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Quadratic eigenparameter-dependent quantum difference equations

Yelda AYGAR*
Department of Mathematics, Faculty of Science, Ankara University, Ankara, Turkey

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Abstract: The main aim of this paper is to construct quantum extension of the discrete Sturm–Liouville equation consisting of second-order difference equation and boundary conditions that depend on a quadratic eigenvalue parameter. We consider a boundary value problem (BVP) consisting of a second-order quantum difference equation and boundary conditions that depend on the quadratic eigenvalue parameter. We present a condition that guarantees that this BVP has a finite number of eigenvalues and spectral singularities with finite multiplicities.

Key words: q-difference equation, Jost solution, spectral analysis, eigenvalue, spectral singularity

1. Introduction

Boundary value problems (BVPs) for difference equations have been intensively studied in the last decade. The modelings of certain problems in engineering, economics, control theory, and other areas of study have led to the rapid development of the theory and also spectral theory of difference equations. Some problems of spectral theory for difference equations were treated by various authors [1, 9, 4, 8, 5]. Furthermore, quantum calculus received a lot of attention, and most of the published work has been interested in some problems of q-difference equations (see [2, 6, 3, 11, 10]). The spectral analysis of eigenparameter-dependent nonselfadjoint difference equations and q-difference equation were studied in [7, 15, 6, 13]. A BVP for the discrete Sturm–Liouville equation consisting of second-order difference equation and boundary conditions that depend on a quadratic eigenvalue parameter was first studied by Koprubasi et al. [13]. We extended their results to the case of quantum difference equations. In this paper, we let $q > 1$ and use the notation $q^{\mathbb{N}_0} := \{q^n : n \in \mathbb{N}_0\}$, where $\mathbb{N}_0$ denotes the set of nonnegative integers. Let us consider the nonselfadjoint BVP consisting of the second-order q-difference equation

$$qa(t)y(qt) + b(t)y(t) + a\left(\frac{t}{q}\right) y\left(\frac{t}{q}\right) = \lambda y(t), \quad t \in q^{\mathbb{N}} \tag{1.1}$$

and the boundary conditions

$$\begin{gathered}(\gamma_0 + \gamma_1\lambda + \gamma_2\lambda^2)y(q) + (\beta_0 + \beta_1\lambda + \beta_2\lambda^2)y(1) = 0,\\ \gamma_0\beta_1 - \gamma_1\beta_0 \neq 0, \quad |\gamma_2| + |\beta_2| \neq 0, \quad \gamma_2 \neq \tfrac{-\beta_1}{a(1)},\end{gathered} \tag{1.2}$$

where $\gamma_i, \beta_i \in \mathbb{C}$, $i = 0, 1, 2$, $\{a(t)\}_{t\in q^{\mathbb{N}_0}}$ and $\{b(t)\}_{t\in q^{\mathbb{N}}}$ are complex sequences, $a(t) \neq 0$ for all $t \in q^{\mathbb{N}_0}$, and λ is a spectral parameter. Differently from other studies in the literature, in this paper, which is one of the articles

*Correspondence: yaygar@science.ankara.edu.tr
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that have applications in mathematics and other disciplines such as medicine, economics, biology, and control theory, we consider the existence of the spectral parameter not only in the q-difference equation but also in the boundary condition for quadratic form. The organization of this paper is as follows: in Section 2, we give the Jost solution, Jost function, Green function, and resolvent of the BVP (1.1)–(1.2). In Section 3, we investigate the eigenvalues and the spectral singularities of this BVP, and we get some properties of the eigenvalues and the spectral singularities of this BVP under the condition

$$\sup_{t\in q^{\mathbb{N}}}\left\{\exp\left[\varepsilon\left(\frac{\ln t}{\ln q}\right)^{\delta}\right](|1-a(t)|+|b(t)|)\right\}<\infty,\quad \varepsilon>0,\quad \frac{1}{2}\leq\delta\leq 1. \tag{1.3}$$

Section 4 contains the main result, where we deal with condition (1.3) for $\delta=1$ and $\delta\neq 1$. For both cases, we prove that the BVP (1.1)–(1.2) has a finite number of eigenvalues and spectral singularities with finite multiplicities. Since the second case is weaker than the first, we have to use a different way for each to prove the theorem.

