Turkish Journal of Mathematics

http://journals.tubitak.gov.tr/math/

Research Article

Turk J Math
(2016) 40: 428 – 437
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doi:10.3906/mat-1502-27

Representations for generalized Drazin inverse of operator matrices over a Banach space

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Received: 09.02.2015 • **Accepted/Published Online:** 28.08.2015 • **Final Version:** 10.02.2016

Abstract: In this paper we give expressions for the generalized Drazin inverse of a (2,2,0) operator matrix and a 2×2 operator matrix under certain circumstances, which generalizes and unifies several results in the literature.

Key words: Generalized Drazin inverse, operator matrix, Banach space

1. Introduction

The concept of the generalized Drazin inverse (GD-inverse) in a Banach algebra was introduced by Koliha [21]. Let $\mathcal{B}$ be a complex unital Banach algebra. An element a of $\mathcal{B}$ is generalized Drazin invertible in the case that there is an element $b \in \mathcal{B}$ satisfying

$$ab = ba, \quad bab = b, \quad \text{and} \quad a - a^2 b \quad \text{is quasinilpotent.}$$

Such b, if it exists, is unique; it is called a generalized Drazin inverse of a and will be denoted by a^d. Then the spectral idempotent a^π of a corresponding to 0 is given by $a^\pi = 1 - aa^d$.

The GD-inverse was extensively investigated for matrices over complex Banach algebras and matrices of bounded linear operators over complex Banach spaces. The GD-inverse of the operator matrix has various applications in singular differential equations and singular difference equations, Markov chains and iterative methods, and so on (see [1, 2, 3, 4, 8, 10, 14, 16, 26, 27, 28]).

The generalized Drazin inverse is a generalization of Drazin inverses and group inverses. The study of representations for the Drazin inverse of block matrices essentially originated from finding the general expressions for the solutions to singular systems of differential equations [4, 5, 6]. Until now, there have been many formulae for the Drazin inverse of general 2×2 block matrices under some restrictive assumptions (see [12, 14, 15, 17, 18, 20, 19, 23, 24]).

Some results of the Drazin inverse have been developed in the GD-inverse of operator matrices over Banach spaces (see [7, 11, 13, 14, 17]). Assume that both X and Y are complex Banach spaces. Denote by $\mathcal{B}(X,Y)$ the set of all bounded linear operators from X to Y, and write $\mathcal{B}(X,X) = \mathcal{B}(X)$. Let an operator matrix $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$, where $A \in \mathcal{B}(X), D \in \mathcal{B}(Y), B \in \mathcal{B}(Y,X), C \in \mathcal{B}(X,Y)$.

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This work was supported by the NSFC (No. 11371165; No. 11401249) and the Doctor Research Start-up Fund of Northeast Dianli University (No. BSJXM-201525).
2010 *AMS Mathematics Subject Classification:* 46H05, 47A05, 15A09.

Castro-González [7] derived explicit expressions of the GD-inverse of M under certain conditions, which extended some results of [13, 14]. Cvetković [11] also extended some results of [13, 14]. Recently, Mosić [25] gave the new formulae for the GD-inverse of 2×2 matrices in a Banach algebra.

In this paper, we derive new formulae for the GD-inverse of a (2,2,0) operator matrix N under certain circumstances. Furthermore, we apply N^d to give representations of M^d under weaker restrictions, which generalizes and unifies several results of [7, 11, 13, 14, 17, 25].

Since $(a^d)^n = (a^n)^d$ for any $a \in \mathcal{B}$ we adopt the convention that $a^{dn} = (a^d)^n$ and $a^0 = 1$ and $\sum_{i=0}^{k} * = 0$ in the case of $k<0$. Moreover, we define $\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{\top} = \begin{pmatrix} a & c \\ b & d \end{pmatrix}$ for $a,b,c,d \in \mathcal{B}$.

