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The sharpening Hölder inequality via abstract convexity

Gültekin TINAZTEPE*
Vocational School of Technical Sciences, Akdeniz University, Antalya, Turkey

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Abstract: In this work, a new inequality by sharpening the well-known Hölder inequality by means of a theorem based on abstract convexity is derived.

Key words: Abstract convexity, functional inequalities, Hölder inequality, global optimization

1. Introduction

The applications of abstract convexity in different areas are known (see [2, 3, 4, 5, 6, 7, 8, 9]). One of them is the application to inequality theory. For instance, for different function classes, Hermite–Hadamard type inequalities were derived by different authors in [2, 3, 4, 8]. Another application of abstract convexity to inequality theory is to sharpen known inequalities in [7] and [1]. In [1], the sharper versions for well-known inequalities among the generalized arithmetic, geometric, and harmonic means is given by using abstract convexity, and it is shown that the presented sharping scheme does not derive a sharper inequality for every inequality satisfying related conditions, such as, for example, Cauchy–Schwarz and Minkowski inequalities.

In this paper, the Hölder inequality is studied and investigated in the frame of abstract convexity in light of [1]. Sharper inequality for the Hölder inequality is derived, and also by using this result, we present sharper inequality for the Cauchy–Schwarz inequality.

The structure of the paper is as follows: in the second section, certain concepts of abstract convexity, an important theorem to be applied to optimization theory and the Hölder inequality, are given. In the third section, the Hölder inequality is considered, results are presented as theorems, and, also by using this result, sharper inequality is given for the Cauchy–Schwarz inequality.

We shall use the following notations:

R is the real line; $R_{+\infty} := R \cup \{+\infty\}\,; R_{-\infty} := R \cup \{-\infty\}\,; \bar{R} := R \cup \{-\infty, +\infty\}\,;$

R^n is an n-dimensional Euclidean space;

R^n_+ is the set of points with nonnegative coordinates;

R^n_{++} is the set of points with strictly positive coordinates;

X is a Hilbert space with the inner product $[.,.]$ and the norm $\|x\| = \sqrt{[x,x]}$;

$B(y,r) = \{x \in X : \|x - y\| \leq r\}\,;$

*Correspondence: gtinaztepe@akdeniz.edu.tr
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If $f : \Omega \to \bar{R}$, then $domf := \{x \in \Omega : -\infty < f(x) < +\infty\}$;

If $f : \Omega \to \bar{R}$ and $g : \Omega \to \bar{R}$, then $f \leq g$ means that $f(x) \leq g(x)$ for all $x \in \Omega$.

2. Preliminaries

2.1. Abstract convexity, abstract concavity, and an application to optimization theory

Let Ω be a set and H be a set of functions $h : \Omega \to R_{-\infty}$. A function $f : \Omega \to R_{+\infty}$ is called abstract convex with respect to H (or $H-convex$) if there exists a set $U \subset H$ such that

$$f(x) = \sup_{h \in U} h(x)$$

for all $x \in \Omega$.

Let H be a set of functions $h : \Omega \to R_{+\infty}$. A function $f : \Omega \to R_{-\infty}$ is called abstract concave with respect to H (or $H-concave$) if there exists a set $U \subset H$ such that

$$f(x) = \inf_{h \in U} h(x)$$

for all $x \in \Omega$.

The set H is called the set of elementary functions.

Let X be Hilbert space, and let $\Omega \subset \Omega' \subset X$, $f : \Omega \to R_{+\infty}$ and $x_0 \in domf$ and L be a set of functions $l : \Omega' \to R_{-\infty}$. An element $l \in L$ is called an L-subgradient of f at the point x_0 if $x_0 \in doml$ and

$$f(x) \geq f(x_0) + l(x) - l(x_0).$$

The set $\partial_L f(x_0)$ of all L-subgradients of f at x_0 is referred to as the L-subdifferential of f at x_0.

