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Some results and examples on difference cordial graphs

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Abstract: In this paper we introduce some results on difference cordial graphs and describe the difference cordial labeling for some families of graphs.

Key words: Difference cordial graph, labeling, families of graphs

1. Introduction

In this paper we will deal with finite simple undirected graphs. By $G = (V, E)$ we mean a finite undirected graph with p vertices and q edges where $p = |V|$ and $q = |E|$. For standard terminology and notations we follow Harary [4], and for more details of labeling see [3].

Ponraj et al. [8] first introduced the concept of difference cordial labeling in 2013. After that, they introduced many concepts and studied some types of graphs that have this kind of labeling, such as path, cycle, complete graph, complete bipartite graph, bistar, wheel, web, sunflower graph, lotus inside a circle, pyramid, permutation graph, book with n pentagonal pages, t-$fold$ wheel, and double fan, and some more standard graphs were investigated in [6, 7, 8, 9, 10]. Within this area Seoud and Salman introduced some results and investigated some difference cordial graphs: ladder, step ladder, two-sided step ladder, diagonal ladder, triangular ladder, grid graph, and some types of one-point union graphs[11].

Definition 1.1 *[8] Let $G = (V, E)$ be a (p, q) graph, and f be a map from $V(G)$ to $\{1, 2, ..., p\}$. For each edge uv assign the label $|f(v) - f(u)|$; f is called a difference cordial labeling if f is a one-to-one map and $|e_f(0) - e_f(1)| \le 1$ where $e_f(1)$ denotes the number of edges labeled with 1 while $e_f(0)$ denotes the number of edges not labeled with 1. A graph with a difference cordial labeling is called a difference cordial graph.*

Proposition 1.2 *[8] If G is a (p, q) difference cordial graph, then $q \le 2p - 1$.*

Definition 1.3 *[4] The number $\delta(G) = \min\{d(v) \mid v \in V\}$ is the minimum degree of the vertices in the graph G, the number $\Delta(G) = \max\{d(v) \mid v \in V\}$ is the maximum degree of the vertices in the graph G, and the number $d(G) = \frac{1}{|V|}\sum_{v \in V} d(v)$ is the average degree of the vertices in the graph G.*

Proposition 1.4 *[11] The graph $G(p, q)$ is not a difference cordial graph if $\delta(G) \ge 4$.*

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Proposition 1.5 *[11] The graph $G(p,q)$ is not difference cordial if $d(G) \geq 4$.*

Definition 1.6 *[4] The line graph $L(G)$ of a graph G has a vertex for each edge of G, and two of these vertices are adjacent if and only if the corresponding edges in G have a common vertex.*

Definition 1.7 *[4] A planar graph is outerplanar if it can be embedded in the plane so that all its vertices lie on the same face; we usually choose this face to be exterior. An outerplanar graph G is maximal outerplanar if no line can be added without losing outerplanarity.*

The maximal outerplanar graph is denoted by MOG in this paper. The MOG has the following properties:

Lemma 1.8 *[4] Let G be a MOG with n vertices, $n \geq 3$, and then:*

1. *there are $2n-3$ edges, in which there are $n-3$ chords;*
2. *there are $n-2$ inner faces and each inner face is triangular;*
3. *there are at least two vertices with degree 2;*
4. *connectivity of G, $k(G)$ is equal to 2.*

Definition 1.9 *[1] Let $G=(V,E)$ be a graph and $u \in V$ a vertex of G. The open neighborhood of u or just the neighborhood of u, denoted by $N_G(u)$ or just $N(u)$, is the set of all of the neighbors of u in G. Likewise, the closed neighborhood of u, denoted by $N_G[u]$ or just $N[u]$, is the set of neighbors of u together with u itself.*

Definition 1.10 *[13] Duplication of a vertex v_i by a new edge $e=v_i^{'}v_i^{''}$ in a graph G produces a new graph $G^{'}$ such that $N(v_i^{'}) \cap N(v_i^{''}) = \{v_i\}$.*

Definition 1.11 *[2] A shell graph is defined as a cycle C_n with $(n-3)$ chords sharing a common end point called the apex, and shell graphs are denoted as $C(n,n-3)$.*

Definition 1.12 *[12] A bow graph is defined to be a double shell in which each shell has any order.*

Definition 1.13 *[12] Define a butterfly graph as a bow graph with exactly two pendent edges at the apex.*