2. Jost solution and Jost function

Assume (1.3). Then (1.1) has the solution

$$e(t,z)=\alpha(t)\frac{e^{i\frac{\ln t}{\ln q}z}}{\sqrt{\mu(t)}}\left(1+\sum_{r\in q^{\mathbb{N}}}A(t,r)e^{i\frac{\ln r}{\ln q}z}\right),\quad t\in q^{\mathbb{N}_0}, \tag{2.1}$$

for $\lambda=2\sqrt{q}\cos z$, where $z\in\overline{\mathbb{C}}_+:=\{z\in\mathbb{C}:\operatorname{Im}z\geq 0\}$, $\mu(t)=(q-1)t$ for all $t\in q^{\mathbb{N}_0}$ and $\alpha(t)$, $A(t,r)$ are given in terms of $\{a(t)\}$ and $\{b(t)\}$ [2]. Moreover, $A(t,r)$ satisfies

$$|A(t,r)|\leq C\sum_{s\in\left[tq^{\lfloor\frac{\ln r}{2\ln q}\rfloor},\infty\right)\cap q^{\mathbb{N}}}(|1-a(s)|+|b(s)|), \tag{2.2}$$

where $C>0$ is a constant and $\lfloor\frac{\ln r}{2\ln q}\rfloor$ is the integer part of $\frac{\ln r}{2\ln q}$. Therefore, $e(\cdot,z)$ is analytic with respect to z in $\mathbb{C}_+:=\{z\in\mathbb{C}:\operatorname{Im}z>0\}$ and continuous in $\overline{\mathbb{C}}_+$. Using (2.1) and the boundary condition (1.2), we define the function f by

$$\begin{aligned}f(z)&=(\gamma_0+2\sqrt{q}\gamma_1\cos z+4q\gamma_2\cos^2 z)e(q,z)\\&+(\beta_0+2\sqrt{q}\beta_1\cos z+\beta_2\cos^2 z)e(1,z).\end{aligned} \tag{2.3}$$

The function f is analytic in $\mathbb{C}_+$, $f(z)=f(z+2\pi)$, and continuous in $\overline{\mathbb{C}}_+$. Similar to the Sturm–Liouville differential equation, the solution $e(\cdot,z)$ and the function f are called the Jost solution and Jost function of (1.1)–(1.2), respectively [14]. Let $\varphi(\lambda)=\{\varphi(t,\lambda)\}$ $t\in q^{\mathbb{N}_0}$ be the solution of (1.1) subject to the initial conditions

$$\varphi(1,\lambda)=-(\gamma_0+\gamma_1\lambda+\gamma_2\lambda^2),\quad \varphi(q,\lambda)=(\beta_0+\beta_1\lambda+\beta_2\lambda^2).$$

If we characterize $\phi(t,z)=\varphi(2\sqrt{q}\cos z)=\{\varphi(t,2\sqrt{q}\cos z)\}_{t\in q^{\mathbb{N}_0}}$, then ϕ is an entire function and $\phi(z)=\phi(z+2\pi)$. Let us take the semistrips $P_0=\{z\in\mathbb{C}_+:0\leq\operatorname{Re}z\leq 2\pi\}$ and $P=P_0\cup[0,2\pi]$. For all $z\in P$ with

$f(z) \neq 0$, we define

$$G_{t,z}(z) := \begin{cases} -\frac{\phi(r,z)e(t,z)}{qa(1)f(z)}, & r = tq^{-k}, \quad k \in \mathbb{N}_0 \\ -\frac{e(r,z)\phi(t,z)}{qa(1)f(z)}, & r = tq^{k}, \quad k \in \mathbb{N}. \end{cases} \tag{2.4}$$

The function $G_{t,z}$ is called the Green function of the BVP (1.1)–(1.2). It is clear that

$$(Rh)(t) := \sum_{r \in q^{\mathbb{N}}} G(t,r)h(r), \qquad h \in \ell_2(q^{\mathbb{N}}) \tag{2.5}$$

is the resolvent of the BVP (1.1)–(1.2), where $\ell_2(q^{\mathbb{N}})$ is the Hilbert space of complex-valued functions with the inner product

$$\langle f, g \rangle_q := \sum_{t \in q^{\mathbb{N}}} \mu(t) f(t) \overline{g(t)} \quad f, g : q^{\mathbb{N}} \to \mathbb{C}.$$