2. Generalized Drazin inverse of a (2,2,0) operator matrix

Let $\mathcal{B}$ be a complex unital Banach algebra. An element $a\in\mathcal{B}$ is called quasinilpotent if $\lim_{n\to\infty}\|a^n\|^{\frac{1}{n}} = 0$. Let $\mathcal{M}_2(\mathcal{B})$ be the 2×2 matrix algebra over $\mathcal{B}$. Given an idempotent e in $\mathcal{B}$, we consider the set $\mathcal{M}_2(\mathcal{B},e) = \begin{pmatrix} e\mathcal{B}e & e\mathcal{B}(1-e) \\ (1-e)\mathcal{B}e & (1-e)\mathcal{B}(1-e) \end{pmatrix} \subset \mathcal{M}_2(\mathcal{B})$. Then $\mathcal{M}_2(\mathcal{B},e)$ is a unital Banach algebra with respect to the norm

$$\left\| \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \right\| = \|a_{11}+a_{12}+a_{21}+a_{22}\|.$$

Lemma 2.1 *Let e be an idempotent of $\mathcal{B}$. For any $a\in\mathcal{B}$ let*

$$\sigma(a) = \begin{pmatrix} eae & ea(1-e) \\ (1-e)ae & (1-e)a(1-e) \end{pmatrix} \in \mathcal{M}_2(\mathcal{B},e).$$

Then the mapping σ is an isometric Banach algebra isomorphism from $\mathcal{B}$ to $\mathcal{M}_2(\mathcal{B},e)$ such that:

1. $(\sigma(a))^d = \sigma(a^d)$*;*
2. *if* $(\sigma(a))^d = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$*, then* $a^d = \alpha+\beta+\gamma+\delta$*.*

Proof By [9, Lemma 2.1] we have that the mapping σ is an isometric Banach algebra isomorphism from $\mathcal{B}$ to $\mathcal{M}_2(\mathcal{B},e)$. The rest of the proof is obvious. □

Lemma 2.2 *([14]) Let $x = \begin{pmatrix} a & b \\ 0 & d \end{pmatrix}$ and let $y = \begin{pmatrix} d & 0 \\ b & a \end{pmatrix}$ for $a,b,d\in\mathcal{B}$. Then*

$$x^d = \begin{pmatrix} a^d & X \\ 0 & d^d \end{pmatrix}, \quad y^d = \begin{pmatrix} d^d & 0 \\ X & a^d \end{pmatrix},$$

where

$$X = a^{\pi}\sum_{i=0}^{\infty} a^i b d^{d(i+2)} + \sum_{i=0}^{\infty} a^{d(i+2)} b d^i d^{\pi} - a^d b d^d.$$

Lemma 2.3 *Let e be an idempotent of $\mathcal{B}$ and let $a \in \mathcal{B}$ be generalized Drazin invertible such that $ea(1-e) = 0$. Then ea and $a(1-e)$ are both generalized Drazin invertible, and*

$$(ea)^d = ea^d, \quad (a(1-e))^d = a^d(1-e), \quad (ea)^n = ea^n$$

for any positive integer n.

Proof Since $ea(1-e) = 0$, combining Lemma 2.1 and Lemma 2.2, we have $ea^d(1-e) = 0$. Then $eaea^d = eaa^d = ea^d ea$ and $ea^d eaea^d = ea^d$. Furthermore,

$$\lim_{n\to\infty} \|(ea - (ea)^2 ea^d)^n\|^{\frac{1}{n}} = \lim_{n\to\infty} \|(eaa^\pi)^n\|^{\frac{1}{n}}$$
$$= \lim_{n\to\infty} \|ea^n a^\pi\|^{\frac{1}{n}} \leq \lim_{n\to\infty} \|e\|^{\frac{1}{n}} \|a^n a^\pi\|^{\frac{1}{n}} = 0.$$

Hence, ea is generalized Drazin invertible and $(ea)^d = ea^d$. Similarly, we can prove that $a(1-e)$ is generalized Drazin invertible and $(a(1-e))^d = a^d(1-e)$. Using $ea(1-e) = 0$ we easily get $(ea)^n = ea^n$ for any positive integer n. □

Lemma 2.4 *([22])(Cline's Formula) For $a, b \in \mathcal{B}$, ab is generalized Drazin invertible if and only if so is ba. Furthermore, if ab is generalized Drazin invertible, then*

$$(ba)^d = b(ab)^{2d}a.$$

The following lemma is an immediate corollary of [17, Corollary 3.3.7].

