Given any two distinct points  $x, y \in B$ , then

$\ln \frac{x}{1-x} - \ln \frac{y}{1-y}$  is an irrational number. Consequently, there exist two sequences of strictly increasing positive integers  $\{m_k\}$  and  $\{n_k\}$  such that

$$\lim_{k \rightarrow +\infty} e^{2\pi i m_k (\ln \frac{x}{1-x} - \ln \frac{y}{1-y})} = 1 \quad \text{and} \quad \lim_{k \rightarrow +\infty} e^{2\pi i n_k (\ln \frac{x}{1-x} - \ln \frac{y}{1-y})} = -1.$$

Since for any  $n \in \mathbb{N}$ ,

$$f^n(z) = \frac{e^n z}{e^n |z| - |z| + 1} e^{2\pi i n \ln \frac{|z|}{1-|z|}} \quad \text{for all } 0 \neq z \in \mathbb{D},$$

and

$$\lim_{n \rightarrow +\infty} \frac{e^n |z|}{e^n |z| - |z| + 1} = 1 \quad \text{for all } 0 \neq z \in \mathbb{D},$$

then

$$\begin{aligned} & \liminf_{n \rightarrow +\infty} |f^n(x) - f^n(y)| \\ & \leq \lim_{k \rightarrow +\infty} |f^{m_k}(x) - f^{m_k}(y)| \\ & = \lim_{k \rightarrow +\infty} \left| \frac{e^{m_k} x}{e^{m_k} x - x + 1} e^{2\pi i m_k \ln \frac{x}{1-x}} - \frac{e^{m_k} y}{e^{m_k} y - y + 1} e^{2\pi i m_k \ln \frac{y}{1-y}} \right| \\ & \leq \lim_{k \rightarrow +\infty} \left| \frac{e^{m_k} x}{e^{m_k} x - x + 1} e^{2\pi i m_k \ln \frac{x}{1-x}} - \frac{e^{m_k} y}{e^{m_k} y - y + 1} e^{2\pi i m_k \ln \frac{x}{1-x}} \right| + \\ & \quad \lim_{k \rightarrow +\infty} \left| \frac{e^{m_k} y}{e^{m_k} y - y + 1} e^{2\pi i m_k \ln \frac{x}{1-x}} - \frac{e^{m_k} y}{e^{m_k} y - y + 1} e^{2\pi i m_k \ln \frac{y}{1-y}} \right| \\ & = \lim_{k \rightarrow +\infty} \left| \frac{e^{m_k} x}{e^{m_k} x - x + 1} - \frac{e^{m_k} y}{e^{m_k} y - y + 1} \right| |e^{2\pi i m_k \ln \frac{x}{1-x}}| + \\ & \quad \lim_{k \rightarrow +\infty} \left| \frac{e^{m_k} y}{e^{m_k} y - y + 1} \right| |e^{2\pi i m_k \ln \frac{y}{1-y}}| |e^{2\pi i m_k (\ln \frac{x}{1-x} - \ln \frac{y}{1-y})} - 1| \\ & = 0 \end{aligned}$$

and

$$\begin{aligned} & \limsup_{n \rightarrow +\infty} |f^n(x) - f^n(y)| \\ & \geq \lim_{k \rightarrow +\infty} |f^{n_k}(x) - f^{n_k}(y)| \\ & = \lim_{k \rightarrow +\infty} \left| \frac{e^{n_k} x}{e^{n_k} x - x + 1} e^{2\pi i n_k \ln \frac{x}{1-x}} - \frac{e^{n_k} y}{e^{n_k} y - y + 1} e^{2\pi i n_k \ln \frac{y}{1-y}} \right| \\ & \geq \lim_{k \rightarrow +\infty} \left| \frac{e^{n_k} y}{e^{n_k} y - y + 1} e^{2\pi i n_k \ln \frac{x}{1-x}} - \frac{e^{n_k} y}{e^{n_k} y - y + 1} e^{2\pi i n_k \ln \frac{y}{1-y}} \right| - \\ & \quad \lim_{k \rightarrow +\infty} \left| \frac{e^{n_k} x}{e^{n_k} x - x + 1} e^{2\pi i n_k \ln \frac{x}{1-x}} - \frac{e^{n_k} y}{e^{n_k} y - y + 1} e^{2\pi i n_k \ln \frac{x}{1-x}} \right| \\ & = \lim_{k \rightarrow +\infty} \left| \frac{e^{n_k} y}{e^{n_k} y - y + 1} \right| |e^{2\pi i n_k \ln \frac{y}{1-y}}| |e^{2\pi i n_k (\ln \frac{x}{1-x} - \ln \frac{y}{1-y})} - 1| - \\ & \quad \lim_{k \rightarrow +\infty} \left| \frac{e^{n_k} x}{e^{n_k} x - x + 1} - \frac{e^{n_k} y}{e^{n_k} y - y + 1} \right| |e^{2\pi i n_k \ln \frac{x}{1-x}}| = 2. \end{aligned}$$

Thus,  $\{x, y\}$  is a Li-Yorke chaotic pair and hence  $f$  is Li-Yorke chaotic. □

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