3. Eigenvalues and spectral singularities

We will denote the set of all eigenvalues and spectral singularities of BVP (1.1)–(1.2) by σ_d and σ_{ss}, respectively. By using (2.4), (2.5), and the definition of the eigenvalues and the spectral singularities, we have [14]:

$$\begin{aligned} \sigma_d &= \left\{ \lambda \in \mathbb{C} : \lambda = 2\sqrt{q}\cos z, \ z \in P_0, \ f(z) = 0 \right\}, \\ \sigma_{ss} &= \left\{ \lambda \in \mathbb{C} : \lambda = 2\sqrt{q}\cos z, \ z \in [0, 2\pi], \ f(z) = 0 \right\} \setminus \{0\}. \end{aligned} \tag{3.1}$$

From (2.1) and (2.3), we get

$$\begin{aligned} f(z) \quad = \quad & \frac{q}{\sqrt{q-1}} \alpha(1)\beta_2 e^{-2iz} + \sqrt{\frac{q}{q-1}} \left[\alpha(q)\gamma_2 + \alpha(1)\beta_1 \right] e^{-iz} \\ & + \frac{1}{\sqrt{q-1}} \left[\alpha(1)\beta_0 + 2q\alpha(1)\beta_2 + \alpha(q)\gamma_1 \right] \\ & + \frac{1}{\sqrt{q(q-1)}} \left[\alpha(q)\gamma_0 + 2\alpha(q)\gamma_2 + q\alpha(1)\beta_1 \right] e^{iz} \\ & + \frac{1}{\sqrt{q-1}} \left[\alpha(q)\gamma_1 + q\alpha(1)\beta_2 \right] e^{2iz} + \alpha(q)\gamma_2 \sqrt{\frac{q}{q-1}} e^{3iz} \\ & + \sum_{r \in q^{\mathbb{N}}} \alpha(1) \frac{q}{\sqrt{q-1}} \beta_2 A(1,r) e^{i\left(\frac{\ln r}{\ln q} - 2\right)z} \\ & + \sum_{r \in q^{\mathbb{N}}} \sqrt{\frac{q}{q-1}} \left[\alpha(1)\beta_1 A(1,r) + \alpha(q)\gamma_2 A(q,r) \right] e^{i\left(\frac{\ln r}{\ln q} - 1\right)z} \\ & + \sum_{r \in q^{\mathbb{N}}} \frac{1}{\sqrt{(q-1)}} \left[\alpha(q)\gamma_1 A(q,r) + \alpha(1)\beta_0 A(1,r) + \alpha(1)2q\beta_2 A(1,r) \right] e^{i\frac{\ln r}{\ln q}z} \\ & + \sum_{r \in q^{\mathbb{N}}} \sqrt{\frac{q}{q-1}} \left[\alpha(q)2\gamma_2 A(q,r) + \alpha(1)\beta_1 A(1,r) + \frac{\gamma_0}{q} A(q,r) \right] e^{i\left(\frac{\ln r}{\ln q} + 1\right)z} \end{aligned}$$

$$+\sum_{r\in q^{\mathbb{N}}}\frac{1}{\sqrt{q-1}}\left[\alpha(q)\gamma_1A(q,r)+q\alpha(1)\beta_2A(1,r)\right]e^{i\left(\frac{\ln r}{\ln q}+2\right)z}$$

$$+\sum_{r\in q^{\mathbb{N}}}\sqrt{\frac{q}{q-1}}\alpha(q)\gamma_2A(q,r)e^{i\left(\frac{\ln r}{\ln q}+3\right)z}.$$

Let us define

$$F(z):=f(z)e^{2iz}, \tag{3.2}$$

and then the function F is analytic in $\mathbb{C}_+$, continuous in $\overline{\mathbb{C}}_+$,