Lemma 2.5 *Let $x = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathcal{M}_2(\mathcal{B})$ with a and d generalized Drazin invertible. If $abc = 0$, $bd = 0$, $(bc)^d = 0$, then x is generalized Drazin invertible, and*

$$x^d = \begin{pmatrix} \phi_1 a & \phi_1 b \\ \tau a + \psi_1 & d^d + \tau b \end{pmatrix},$$

where

$$\begin{aligned}
\phi_n &= \Sigma_{j=0}^{\infty} (bc)^j a^{d(2j+2n)}, \\
\psi_n &= \Sigma_{j=0}^{\infty} d^{d(2j+2n)} (cb)^j c, \\
\tau &= \Sigma_{i=0}^{\infty} (cb + d^2)^i c a^{d(2i+3)} + \Sigma_{i=0}^{\infty} d^\pi d^{2i+1} c \phi_{i+2} \\
&\quad - \Sigma_{i=0}^{\infty} d^2 (cb + d^2)^i \psi_1 a^{d(2i+3)} + \Sigma_{i=0}^{\infty} \psi_{i+2} a^{2i+1} a^\pi \\
&\quad + \Sigma_{i=0}^{\infty} d^{d(2i+3)} c (a^2 + bc)^i a^\pi - \Sigma_{i=0}^{\infty} d^{d(2i+1)} c (bc)^i \phi_1 - \psi_1 a^d.
\end{aligned}$$

Now we can give our first main result. Recall that $a^{dn} = (a^d)^n$ and $\begin{pmatrix} a & b \\ c & d \end{pmatrix}^\top = \begin{pmatrix} a & c \\ b & d \end{pmatrix}$ for $a, b, c, d \in \mathcal{B}$.

Theorem 2.6 *Let* $N = \begin{pmatrix} E & I \\ F & 0 \end{pmatrix}$ *be an operator matrix with* E *and* F *generalized Drazin invertible. If* $F^dEF^\pi = 0$ *and* $F^\pi FE = 0$, *then* N *is generalized Drazin invertible, and*

$$N^d = \sum_{i=0}^{\infty} \begin{pmatrix} E^{2i+1}E^\pi F^\pi EF^d & 0 \\ E^{2i}E^\pi(F^\pi - EF^\pi EF^d)EF^d & 0 \end{pmatrix}^\top \begin{pmatrix} 0 & F^d \\ I & -EF^d \end{pmatrix}^{2i+1}$$

$$+ \begin{pmatrix} E^dF^\pi + \sum_{i=0}^{\infty} E^{d(2i+3)}F^\pi F^{i+1} & FF^d \\ F^d - E^dF^\pi EF^d + \sum_{i=0}^{\infty} E^{d(2i+2)}F^\pi F^i & -FF^dEF^d \end{pmatrix}^\top.$$

Proof We adopt the convention that $F^e = FF^d$. Let $e = \begin{pmatrix} F^e & 0 \\ 0 & I \end{pmatrix}$, σ as in Lemma 2.1, and $\sigma(N) = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Since $F^dEF^\pi = 0$, we have

$$a = \begin{pmatrix} F^eE & F^e \\ FF^e & 0 \end{pmatrix}, \quad b = \begin{pmatrix} 0 & 0 \\ FF^\pi & 0 \end{pmatrix}, \quad c = \begin{pmatrix} F^\pi EF^e & F^\pi \\ 0 & 0 \end{pmatrix}, \quad d = \begin{pmatrix} EF^\pi & 0 \\ 0 & 0 \end{pmatrix}.$$