If $f : \Omega \to R_{+\infty}$ is a lower semicontinuous convex function and $x \in domf$, then $\partial_L f(x) = \partial f(x)$, where $\partial f(x)$ is the subdifferential in the sense of convex analysis.

Let H be the set of all quadratic functions h of the form

$$h(x) = a\|x\|^2 + [l, x] + c, \quad x \in X$$

where $a > 0$, $l \in X$ and $c \in R$. We say that a function $f : \Omega \to R_{-\infty}$ is majorized by H if there exists $h \in H$ such that $h \geq f$.

Let $\Omega \subset X$ and let H be the set of quadratic functions. Then a function $f : \Omega \to R_{-\infty}$ is $H-concave$ if and only if f is majorized by H and f is upper semicontinuous (see [6]).

The following result holds (see [7]).

Proposition 1 *Let $\Omega \subset X$ be a convex set and let f be a differentiable function defined on an open set containing Ω. Assume that the mapping $x \mapsto \nabla f(x)$ is Lipschitz continuous on Ω:*

$$K := \sup_{x,y \in X, x \neq y} \frac{\|\nabla f(x) - \nabla f(y)\|}{\|x - y\|} < +\infty.$$

Let $a \geq K$. For each $t \in \Omega$, consider the function

$$f_t(x) = f(t) + [\nabla f, x - t] + a\|x - t\|^2, \quad x \in X.$$

Then $f(x) = \min_{t \in \Omega} f_t(x)$, $x \in \Omega$.

From [7], consider the global minimization of a function f over a convex set that can be represented as the infimum of a family $(f_t)_{t\in T}$ of convex functions and derive necessary and sufficient (or only sufficient) conditions for the global minimum.

In the simplest case of the unconstrained minimization of a function $f : X \to R$ such that $\|\nabla f(x) - \nabla f(y)\| \leq a\|x-y\|$ for all $x, y \in X$, the following result is obtained: if a point x^* is a global minimizer of f over X, then

$$f(x) - f(x^*) \geq \frac{1}{4a}\|\nabla f(x)\|^2 \tag{1}$$

for all $x \in X$.

The following theorem, which gives the more general case of inequality (1), is proved in [7].

Theorem 1 *Consider an n-dimensional space $\mathbb{R}^n$ with norms $\|.\|$ and $\|.\|_\circ$. Let $\Omega \subset \mathbb{R}^n$ be a set with int $\Omega \neq \emptyset$ and let $f \in C^1(\Omega)$. Assume that the mapping $x \longmapsto \nabla f(x)$ is Lipschitz on Ω:*

$$K := \sup_{\substack{x,y\in\Omega \\ x\neq y}} \frac{\|\nabla f(x) - \nabla f(y)\|}{\|x-y\|} < \infty.$$

Let $x^* \in$ int Ω be a global minimizer of f over Ω. Consider the ball

$$B_\circ(x^*, r) = \{x : \|x - x^*\|_\circ \leq r\} \subset int\Omega$$

and let

$$M := \max\{\|\nabla f(x)\|_\circ : x \in B_\circ(x^*, r)\}.$$

Let $q > 0$ be a number such that $B_\circ(x^*, r+q) \subset \Omega$ and let $a \geq \max\left(K, \dfrac{M}{2q}\right)$. Then

$$\frac{1}{4a}\|\nabla f(x)\|^2 \leq f(x) - f(x^*),\ x \in B_\circ(x^*, r).$$

2.1.1. Hölder mean

Let $x_1, x_2, ...x_n$ and $y_1, y_2, ..., y_n$ be positive numbers and $p, q > 1$ be such that $\frac{1}{p} + \frac{1}{q} = 1$. Then

$$\sum_{i=1}^{n} x_i y_i \leq \left(\sum_{i=1}^{n} x_i^p\right)^{\frac{1}{p}} \left(\sum_{i=1}^{n} y_i^q\right)^{\frac{1}{q}}.$$

Equality occurs if and only if $\dfrac{x_1^p}{y_1^q} = \dfrac{x_2^p}{y_2^q} = \cdots = \dfrac{x_n^p}{y_n^q}$.