$$\begin{aligned}
F(z) \quad = \quad & \frac{q}{\sqrt{q-1}}\alpha(1)\beta_2+\sqrt{\frac{q}{q-1}}\left[\alpha(q)\gamma_2+\alpha(1)\beta_1\right]e^{iz}\\
&+\frac{1}{\sqrt{q-1}}\left[\alpha(1)\beta_0+2q\alpha(1)\beta_2+\alpha(q)\gamma_1\right]e^{2iz}\\
&+\frac{1}{\sqrt{q(q-1)}}\left[\alpha(q)\gamma_0+2\alpha(q)\gamma_2+q\alpha(1)\beta_1\right]e^{3iz}\\
&+\frac{1}{\sqrt{q-1}}\left[\alpha(q)\gamma_1+q\alpha(1)\beta_2\right]e^{4iz}+\sqrt{\frac{q}{q-1}}\alpha(q)\gamma_2e^{5iz}\\
&+\sum_{r\in q^{\mathbb{N}}}\frac{q}{\sqrt{q-1}}\alpha(1)\beta_2A(1,r)e^{i\frac{\ln r}{\ln q}z}\\
&+\sum_{r\in q^{\mathbb{N}}}\sqrt{\frac{q}{q-1}}\left[\alpha(1)\beta_1A(1,r)+\alpha(q)\gamma_2A(q,r)\right]e^{i\left(\frac{\ln r}{\ln q}+1\right)z}\\
&+\sum_{r\in q^{\mathbb{N}}}\frac{1}{\sqrt{q-1}}\left[\alpha(q)\gamma_1A(q,r)+\alpha(1)\beta_0A(1,r)+2q\alpha(1)\beta_2A(1,r)\right]e^{i\left(\frac{\ln r}{\ln q}+2\right)z}\\
&+\sum_{r\in q^{\mathbb{N}}}\sqrt{\frac{q}{q-1}}\left[2\alpha(q)\gamma_2A(q,r)+\alpha(1)\beta_1A(1,r)+\frac{\alpha(q)}{q\sqrt{q-1}}\gamma_0A(q,r)\right]e^{i\left(\frac{\ln r}{\ln q}+3\right)z}\\
&+\sum_{r\in q^{\mathbb{N}}}\frac{1}{\sqrt{q-1}}\left[\alpha(q)\gamma_1A(q,r)+q\alpha(1)\beta_2A(1,r)\right]e^{i\left(\frac{\ln r}{\ln q}+4\right)z}\\
&+\sum_{r\in q^{\mathbb{N}}}\sqrt{\frac{q}{q-1}}\left[\alpha(q)\gamma_2A(q,r)\right]e^{i\left(\frac{\ln r}{\ln q}+5\right)z},
\end{aligned}$$

and $F(z)=F(z+2\pi)$. It follows from (3.1) and (3.2) that

$$\begin{aligned}\sigma_d&=\left\{\lambda\in\mathbb{C}:\lambda=2\sqrt{q}\cos z,\ z\in P_0,\ F(z)=0\right\},\\ \sigma_{ss}&=\left\{\lambda\in\mathbb{C}:\lambda=2\sqrt{q}\cos z,\ z\in[0,2\pi],\ F(z)=0\right\}\setminus\{0\}.\end{aligned} \tag{3.3}$$

Definition 3.1 *The multiplicity of a zero of F in P is called the multiplicity of the corresponding eigenvalue or spectral singularity of BVP (1.1)–(1.2).*

By using (3.3), we obtain that, in order to investigate the quantitative properties of the sets σ_d and σ_{ss}, we need to discuss the quantitative properties of the zeros of F in P. Let us define

$$M_1:=\{z\in P_0:\ F(z)=0\},\quad M_2:=\{z\in[0,2\pi]:\ F(z)=0\}. \tag{3.4}$$

We also denote the set of all limit points of M_1 by M_3 and the set of all zeros of F with infinite multiplicity in P by M_4. From (3.3) and (3.4), we find that

$$\begin{aligned}\sigma_d &= \left\{\lambda \in \mathbb{C} : \lambda = 2\sqrt{q}\cos z,\ z \in M_1\right\}, \\ \sigma_{ss} &= \left\{\lambda \in \mathbb{C} : \lambda = 2\sqrt{q}\cos z,\ z \in M_2\right\} \backslash \{0\}.\end{aligned} \tag{3.5}$$

Theorem 3.2 *Assume* (1.3). *Then:*

i) the set M_1 is bounded and countable;

ii) $M_1 \cap M_3 = \emptyset$, $M_1 \cap M_4 = \emptyset$;

iii) the set M_2 is compact and the Lebesgue measure of M_2 in the real axis is zero;

iv) $M_3 \subset M_2$, $M_4 \subset M_2$, the Lebesgue measure of M_3 and M_4 is also zero;

v) $M_3 \subset M_4$.