Note that a has the group inverse

$$a^\sharp = \begin{pmatrix} 0 & F^d \\ F^e & -F^eEF^d \end{pmatrix} \tag{2.1}$$

and so $aa^\pi = 0$. Using Lemma 2.3 we have $(EF^\pi)^d = E^dF^\pi$, and

$$(a^\sharp)^n = \left(\begin{pmatrix} F^e & 0 \\ 0 & F^e \end{pmatrix} \begin{pmatrix} 0 & F^d \\ I & -EF^d \end{pmatrix} \right)^n = \begin{pmatrix} F^e & 0 \\ 0 & F^e \end{pmatrix} \begin{pmatrix} 0 & F^d \\ I & -EF^d \end{pmatrix}^n$$

for any positive integer n. Hence,

$$d^d = \begin{pmatrix} E^dF^\pi & 0 \\ 0 & 0 \end{pmatrix}. \tag{2.2}$$

Note that $(F^\pi F)^d = 0$, and so

$$bc = \begin{pmatrix} 0 & 0 \\ F^\pi FE & F^\pi F \end{pmatrix} \quad \text{and} \quad cb = \begin{pmatrix} F^\pi F & 0 \\ 0 & 0 \end{pmatrix}. \tag{2.3}$$

Using Lemma 2.2 we get $(bc)^d = (cb)^d = 0$. Since $F^dE^{i+1}F^\pi = F^d(EF^\pi)^{i+1} = 0$ for any nonnegative integer i, we have

$$ab = 0, \quad bd = 0, \quad bca = 0.$$

By Lemma 2.5 we have

$$(\sigma(N))^d = \begin{pmatrix} a^d & 0 \\ \Sigma_0 & d^d + \Lambda \end{pmatrix}, \tag{2.4}$$

where

$$\begin{aligned} \Lambda = {} & \sum_{i=0}^{\infty} d^{d(2i+3)}c(bc)^ib, \\ \Sigma_0 = {} & \sum_{i=0}^{\infty} d^{2i}ca^{d(2i+2)} + \sum_{i=0}^{\infty} d^\pi d^{2i+1}ca^{d(2i+3)} - \sum_{i=0}^{\infty} d^{2i+1}d^dca^{d(2i+2)} \\ & + \sum_{i=0}^{\infty} d^{d(2i+2)}(cb)^ic - d^dca^d - d^{2d}ca^da. \end{aligned}$$

Substituting (2.1), (2.2), and (2.3) into (2.4) and using Lemma 2.1 will give the expression of N^d that we wanted. □

Corollary 2.7 *Let* $N = \begin{pmatrix} E & I \\ F & 0 \end{pmatrix}$ *be an operator matrix with* E *and* F *generalized Drazin invertible. If* $F^dEF^\pi = 0$ *and* $F^\pi FE = 0$, *then* N *is generalized Drazin invertible, and*

$$N^{dn} = \sum_{i=0}^{\infty} \begin{pmatrix} E^{2i+1}E^\pi F^\pi EF^d & 0 \\ E^{2i}E^\pi(F^\pi - EF^\pi EF^d)EF^d & 0 \end{pmatrix}^\top \begin{pmatrix} 0 & F^d \\ I & -EF^d \end{pmatrix}^{2i+n}$$
$$+ \sum_{j=1}^{n} \begin{pmatrix} E^{dj}F^\pi + \sum_{i=0}^{\infty} E^{d(2i+j+2)}F^{i+1}F^\pi & 0 \\ E^{dj}(\sum_{i=0}^{\infty} E^{d(2i+1)}F^iF^\pi - F^\pi EF^d) & 0 \end{pmatrix}^\top \begin{pmatrix} 0 & F^d \\ FF^d & -FF^dEF^d \end{pmatrix}^{n-j}$$
$$+ \begin{pmatrix} 0 & F^d \\ FF^d & -FF^dEF^d \end{pmatrix}^n$$

for any positive integer n.