3. Main results

Many inequalities can be represented in the form $f(x) \geq 0$, where f is a certain function. We say that the inequality $f(x) \geq u(x)$ with $u(x) \geq 0$ is sharper than the inequality $f(x) \geq 0$ if there exists x with $u(x) > 0$.

Certain conditions for the global minimum can be used for sharpening some special inequalities. Using the optimality conditions that were obtained with the help of abstract convexity in the previous section, we will study the Hölder inequality in terms of sharpening.

Theorem 2 *Let* $\lambda > r > 0,\ y \in R^n_{++},\ \mathbf{y} = y^{\frac{q}{p}} = (y_1^{\frac{q}{p}}, y_2^{\frac{q}{p}}, \cdots, y_n^{\frac{q}{p}}) \in R^n_{++}$ *and*

$$a_{\lambda',r} = \min_{r<d<\lambda'} \max\left\{ \left(\sum_{i=1}^n y_i^q\right)^{\frac{1}{q}} \left(\sum_{i=1}^n s_i^2\right)^{\frac{1}{2}}, \frac{M_0}{2(d-r)} \right\}$$

where

$$s_i = \frac{p(\lambda y_i^{\frac{q}{p}} + d)^{p-2}}{q\left(\sum\limits_{k=1}^n (\lambda y_k^{\frac{q}{p}} - d)^q\right)^{1+\frac{1}{q}}} \times \left[\left(\sum_{\substack{k=1\\k\neq i}}^n (\lambda y_k^{\frac{q}{p}} + d)^p \right)^2 + (\lambda y_i^{\frac{q}{p}} + d)^2 \sum_{\substack{k=1\\k\neq i}}^n (\lambda y_k^{\frac{q}{p}} + d)^{2p-2} \right]^{\frac{1}{2}},$$

$$M_0 = \max_{1\leq i\leq n} \left\{ \left| \left(\frac{\sum\limits_{i=1}^n y_i^q}{\sum\limits_{i=1}^n (\lambda y_i^{\frac{q}{p}} - r)^p} \right)^{\frac{1}{q}} (\lambda y_i^{\frac{q}{p}} + r)^{p-1} - y_i \right| \right\},\ \lambda' = \min_{i\in\{1,2,\dots,n\}} \left\{ \lambda y_i^{\frac{q}{p}} \right\}.$$

Then for all $x \in \mathbb{R}^n_{++}$ *such that* $\|x - \lambda\mathbf{y}\|_\infty \leq r$ *it holds that:*

$$\sum_{i=1}^n x_i y_i + \frac{1}{4a_{\lambda',r}} \sum_{i=1}^n \left(\left(\frac{\sum\limits_{i=1}^n y_i^q}{\sum\limits_{i=1}^n x_i^p} \right)^{\frac{1}{q}} x_i^{p-1} - y_i \right)^2 \leq \left(\sum_{i=1}^n x_i^p\right)^{\frac{1}{p}} \left(\sum_{i=1}^n y_i^q\right)^{\frac{1}{q}}.$$

Proof Let $y \in \mathbb{R}^n_{++}$ and

$$f_y(x) = \left(\sum_{i=1}^n x_i^p\right)^{\frac{1}{p}} \left(\sum_{i=1}^n y_i^q\right)^{\frac{1}{q}} - \sum_{i=1}^n x_i y_i$$