Definition 1.14 *[5] Define the shell-flower graph as k copies of the union of the shell $C(n,n-3)$ and K_2 where one end vertex of K_2 is joined to the apex of the shell. This type of graph is illustrated in Section 3.4.*

2. Some results

Proposition 2.1 *The graph $G=(V,E)$ is difference cordial if and only if there exist some disjoint paths, such that their total length is more than or equal to $\left\lfloor \frac{1}{2}|E| \right\rfloor$.*

Proof Let $G=(V,E)$ be a difference cordial graph with a mapping f; then $|e_f(1)-e_f(0)| \leq 1$, i.e. $e_f(1)-1 \leq e_f(0) \leq e_f(1)+1$ and $e_f(1)+e_f(0)=|E|$. If $|E|$ is even, then $e_f(1)=e_f(0)$ *and* $e_f(1)=\frac{1}{2}|E|$, and this means $\frac{1}{2}|E|$ edges join the vertices labeled $i, i+1$.

Then there are some paths such that the sum of their lengths is $\frac{1}{2}|E|$, which is more than or equal to $\left\lfloor \frac{1}{2}|E| \right\rfloor$.

If $|E|$ is odd, then $e_f(1) = e_f(0) + 1$ or $e_f(1) = e_f(0) - 1$. If $e_f(1) = e_f(0) + 1$, then $e_f(1) = \left\lceil \frac{1}{2}|E| \right\rceil$; thus, there are $\left\lceil \frac{1}{2}|E| \right\rceil$ edges joining the vertices labeled $i, i+1$. In other words, there are some paths such that the sum of their lengths is more than or equal to $\left\lfloor \frac{1}{2}|E| \right\rfloor$.

If $e_f(1) = e_f(0) - 1$, then $e_f(1) = \left\lfloor \frac{1}{2}|E| \right\rfloor$, and thus there are $\left\lfloor \frac{1}{2}|E| \right\rfloor$ edges joining the vertices labeled $i, i+1$. This means the existence of some disjoint paths, as above, for which the sum of their lengths is more than or equal to $\left\lfloor \frac{1}{2}|E| \right\rfloor$.

Thus, if $G = (V, E)$ is a difference cordial graph, then there exist some disjoint paths, the sum of their lengths being more than or equal to $\left\lfloor \frac{1}{2}|E| \right\rfloor$. It is necessary that the paths are disjoint; otherwise, there are three paths having a common vertex, which is impossible, since if this common vertex has the label x, two of the adjacent vertices should have the labels $x-1, x+1$. However, the third vertex cannot take either $x-1$ or $x+1$, but something else.

Suppose there exist some disjoint paths, the sum of their lengths being more than or equal to $\left\lfloor \frac{1}{2}|E| \right\rfloor$ on the graph $G = (V, E)$. If there is only one such path and we label its vertices by $i, i+1, ..., \left\lfloor \frac{1}{2}|E| \right\rfloor$, then all edges of this path are labeled 1.

If there are two disjoint paths their lengths are k and h where $k + h$ is more than or equal to $\left\lfloor \frac{1}{2}|E| \right\rfloor$, and then we label these paths by $j, j+1, ..., j+k$ and $t, t+1, ..., t+h$, respectively. Hence, there are $k+h$ edges labeled 1, and we continue this procedure with all paths.

Then $G = (V, E)$ is difference cordial. □

Proposition 2.2 *If the graph $G = (V, E)$ is semi-Hamiltonian, then G is a difference cordial graph if and only if the length of the semi-Hamiltonian path is more than or equal to $\left\lfloor \frac{1}{2}|E| \right\rfloor$.*

Proof Let G be a (p, q) graph containing a semi-Hamiltonian path where $q = |E|$, and let G be a difference cordial graph. Since G contains a semi-Hamiltonian path, we can label it such that $e_f(1) = p-1$, and according to Proposition 1.2:

$$q \leq 2p - 1 \Longrightarrow p - 1 > \left\lfloor \frac{1}{2} q \right\rfloor - 1.$$

G being difference cordial implies that the length of the semi-Hamiltonian path is more than or equal to $\left\lfloor \frac{1}{2}|E| \right\rfloor$.

Suppose G contains a semi-Hamiltonian path whose length is more than or equal to $\left\lfloor \frac{1}{2}|E| \right\rfloor$; then G is difference cordial by Proposition 2.1. □

Proposition 2.3 *Every connected graph $G(p, q)$ with $q = 2p - 1$ is difference cordial if and only if G is semi-Hamiltonian.*

Proof Let $G(p, q)$ be undirected simple connected with $q = 2p - 1$.