Proof Let $N^d = P + Q + R$ by Theorem 2.6, where

$$P = \sum_{i=0}^{\infty} \begin{pmatrix} E^{2i+1}E^\pi F^\pi EF^d & 0 \\ E^{2i}E^\pi(F^\pi - EF^\pi EF^d)EF^d & 0 \end{pmatrix}^\top \begin{pmatrix} 0 & F^d \\ I & -EF^d \end{pmatrix}^{2i+1},$$
$$Q = \begin{pmatrix} E^dF^\pi + \sum_{i=0}^{\infty} E^{d(2i+3)}F^\pi F^{i+1} & 0 \\ -E^dF^\pi EF^d + \sum_{i=0}^{\infty} E^{d(2i+2)}F^\pi F^i & 0 \end{pmatrix}^\top,$$
$$R = \begin{pmatrix} 0 & F^d \\ FF^d & -FF^dEF^d \end{pmatrix}.$$

Since $F^dE^{i+1}F^\pi = F^d(EF^\pi)^{i+1} = 0$ for any nonnegative integer i, and since $F^\pi FE = 0$, we have $P^2 = 0$, $RP = 0$, $RQ = 0$, $PQ = QP = 0$, and

$$Q^n = \sum_{j=1}^{n} \begin{pmatrix} E^{dn}F^\pi + \sum_{i=0}^{\infty} E^{d(2i+n+2)}F^{i+1}F^\pi & 0 \\ E^{dn}(\sum_{i=0}^{\infty} E^{d(2i+1)}F^iF^\pi - F^\pi EF^d) & 0 \end{pmatrix}^\top$$

for any positive integer n. Then $N^{dn} = Q^n + R^n + PR^{n-1} + \sum_{j=1}^{n-1} Q^jR^{n-j}$, and by a routine computation, we get the expression of N^{dn} as shown in Corollary 2.7. □

The following theorem, which is a dual version of Theorem 2.6, can be proved similarly.

Theorem 2.8 *Let* $N = \begin{pmatrix} E & I \\ F & 0 \end{pmatrix}$ *be an operator matrix with* E *and* F *generalized Drazin invertible. If* $F^\pi EF^d = 0$ *and* $EFF^\pi = 0$, *then* N *is generalized Drazin invertible, and*

$$N^d = \sum_{i=0}^{\infty} \begin{pmatrix} 0 & F^d \\ FF^d & -EF^d \end{pmatrix}^{2i+2} \begin{pmatrix} F^dEF^\pi E^{2i+2}E^\pi & F^dEF^\pi E^{2i+1}E^\pi \\ GE & G \end{pmatrix}$$
$$+ \begin{pmatrix} \sum_{i=0}^{\infty} F^iF^\pi E^{d(2i+1)} + F^dEF^\pi E^\pi & \sum_{i=0}^{\infty} F^iF^\pi E^{d(2i+2)} + F^d - F^dEF^\pi E^d \\ H + FF^d - EF^dEF^\pi & HE^d + EF^dEF^\pi E^d - EF^d \end{pmatrix}$$

such that

$$G = (FF^dEF^\pi - EF^dEF^\pi E)E^{2i}E^\pi,$$

and

$$H = \sum_{i=1}^{\infty} F^{i+1} F^{\pi} E^{d(2i+2)} - F F^d E F^{\pi} E^d.$$

Proof We adopt the convention that $F^e = FF^d$. Let $e = \begin{pmatrix} F^{\pi} & 0 \\ 0 & 0 \end{pmatrix}$. Using Lemma 2.3 we have $(EF^{\pi})^d = E^d F^{\pi}$. The proof is similar in spirit to that of Theorem 2.6. Since $F^{\pi} E F^d = 0$, we have

$$a = \begin{pmatrix} F^{\pi} E & 0 \\ 0 & 0 \end{pmatrix}, \quad b = \begin{pmatrix} 0 & F^{\pi} \\ 0 & 0 \end{pmatrix}, \quad c = \begin{pmatrix} F^e E F^{\pi} & 0 \\ F F^{\pi} & 0 \end{pmatrix}, \quad d = \begin{pmatrix} E F^e & F^e \\ F F^e & 0 \end{pmatrix}.$$

Then

$$(bc)^d = 0 = (cb)^d, \quad d^{\sharp} = \begin{pmatrix} 0 & F^d \\ F^e & -EF^d \end{pmatrix} \quad \text{and} \quad dd^{\pi} = 0. \tag{2.5}$$