Proof From (2.2) and the definition of F, for all $z \in P$, we have

$$F(z) = \frac{q}{\sqrt{q-1}}\alpha(1)\beta_2 + O(e^{-\operatorname{Im} z}), \quad \beta_2 \neq 0, \quad \operatorname{Im} z \to \infty,$$

and

$$F(z) = \sqrt{\frac{q}{q-1}}\left[\alpha(q)\gamma_2 + \alpha(1)\beta_1\right] e^{iz} + O(e^{-2\operatorname{Im} z}), \quad \beta_2 = 0, \quad \operatorname{Im} z \to \infty.$$

By using these equations, we get that the set M_1 is bounded. Since F is analytic in $\mathbb{C}_+$ and is a 2π-periodic function, we also get that M_1 has at most a countable number of elements. ii)–iv) can be obtained from the boundary uniqueness theorem of analytic functions [12]. We can also easily get v) using the continuity of all derivatives of F on $[0, 2\pi]$. □

Now we can give the following theorem as a result of Theorem 3.2 and (3.5).

Theorem 3.3 *Assume* (1.3). *Then:*

i) the set σ_d is bounded and has at most a countable number of elements, and its limit points can lie only in $[-2\sqrt{q}, 2\sqrt{q}]$;

ii) $\sigma_{ss} \subset [-2\sqrt{q}, 2\sqrt{q}]$ and the Lebesgue measure of the set σ_{ss} in the real axis is zero.

4. Main result

Let us suppose that the complex sequences $\{a(t)\}_{t\in q^{\mathbb{N}_0}}$ and $\{b(t)\}_{t\in q^{\mathbb{N}}}$ satisfy

$$\sup_{t\in q^{\mathbb{N}}}\left\{\exp\left(\varepsilon\frac{\ln t}{\ln q}\right)(|1-a(t)| + |b(t)|)\right\} < \infty, \quad \varepsilon > 0. \tag{4.1}$$

It is clear that (1.3) reduces to (4.1) for $\delta = 1$.

Theorem 4.1 *The BVP (1.1)–(1.2) has a finite number of eigenvalues and spectral singularities, and each of them is of finite multiplicity under condition (4.1).*

Proof It follows from (2.2) and (4.1) that

$$|A(t,r)| \leq C\exp\left(-\frac{\varepsilon}{4}\frac{\ln r}{\ln q}\right), \ t\in\{1,q\}, \quad r\in q^{\mathbb{N}}. \tag{4.2}$$

Using (4.2), we obtain that the function F has an analytic continuation to the half-plane $\operatorname{Im} z > -\frac{\varepsilon}{4}$. Since F is a 2π periodic function, the limit points of its zeros in P cannot lie in $[0,2\pi]$. From Theorem 3.2, we get that the bounded sets M_1 and M_2 have no limit points, i.e. the sets M_1 and M_2 have a finite number of elements. From the analyticity of F in $\operatorname{Im} z > -\frac{\varepsilon}{4}$, we find that all zeros of F in P have finite multiplicity. Consequently, we get the finiteness of the eigenvalues and the spectral singularities of the BVP (1.1)–(1.2). □

In the following, we will assume that

$$\sup_{t\in q^{\mathbb{N}}}\left\{\exp\left[\varepsilon\left(\frac{\ln t}{\ln q}\right)^{\delta}\right](|1-a(t)|+|b(t)|)\right\}<\infty, \quad \varepsilon>0, \quad \frac{1}{2}\leq\delta<1, \tag{4.3}$$

which is weaker than (4.1). It is seen that condition (4.1) guarantees the analytic continuation of F from the real axis to the lower half-plane. Thus, we get the finiteness of the eigenvalues and the spectral singularities of BVP (1.1)–(1.2) as a result of this analytic continuation. It is evident that the function F is analytic in $\mathbb{C}_+$ and infinitely differentiable on the real axis under condition (4.3), but F does not have an analytic continuation from the real axis to the lower half-plane. Therefore, under condition (4.3), the finiteness of the eigenvalues and the spectral singularities of the BVP (1.1)–(1.2) cannot be proved by the same technique used in Theorem 4.1. We will use the following uniqueness theorem for analytic functions in order to prove the next theorem.

Theorem 4.2 (See[9, Lemma 4.4]) *Assume that the 2π-periodic function g is analytic in $\mathbb{C}_+$, all of its derivatives are continuous in $\overline{\mathbb{C}}_+$, and*

$$\sup_{z\in P}|g^{(k)}(z)|\leq\eta_k, \quad k\in\mathbb{N}_0.$$

If the set $G\subset[0,2\pi]$ with Lebesgue measure zero is the set of all zeros of the function g with infinity multiplicity in P, and if

$$\int_0^w \ln t(s)\,\mathrm{d}\mu(G_s)=-\infty,$$

where $t(s)=\inf_{k\in\mathbb{N}_0}\frac{\eta_k s^k}{k!}$ and $\mu(G_s)$ is the Lebesgue measure of the s-neighborhood of G, and $w>0$ is an arbitrary constant, then $g\equiv 0$ in $\overline{\mathbb{C}}_+$.