where $p, q > 0$, $\frac{1}{p} + \frac{1}{q} = 1$ and $x = (x_1, x_2, ..., x_n) \in \mathbb{R}^n_{++}$. Then $f_y(x) \geq 0$ and $f_y(x) = 0$ if and only if $x = \lambda\mathbf{y}$ where $\mathbf{y} = (y_1^{\frac{q}{p}}, y_2^{\frac{q}{p}}, \cdots, y_n^{\frac{q}{p}})$, $\lambda > 0$. Thus, the vectors $\lambda\mathbf{y}$ are global minimizers of f over $\mathbb{R}^n_{++}$. We will sharpen the Hölder inequality, applying Theorem 1 to the inequality $f_y(x) \geq 0$. Necessary calculations show that

$$\nabla f_y(x) = \left[\left(\frac{\sum\limits_{i=1}^n y_i^q}{\sum\limits_{i=1}^n x_i^p} \right)^{\frac{1}{q}} x_1^{p-1} - y_1, \left(\frac{\sum\limits_{i=1}^n y_i^q}{\sum\limits_{i=1}^n x_i^p} \right)^{\frac{1}{q}} x_2^{p-1} - y_2, \cdots, \left(\frac{\sum\limits_{i=1}^n y_i^q}{\sum\limits_{i=1}^n x_i^p} \right)^{\frac{1}{q}} x_n^{p-1} - y_n \right].$$

Hence:

$$\|\nabla f(x)\|^2 = \sum_{i=1}^n \left[\left(\frac{\sum\limits_{i=1}^n y_i^q}{\sum\limits_{i=1}^n x_i^p} \right)^{\frac{1}{q}} x_i^{p-1} - y_i \right]^2.$$

Later we will use not only the norm $\|.\| = \|.\|_2$ but also the norm $\|.\|_\infty$. Let us consider the ball

$$\begin{aligned} V_{\lambda,d} &= B_\infty(\lambda\mathbf{y}, d) = \{x \in \mathbb{R}^n : \|x - \lambda\mathbf{y}\|_\infty \leq d\} \\ &= \left\{ x \in \mathbb{R}^n : \lambda y_i^{\frac{q}{p}} - d \leq x_i \leq \lambda y_i^{\frac{q}{p}} + d, i = 1, ..., n \right\} \end{aligned}$$

where $\lambda' = \min_i \left\{ \lambda y_i^{\frac{q}{p}} \right\} > d > 0$. Since $d < \lambda y_i^{\frac{q}{p}}$, it follows that $V_{\lambda',d} \subset \mathbb{R}^n_{++}$. Let $\rho_i(x) = \frac{x_i^{p-1}}{\left(\sum_{i=1}^n x_i^p\right)^{\frac{1}{q}}}$. We need to estimate $\|\nabla \rho_i(x)\|$ for $x \in V_{\lambda',d}$. We have

$$\left|\frac{\partial \rho_i}{\partial x_i}(x)\right| = \left|\frac{p x_i^{p-2} \sum\limits_{\substack{k=1\\k\neq i}}^n x_k^p}{q\left(\sum\limits_{k=1}^n x_k^p\right)^{1+\frac{1}{q}}}\right| \leq \frac{p(\lambda y_i^{\frac{q}{p}} + d)^{p-2} \sum\limits_{\substack{k=1\\k\neq i}}^n (\lambda y_k^{\frac{q}{p}} + d)^p}{q\left(\sum\limits_{k=1}^n (\lambda y_k^{\frac{q}{p}} - d)^q\right)^{1+\frac{1}{q}}}$$

$$\left|\frac{\partial \rho_i}{\partial x_j}(x)\right| = \left|\frac{p(x_i x_j)^{p-1}}{q\left(\sum\limits_{k=1}^n x_k^p\right)^{1+\frac{1}{q}}}\right| \leq \frac{p(\lambda y_i^{\frac{q}{p}} + d)^{p-1}(\lambda y_j^{\frac{q}{p}} + d)^{p-1}}{q\left(\sum\limits_{k=1}^n (\lambda y_k^{\frac{q}{p}} - d)^q\right)^{1+\frac{1}{q}}}$$