Since $F^{\pi} E^{i+1} F^d = (F^{\pi} E)^{i+1} F^d = 0$ for any nonnegative integer i, we have

$$abc = 0, \quad bd = 0, \quad dcbc = 0. \tag{2.6}$$

Combining (2.5) and (2.6), in a way exactly similar to Theorem 2.6, we get the result. $\square$

3. Applications to a 2×2 operator matrix

Lemma 3.1 *[17, Theorem 3.2.2] Let $x = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathcal{M}_2(\mathcal{B})$ with a and d generalized Drazin invertible. If $ca^d = 0$ and $ca^i b = 0$ for any nonnegative integer i, then x is generalized Drazin invertible, and*

$$x^d = \begin{pmatrix} a^d + \varphi & \phi \\ \psi & d^d \end{pmatrix},$$

where

$$\begin{aligned} \psi = &\ \textstyle\sum_{i=0}^{\infty} d^{d(i+2)} c a^i, \\ \phi = &\ a^{\pi} \textstyle\sum_{i=0}^{\infty} a^i b d^{d(i+2)} + \sum_{i=0}^{\infty} a^{d(i+2)} b d^i d^{\pi} - a^d b d^d, \\ \varphi = &\ a^{\pi} \textstyle\sum_{i=0}^{\infty} \sum_{j=0}^{\infty} a^i b d^{d(i+j+3)} c a^j - \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} a^{d(i+1)} b d^i d^{d(j+2)} c a^j \\ &+ \textstyle\sum_{i=0}^{\infty} \sum_{j=0}^{i} a^{d(i+3)} b d^j c a^{i-j}. \end{aligned}$$

Now, based on an observation of the matrix decomposition, we apply the representations of the generalized Drazin inverse of the (2,2,0) operator matrix to give our another main result. Recall that $a^e = a^d a$, $a^{dn} = (a^d)^n$, and $\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{\top} = \begin{pmatrix} a & c \\ b & d \end{pmatrix}$ for $a, b, c, d \in \mathcal{B}$.

Theorem 3.2 *Let $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ be an operator matrix with A and D generalized Drazin invertible. If $(BC)^d A (BC)^{\pi} = 0$, $(BC)^{\pi} BCA = 0$, $BD^d = 0$, and $BD^i C = 0$ for any positive integer i, then M is* *generalized Drazin invertible, and*

$$M^d = \left(\begin{pmatrix} 0 \\ D^e + D\Omega \end{pmatrix} + \begin{pmatrix} A & I \\ C & 0 \end{pmatrix}\Phi\right)\left(\begin{pmatrix} 0 & D^d + \Omega \end{pmatrix} + \Psi\begin{pmatrix} I & 0 \\ 0 & B \end{pmatrix}\right)$$
$$+ \left(\begin{pmatrix} 0 \\ D\Psi \end{pmatrix} + \begin{pmatrix} A & I \\ C & 0 \end{pmatrix}\Delta^d\right)\left(\begin{pmatrix} 0 & \Phi \end{pmatrix} + \Delta^d\begin{pmatrix} I & 0 \\ 0 & B \end{pmatrix}\right),$$

where

$$\Phi = \sum_{k=0}^{\infty} \Delta^{d(k+2)} \begin{pmatrix} 0 \\ BD^{k+1} \end{pmatrix},$$

$$\Psi = \sum_{k=0}^{\infty} \begin{pmatrix} D^k D^\pi C & 0 \end{pmatrix} \Delta^{d(k+2)} + \sum_{k=0}^{\infty} \begin{pmatrix} D^{d(k+2)} C & 0 \end{pmatrix} \Delta^k \Delta^\pi - \begin{pmatrix} D^d C & 0 \end{pmatrix} \Delta^d,$$