Lemma 4.3 *If (4.3) holds, then we have*

$$|F^{(k)}(z)|\leq\eta_k, \quad z\in P, \quad k\in\mathbb{N}_0, \tag{4.4}$$

where

$$\eta_k\leq C5^k+Dd^k k!k^{k\left(\frac{1}{\delta}-1\right)}, \tag{4.5}$$

and D and d are positive constants depending on C, ε, and δ.

Proof Using (2.2) and (4.3), we get

$$|A(t,r)| \leq C\exp\left(-\frac{\varepsilon}{4}\left(\frac{\ln r}{\ln q}\right)^{\delta}\right), \quad t\in\{1,q\}, \quad r\in q^{\mathbb{N}}. \tag{4.6}$$

It follows from the definition of F and (4.6) that

$$|F^{(k)}(z)| \leq C5^k + D_k, \quad z\in P, \quad k\in\mathbb{N}_0,$$

where

$$D_k = C6^k \sum_{r\in q^{\mathbb{N}}} \left(\frac{\ln r}{\ln q}\right)^k e^{-\frac{\varepsilon}{4}\left(\frac{\ln r}{\ln q}\right)^{\delta}}, \quad k\in q^{\mathbb{N}_0}.$$

Moreover, we get for D_k

$$\begin{aligned} D_k &= C6^k \sum_{m=1}^{\infty} m^k e^{-\frac{\varepsilon}{4}m^{\delta}} \\ &= C6^k \int_0^n t^k e^{-\frac{\varepsilon}{4}t^{\delta}}\mathrm{d}t \leq C6^k \int_0^{\infty} t^k e^{-\frac{\varepsilon}{4}t^{\delta}}\mathrm{d}t. \end{aligned}$$

If we define $y=\frac{\varepsilon}{4}t^{\delta}$, then we obtain

$$D_k \leq C6^k\left(\frac{4}{\varepsilon}\right)^{\frac{k+1}{\delta}}\frac{1}{\delta}\int_0^{\infty} y^{\frac{k+1}{\delta}-1}e^{-y}\mathrm{d}y.$$

Using the gamma function and the inequalities $\left(1+\frac{1}{k}\right)^{\frac{k}{\delta}} < e^{\frac{1}{\delta}}$, $(k+1)^{\frac{1}{\delta}-1} < e^{\frac{k}{\delta}}$, and $k^k < k!e^k$, we have

$$D_k \leq Dd^k k! k^{k\left(\frac{1}{\delta}-1\right)}, \quad k\in N,$$

where D and d are positive constants depending on ε and δ. □

Lemma 4.4 *Assume* (4.3)*. Then* $M_4=\emptyset$.

Proof Using Theorem 4.2, we can write

$$\int_0^w \ln t(s)\mathrm{d}\mu(M_4,s) > -\infty, \tag{4.7}$$

where $t(s)=\inf_{k\in N_0}\frac{\eta_k s^k}{k!}$, $\mu(M_4,s)$ is the Lebesgue measure of the s-neighborhood of M_4, and η_k is defined by (4.5). Substituting (4.5) into the definition of $t(s)$, we find

$$t(s) = D\exp\left\{-\frac{1-\delta}{\delta}e^{-1}(ds)^{\frac{-\delta}{1-\delta}}\right\}. \tag{4.8}$$

It follows from (4.7) and (4.8) that

$$\int_0^w s^{-\frac{\delta}{1-\delta}} \mathrm{d}\mu(M_4, s) < \infty.$$

The last inequality holds for arbitrary s if and only if $\mu(M_4, s) = 0$, i.e. $M_4 = \emptyset$. This completes the proof. □

Theorem 4.5 *Assume* (4.3). *Then the BVP* (1.1)–(1.2) *has a finite number of eigenvalues and spectral singularities, and each of them is of finite multiplicity.*

Proof To be able to prove this, we have to show that the function F has a finite number of zeros with finite multiplicities in P. Using Theorem 3.2 and Lemma 4.4, we obtain that $M_3 = \emptyset$. Thus, the bounded sets M_1 and M_2 have no limit points, i.e. the function F has only a finite number of zeros in P. Since $M_4 = \emptyset$, these zeros are of finite multiplicity. □

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