Suppose G is a difference cordial graph; then there exists labeling f such that $|e_f(1) - e_f(0)| \leq 1$. From Proposition 2.1 there is a path with length at least $p-1$, but by [11] the maximum length is $p-1$ and then this path must pass through all vertices of graph G, i.e. G is semi-Hamiltonian.

If G is a semi-Hamiltonian graph, we label the vertices on a semi-Hamiltonian path by a sequence of integers $1, 2, ..., p$ and we will get $p-1$ edges labeled 1 and the other edges labeled 0. This means that G is difference cordial. □

Corollary 2.4 *The Peterson graph is difference cordial.*

Proof Direct consequence of Proposition 2.2. □

Proposition 2.5 *Every biconnected outerplanar graph is a difference cordial graph.*

Proof Let G (p,q) be a biconnected outerplanar graph; then G is a Hamiltonian graph and thus there is a semi-Hamiltonian path in G.

Case 1: If G is a maximal outerplanar graph, then $q = 2p-3$ and thus $q \leq 2p-1$ and we label the vertices in this path by a sequence of integer numbers. We get $p-1$ of edges labeled 1, and the other edges will be labeled 0; thus, $e_f(0) = p-2$ means that G is difference cordial.

Case 2: If G is not a maximal outerplanar graph, then $q \leq 2p-3$; thus, $q \leq 2p-1$, and then the semi-Hamiltonian path is of length $p-1$, which is more than $\lfloor \frac{1}{2}q \rfloor$. Then, by Proposition 2.2, G is difference cordial.

Then from **Case 1 and Case 2** every biconnected outerplanar graph is difference cordial. □

Example 2.6 *The following outerplanar graphs with their difference cordial labeling are shown in Figures 1 and 2 respectively.*

$p = 12$, $q = 21$ *and* $e_f(0) = 10, e_f(1) = 11$

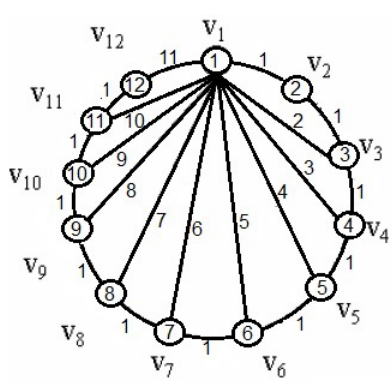

Figure 1. A difference cordial labeling for the maximal outerplanar graph with 12 vertices.

$p = 12$, $q = 20$ *and* $e_f(0) = 10, e_f(1) = 10$.

Proposition 2.7 *1. The line graph $L(G)$ for any graph G with $\delta(G) \geq 3$ cannot be difference cordial.*

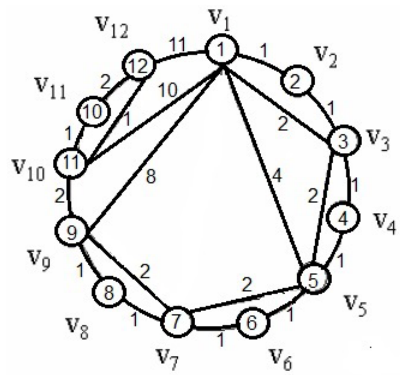

Figure 2. An outerplanar graph with 12 vertices and 20 edges.

2. The line graph $L(G)$ for any graph G with $d(G) \geq 3$ cannot be difference cordial.

Proof

1. Let $G(p,q)$ be a simple undirected graph with $\delta(G) \geq 3$ and $L(G)$ its line graph.

   The number of vertices on $L(G)$ is equal to q and each of these edges in $L(G)$ has on its ends two vertices with degree more than or equal to 3; then the vertex corresponding to this edge in the line graph $L(G)$ will be of degree more than 4. Then $L(G)$ is a graph with $\delta(G) \geq 4$ and then, by Proposition 1.4, it cannot be difference cordial.

2. It follows directly from 1.

□

Remark 2.8 *The union of two disjoint difference cordial graphs need not be difference cordial.*

Proof Let $G_1(p_1,q_1)$ and $G_2(p_2,q_2)$ be two disjoint difference cordial graphs, where f_1 and f_2 are their labelings; then $|e_{f_1}(0) - e_{f_1}(1)| \leq 1$ and $|e_{f_2}(0) - e_{f_2}(1)| \leq 1$.