$$\Omega = \sum_{k=0}^{\infty}\sum_{l=0}^{\infty} \begin{pmatrix} D^k D^\pi C & 0 \end{pmatrix} \Delta^{d(k+l+3)} \begin{pmatrix} 0 \\ BD^{l+1} \end{pmatrix}$$
$$- \sum_{k=0}^{\infty}\sum_{l=0}^{\infty} \begin{pmatrix} D^{d(k+1)} C & 0 \end{pmatrix} \Delta^k \Delta^{d(l+2)} \begin{pmatrix} 0 \\ BD^{l+1} \end{pmatrix}$$
$$+ \sum_{k=0}^{\infty}\sum_{l=0}^{k} \begin{pmatrix} D^{d(k+3)} C & 0 \end{pmatrix} \Delta^l \begin{pmatrix} 0 \\ BD^{k-l+1} \end{pmatrix},$$

$$\Delta^{dn} = \sum_{i=0}^{\infty} \begin{pmatrix} A^{2i+1}A^\pi(BC)^\pi A(BC)^d & 0 \\ A^{2i}A^\pi((BC)^\pi - A(BC)^\pi A(BC)^d)A(BC)^d & 0 \end{pmatrix}^\top$$
$$\times \begin{pmatrix} 0 & (BC)^d \\ I & -A(BC)^d \end{pmatrix}^{2i+n}$$
$$+ \sum_{j=1}^{n} \begin{pmatrix} A^{dj}(BC)^\pi + \sum_{i=0}^\infty A^{d(2i+j+2)}(BC)^{i+1}(BC)^\pi & 0 \\ \sum_{i=0}^\infty A^{d(2i+j+1)}(BC)^i(BC)^\pi - A^{dj}(BC)^\pi A(BC)^d & 0 \end{pmatrix}^\top$$
$$\times \begin{pmatrix} 0 & (BC)^d \\ BC(BC)^d & -BC(BC)^d A(BC)^d \end{pmatrix}^{n-j}$$
$$+ \begin{pmatrix} 0 & (BC)^d \\ BC(BC)^d & -BC(BC)^d A(BC)^d \end{pmatrix}^n$$

for $n \geq 1$ *and* $\Delta = \begin{pmatrix} A & I \\ BC & 0 \end{pmatrix}$.

Proof Note that

$$M = \begin{pmatrix} 0 & A & I \\ I & C & 0 \end{pmatrix}\begin{pmatrix} 0 & D \\ I & 0 \\ 0 & B \end{pmatrix}. \tag{3.1}$$

Let us denote by P and Q the left matrix and the right matrix of the right-hand side in (3.1), respectively. Then $QP = \begin{pmatrix} \alpha & \beta \\ \gamma & \Delta \end{pmatrix}$, where

$$\alpha = D, \quad \beta = \begin{pmatrix} DC & 0 \end{pmatrix}, \quad \gamma = \begin{pmatrix} 0 \\ B \end{pmatrix}, \quad \text{and} \quad \Delta = \begin{pmatrix} A & I \\ BC & 0 \end{pmatrix}. \tag{3.2}$$

Applying Corollary 2.7 to Δ we obtain the expression of Δ^{dn} for any $n \geq 1$ as shown in Theorem 3.2. Since $BD^d = 0$ and $BD^iC = 0$ for $i = 1, 2, \ldots$, we have

$$\alpha^n \beta = \begin{pmatrix} D^{n+1}C & 0 \end{pmatrix}, \quad \alpha^{dn}\beta = \begin{pmatrix} D^{d(n+1)}C & 0 \end{pmatrix}, \quad \text{and} \quad \gamma\alpha^n = \begin{pmatrix} 0 \\ BD^n \end{pmatrix}. \tag{3.3}$$

Moreover, $\gamma\alpha^d = 0$ and $\gamma\alpha^i\beta = 0$ for $i = 0, 1, 2, \ldots$. Substitute (3.2), (3.3), and Δ^{dn} into Lemma 3.1 to obtain $(QP)^d$. By Lemma 2.4 $M^d = P(QP)^{2d}Q$ and a routine computation we get the expression of M^d as shown in Theorem 3.2. □

We now analyze some special cases of the preceding theorem, some of which give results of [7, 13, 14, 17, 25].