and so

$$\|\nabla \rho_i(x)\| \leq \frac{p(\lambda y_i^{\frac{q}{p}} + d)^{p-2}}{q\left(\sum\limits_{k=1}^n (\lambda y_k^{\frac{q}{p}} - d)^q\right)^{1+\frac{1}{q}}} \times \left[\left(\sum_{\substack{k=1\\k\neq i}}^n (\lambda y_k^{\frac{q}{p}} + d)^p\right)^2 + (\lambda y_i^{\frac{q}{p}} + d)^2 \sum_{\substack{k=1\\k\neq i}}^n (\lambda y_k^{\frac{q}{p}} + d)^{2p-2}\right]^{\frac{1}{2}} = s_i. \tag{2}$$

Let $x, z \in V_{\lambda',d}$. Applying the mean value theorem and Cauchy–Schwarz inequality, we conclude that there exist numbers $\theta_i \in (0,1)$, $i = 1, ..., n$ such that

$$\begin{aligned}
\|\nabla f_y(x) - \nabla f_y(z)\| &= \left(\sum_{i=1}^n y_i^q\right)^{\frac{1}{q}} \|[\rho_1(x) - \rho_1(z)], [\rho_2(x) - \rho_2(z)], ..., [\rho_n(z) - \rho_n(z)]\| \\
&= \left(\sum_{i=1}^n y_i^q\right)^{\frac{1}{q}} \left(\sum_{i=1}^n [\rho_i(x) - \rho_i(z)]^2\right)^{\frac{1}{2}} \\
&= \left(\sum_{i=1}^n y_i^q\right)^{\frac{1}{q}} \left(\sum_{i=1}^n [\nabla \rho_i(x + \theta_i(z-x))(x-z)]^2\right)^{\frac{1}{2}} \\
&\leq \left(\sum_{i=1}^n y_i^q\right)^{\frac{1}{q}} \left(\sum_{i=1}^n \|\nabla \rho_i(x + \theta_i(z-x))\|^2 \|x-z\|^2\right)^{\frac{1}{2}} \\
&\leq \left(\sum_{i=1}^n y_i^q\right)^{\frac{1}{q}} \left(\sum_{i=1}^n s_i^2\right)^{\frac{1}{2}} \|(z-x)\|.
\end{aligned}$$

Since $x, z \in V_{\lambda',d}$ it follows that $x + \theta_i(z-x) \in V_{\lambda',d}$ for all i. Applying inequality (2), we conclude that

$$\|\nabla f_y(x) - \nabla f_y(z)\| \leq a_1(\lambda, d)\|x - z\|, \quad x, z \in V_{\lambda',d}$$

where

$$a_1(\lambda, d) = \left(\sum_{i=1}^n y_i^q\right)^{\frac{1}{q}} \left(\sum_{i=1}^n s_i^2\right)^{\frac{1}{2}}.$$

Hence, the mapping $x \to \nabla f(x)$ is Lipschitz continuous on $V_{\lambda',d}$ with the Lipschitz constant $K \leq a_1(\lambda', d)$. We will apply Theorem 1 to a set $\Omega = V_{\lambda',d}$ where $d < \lambda' = \min_i \left\{ \lambda y_i^{\frac{q}{p}} \right\}$ and the global minimizer $x^* = \lambda \mathbf{y}$ of the function f. Assume that the norm $\|.\|_\circ$ that was used in Theorem 1 coincides with $\|.\|_\infty$. Let $r \in (0, d)$ and $q = d - r$. Let us estimate $M = \max\left\{ \|\nabla f_y(x)\|_\infty : x \in V_{\lambda',r} \right\}$ as follows:

$$
\begin{aligned}
M &= \max_{x \in V_{\lambda',r}} \{\|\nabla f(x)\|_\infty\} = \max_{x \in V_{\lambda',r}} \left\{ \max_{1 \leq i \leq n} \left| \left(\frac{\sum_{i=1}^n y_i^q}{\sum_{i=1}^n x_i^p} \right)^{\frac{1}{q}} x_i^{p-1} - y_i \right| \right\} \\
&\leq \max_{1 \leq i \leq n} \left\{ \left| \left(\frac{\sum_{i=1}^n y_i^q}{\sum_{i=1}^n (\lambda y_i^{\frac{q}{p}} - r)^p} \right)^{\frac{1}{q}} (\lambda y_i^{\frac{q}{p}} + r)^{p-1} - y_i \right| \right\} \equiv M_0.
\end{aligned}
$$