Since if $e_{f_1}(0) = e_{f_1}(1) + 1$, and $e_{f_2}(0) = e_{f_2}(1) + 1$, and G_1, G_1 are disjoint graphs, then $G_1 \cup G_2$ has $q = q_1 + q_2$ and $p = p_1 + p_2$. Then $e_{f_1}(0) + e_{f_2}(0) = e_{f_1}(1) + e_{f_2}(1) + 2$, and hence $G_1 \cup G_2$ is not difference cordial. □

Example 2.9 *The following two disjoint difference cordial graphs are shown in Figure 3.*

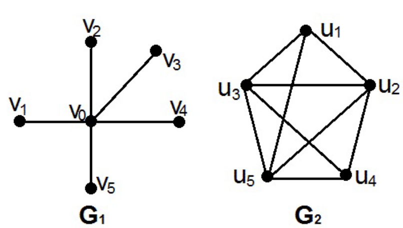

Figure 3. Two disjoint difference cordial graphs.

3. Difference cordial labeling for some families of graphs

In this section we introduce difference cordial labeling for some types of graphs.

3.1. Graph Obtained by Duplication of Vertex by an Edge

Here we discuss only the graph obtained by duplication of each vertex of C_n by an edge.

Proposition 3.1 *The graph obtained by duplication of each vertex of C_n by an edge is difference cordial.*

Proof Let $v_1, v_2, ..., v_n$ be the vertices of the cycle C_n and G be the graph obtained by duplication of each vertex v_i of the cycle C_n by an edge u_iu_{i+1} $(1 \leq i \leq n)$.

Then $V(G) = V(C_n) \cup \{u_1, u_2, ..., u_{2n}\}$ and $E(G) = E(C_n) \cup \{u_{2i-1}v_i, u_{2i}v_i, u_{2i}u_{2i-1}; 1 \leq i \leq n\}$, and then $V(G) = 3n$ and $E(G) = 4n$.

Define the mapping $f : V(G) \longrightarrow \{1, 2, ..., 3n\}$ by:

$$f(u_i) = i \quad \& \quad f(v_i) = \left\{ \begin{array}{cc} i+1+2n & , i \neq n \\ 2n+1 & , i = n \end{array} \right\}.$$

From the definition of $f(u_i)$ there are n edges labeled 1 and from $f(v_i)$ there are $n-1$ edges labeled 1. The edge $u_{2n}v_n$ is labeled 1, and then $e_f(1) = 2n$ and G is a difference cordial graph. □

Example 3.2 *The graph obtained by duplication of the vertex of C_7 by an edge with its difference cordial labeling is shown in Figure 4.*

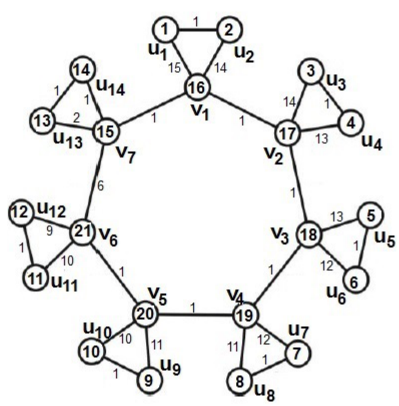

Figure 4.

3.2. Bow graphs

The bow graph $G(p, q)$ could be described as follows: in graph G, the shell that is present to the left of the apex is called the left wing and the shell that is present to the right of the apex is considered as the right wing.

Figure 5 shows the bow graph with shells of orders m and n excluding the apex.

Proposition 3.3 *All bow graphs are difference cordial.*

Proof Let G be a bow graph with two shells of orders m and n excluding the apex. Then the number of vertices in G is $p = m + n + 1$ and the number of edges $q = 2(m + n - 1)$. The apex of the bow graph is denoted by v_0, we denote the vertices in the right wing of the bow graph from bottom to top by $v_1, v_2, ..., v_m$, and the vertices in the left wing of the bow graph are denoted from top to bottom by $v_{m+1}, v_{m+2}, ..., v_{m+n}$.