Corollary 3.3 *Let $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ be an operator matrix with A and D generalized Drazin invertible. If $A(BC)^\pi = 0$, $(BC)^\pi BCA = 0$, $BD = 0$, then M is generalized Drazin invertible, and*

$$M^d = \begin{pmatrix} 0 & 0 \\ 0 & D^d \end{pmatrix} + \begin{pmatrix} 0 \\ DD^d \end{pmatrix} \Psi \begin{pmatrix} I & 0 \\ 0 & B \end{pmatrix} + \begin{pmatrix} 0 \\ D\Psi \end{pmatrix} \Delta^d \begin{pmatrix} I & 0 \\ 0 & B \end{pmatrix} + \begin{pmatrix} A & I \\ C & 0 \end{pmatrix} \Delta^{2d} \begin{pmatrix} I & 0 \\ 0 & B \end{pmatrix},$$

where

$$\Psi = \sum_{k=0}^{\infty} \begin{pmatrix} D^k D^\pi C & 0 \end{pmatrix} \Delta^{d(k+2)} + \sum_{k=0}^{\infty} \begin{pmatrix} D^{d(k+2)}C & 0 \end{pmatrix} \Delta^k \Delta^\pi - \begin{pmatrix} D^d C & 0 \end{pmatrix} \Delta^d,$$

$$\Delta^{dn} = \sum_{i=0}^{\infty} \begin{pmatrix} 0 & (BC)^\pi A (BC)^d \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & (BC)^d \\ I & -A(BC)^d \end{pmatrix}^{2i+n} + \begin{pmatrix} 0 & (BC)^d \\ BC(BC)^d & -BC(BC)^d A (BC)^d \end{pmatrix}^n$$

for $n \geq 1$ and $\Delta = \begin{pmatrix} A & I \\ BC & 0 \end{pmatrix}$.

Proof The result can be deduced by routine computations. □

The corollary above relaxes and removes Theorem 2.3 of [25], in which Mosić consider the conditions $BD = 0$, $A(BC)^\pi = 0$, $C(BC)^\pi = 0$, and $(BC)^\pi B = 0$.

In [7, 13, 14, 17], expressions of the GD-inverse of M are given under the following conditions:

1. $BC = 0, BD = 0$, and $DC = 0$ (see [14]);
2. $BC = 0, BD = 0$ (see [13]);

3. $BCA = 0, BD = 0, DC = 0$ (see [7]);

4. $BCA = 0, BD = 0,$ and BC is nilpotent (see [7]);

5. $BD^d = 0,\ BD^iC = 0$ for $i = 0, 1, ..., n-1$ (see [17]).

Theorem 3.2 relaxes some conditions in each item of (1)–(5) and gives a unified generalization of [14, Theorem 5.3], [13, Theorem 2], [7, Theorem 4.4], [7, Theorem 4.2], and [17, Theorem 3.2.1]

We conclude this paper with some remarks. Using a way similar to Theorem 3.2 we can give an expression of the generalized Drazin inverse M^d under the following condition:

$$(BC)^\pi A(BC)^d = 0, \quad ABC(BC)^\pi = 0, \quad BD^d = 0, \quad BD^iC = 0 \ \ \forall i \geq 1,$$

which gives a unified generalization of [14, Theorem 5.3], [13, Theorem 2], [11, Theorem 1], [17, Theorem 3.2.5], and [25, Theorem 2.5].

Moreover, by using [17, Theorem 3.2.4] instead of Lemma 3.1 and by using the similar argument we can give expressions of the generalized Drazin inverse M^d under the following conditions, respectively:

1. $(BC)^dA(BC)^\pi = 0,\ (BC)^\pi BCA = 0,\ D^dC = 0,$ and $BD^iC = 0 \ \ \forall i \geq 1;$

2. $(BC)^\pi A(BC)^d = 0,\ ABC(BC)^\pi = 0,\ D^dC = 0,$ and $BD^iC = 0 \ \ \forall i \geq 1.$

These give a unified generalization of [14, Theorem 5.3], [13, Theorem 3], [7, Theorem 4.4, Theorem 4.5], [17, Theorem 3.2.3, Theorem 3.2.6], [25, Theorem 2.4, Theorem 2.6], and [11, Theorem 1–3].

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