Let

$$
a_2(\lambda', d, r) = \frac{M_0}{2(d-r)}
$$

and

$$
a(\lambda', d, r) = \max\{a_1(\lambda', d), a_2(\lambda', d, r)\}.
$$

Note that $\lim_{d \to \lambda' - 0} a(\lambda', d, r) = \lim_{d \to r+0} a(\lambda', d, r) = +\infty$ so the function $d \longmapsto a(\lambda', d, r)$ attends its minimum on the segment (r, λ'). Let $a_{\lambda',r} = \min_{r<d<\lambda'} a(\lambda', d, r)$. Applying Theorem 1 we conclude that

$$
\left(\sum_{i=1}^n x_i^p \right)^{\frac{1}{p}} \left(\sum_{i=1}^n y_i^q \right)^{\frac{1}{q}} \geq \sum_{i=1}^n x_i y_i + \frac{1}{4a_{\lambda',r}} \sum_{i=1}^n \left(\left(\frac{\sum_{i=1}^n y_i^q}{\sum_{i=1}^n x_i^p} \right)^{\frac{1}{q}} x_i^{p-1} - y_i \right)^2 \quad \text{for } x \in V_{\lambda',r}.
$$

□

For $p = q = 2$ in the above theorem we can derive a sharp inequality for the Cauchy–Schwarz inequality:

Corollary 1 *Let* $\lambda > r > 0$, $y \in R^n_{++}$ *and*

$$
a_{\lambda',r} = \min_{r<d<\lambda'} \max \left\{ \left(\sum_{i=1}^n y_i^2 \right)^{\frac{1}{2}} \left(\sum_{i=1}^n s_i^2 \right)^{\frac{1}{2}}, \frac{M_0}{2(d-r)} \right\}
$$

where

$$s_i = \frac{1}{\left(\sum\limits_{k=1}^{n}(\lambda y_k - d)^2\right)^{\frac{3}{2}}} \times \left[\left(\sum_{\substack{k=1\\k\neq i}}^{n}(\lambda y_k + d)^2\right)^2 + (\lambda y_i + d)^2 \sum_{\substack{k=1\\k\neq i}}^{n}(\lambda y_k + d)^2\right]^{\frac{1}{2}},$$

$$M_0 = \max_{1\leq i\leq n}\left\{\left|\left(\frac{\sum\limits_{i=1}^{n} y_i^2}{\sum\limits_{i=1}^{n}(\lambda y_i^2 - r)^2}\right)^{\frac{1}{2}}(\lambda y_i + r) - y_i\right|\right\},\ \lambda' = \min_{i\in\{1,2,\ldots,n\}}\{\lambda y_i\}.$$

Then for all $x \in \mathbb{R}^n_{++}$ *such that* $\|x - \lambda y\|_\infty \leq r$ *it holds that:*

$$\sum_{i=1}^{n} x_i y_i + \frac{1}{4a_{\lambda',r}}\sum_{i=1}^{n}\left(\left(\frac{\sum\limits_{i=1}^{n} y_i^2}{\sum\limits_{i=1}^{n} x_i^2}\right)^{\frac{1}{2}} x_i - y_i\right)^2 \leq \left(\sum_{i=1}^{n} x_i^2\right)^{\frac{1}{2}}\left(\sum_{i=1}^{n} y_i^2\right)^{\frac{1}{2}}.$$

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