Define the mapping of labeling $f : V \longrightarrow \{1, 2, ..., m + n + 1\}$ by:

$$f(v_i) = \left\{ \begin{array}{cc} i & , i \neq 0 \\ m+n+1 & , i = 0 \end{array} \right\}.$$

From the above definition we see there are $m - 1$ edges labeled 1 and m edges labeled 0 in the right wing of the bow graph. There are also n edges labeled 1 and $n - 1$ edges labeled 0 in the left wing of the bow graph. Then $|e_f(1)| = |e_f(0)| = m + n - 1$, which implies that the bow graph is difference cordial. □

Example 3.4 *The bow graph G with two wings having m, n vertices respectively, with its difference cordial labeling, is shown in Figure 5.*

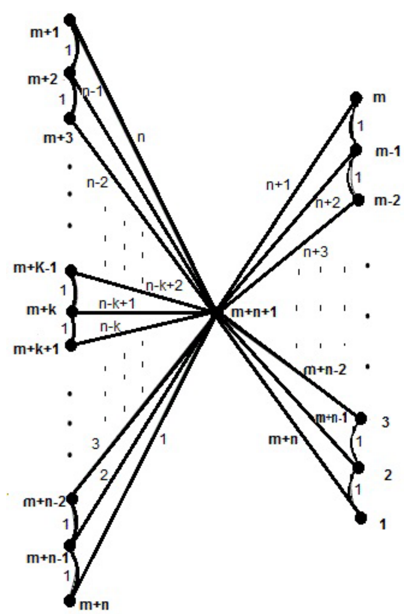

Figure 5. The bow graph with $m + n + 1$ vertices and its difference cordial labeling.

3.3. Butterfly graphs

Proposition 3.5 *The butterfly graphs are difference cordial.*

Proof Let G be a butterfly graph with shells of orders m and n excluding the apex; then the number of vertices in G is $p = m+n+3$ and the number of edges $q = 2(m+n)$. The apex of the butterfly graph is denoted as v_0, we denote the vertices in the right wing of the butterfly graph from bottom to top by $v_1, v_2, ..., v_m$, the vertices in the left wing of the butterfly graph are denoted from top to bottom by $v_{m+1}, v_{m+2}, ..., v_{m+n}$, and the pendant vertices in the pendant edges are denoted by v_{m+n+1}, v_{m+n+2}.

Define the mapping of labeling $f : V \longrightarrow \{1, 2, ..., m + n + 3\}$ by:

$$f(v_i) = \left\{ \begin{array}{cl} i & ,1 \le i \le n+m \\ m+n+1 & ,i=0 \\ i+1 & ,n+m < i \le n+m+2 \end{array} \right\}.$$

From the above definition we see there are $m-1$ edges labeled 1 and m edges labeled 0 in the right wing of the bow graph. There are also n edges labeled 1 and $n-1$ edges labeled 0 in the left wing of the butterfly graph, while the pendant edges are labeled 1 and 0. Then $|e_f(1)| = |e_f(0)| = m+n$, which implies that the butterfly graph is difference cordial. □

Example 3.6 *The butterfly graph G with two wings having m, n vertices respectively, and its difference cordial labeling, is shown in Figure 6.*

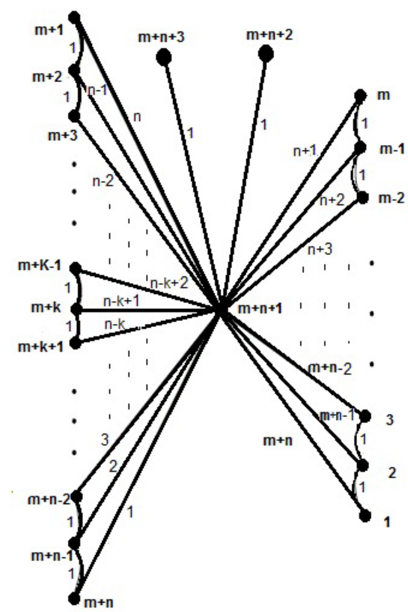

Figure 6. A difference cordial labeling for the butterfly with $m+n+3$ vertices.

3.4. Shell-flower graphs

From the definition of the shell-flower graph it contains k copies of the shell $C(n, n-3)$ and k copies of K_2 where one vertex of K_2 is joined to the apex of the shell and each shell in the shell-flower graph is called a petal; hence, it consists of k petals and k pendant edges.

Proposition 3.7 *The shell-flower graph cannot be difference cordial when $k \ge 3$ for all n.*

Proof Let $G(p,q)$ be the shell-flower graph with k petals, in each one $C(n, n-3)$, and then $p = nk+1$, and $q = 2k(n-1)$ as in Figure 7.

The apex vertex has degree $p-1$, and then for any labeling of vertices there are at least $p-3 = nk-2$ edges labeled 0 but the number of the total other edges is:

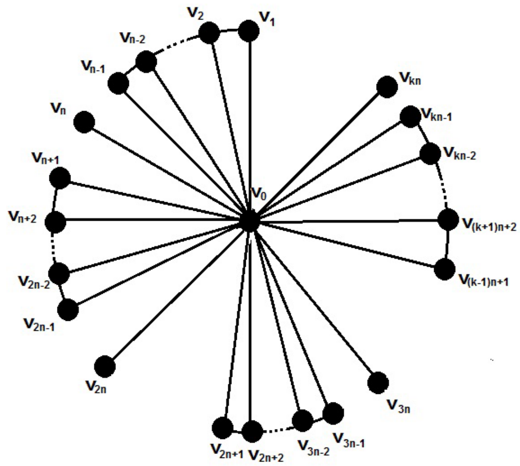

Figure 7. A shell-flower graph with k petals.

$$2k(n-1) - nk + 2$$
$$= (nk-2) + (4-2k),$$

and we assume that all these edges are labeled as 1. Then for all $k \geq 3$, $e_f(0) \geq e_f(1) + 2$. Hence, the shell-flower graph is not difference cordial for all $k \geq 3$. □

Example 3.8 *The shell-flower graph with two petals $C(9,6)$ is shown in Figure 8.*

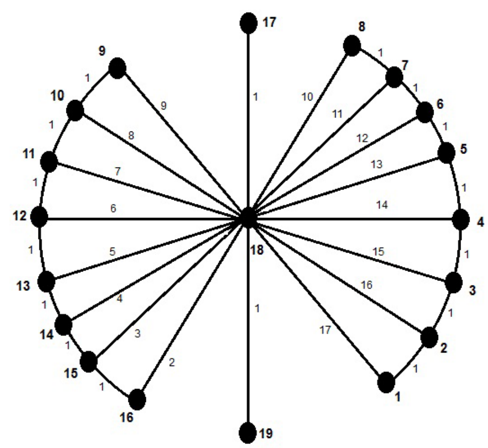

Figure 8. A difference cordial labeling for the shell-flower graph with 2 petals.

3.5. One-point union of complete graphs

In this subsection we discuss the difference cordial labeling of the one-point union of m complete graphs K_n of order n, as in Figure 9.

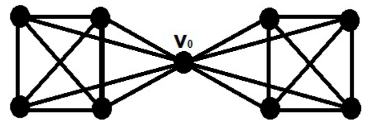

Figure 9. The graph $K_5^{(2)}$.

Proposition 3.9 *Let $K_n^{(m)}$ be the one-point union of m complete graphs K_n.*

1. *$K_2^{(m)}$ is difference cordial when $m \leq 5$.*

2. *$K_3^{(m)}$ is difference cordial when $m \leq 5$.*

3. *$K_4^{(m)}$ is difference cordial when $m \leq 2$.*

4. *$K_n^{(m)}$ is not difference cordial for all $n \geq 5$.*

Proof

1. $K_2^{(m)}$ is a star graph, then from [11] it is difference cordial when $m \leq 5$.

2. $K_3^{(m)}$ is a friendship graph, from [11] it is difference cordial when $m \leq 5$.

3. Let $G = K_4^{(m)}$; then G has $3m+1$ vertices and $6m$ edges, so there is one vertex, say v_0, adjacent to all other vertices in G, so the graph $G - v_0$ consists of m components. Each component is a triangle that consists of at most 2 edges labeled 1 and at least one edge labeled 0, i.e. there are at most $2m$ edges labeled 1 and at least m edges labeled 0. However, v_0 is adjacent to all other vertices and hence its degree is $3m$; then G contains at least $4m-2$ edges labeled 0 and at most $2m+2$ edges labeled 1 and we have

$$\begin{aligned} e_f(0) - e_f(1) &\geq (4m-2) - (2m+2) \\ &\geq 2m - 4. \end{aligned}$$

   If $m \geq 3$ then $e_f(0) - e_f(1) \geq 2$, and thus $K_4^{(m)}$ is not difference cordial for all $m \geq 3$.

4. In graph $K_n^{(m)}$, $\delta(K_n^{(m)}) \geq 4$ when $n \geq 5$, and then by Proposition 1.4, $K_n^{(m)}$ is not difference cordial.

□

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