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# On the finite $p$-groups with unique cyclic subgroup of given order

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**Abstract:** In this paper, we prove that if $G$ is nonabelian and $|G| > p^4$, then $G$ has a unique cyclic subgroup of order $p^m$ with $m \geq 3$ if and only if $G$ has a unique abelian subgroup of order $p^3$ if and only if $G$ is a 2-group of maximal class.

**Key words:** $p$-group of maximal class, extra-special $p$-group

## 1. Introduction

All groups considered in this paper are finite $p$-groups. The terminology and the notation in this paper are standard. The Frattini subgroup, the commutator subgroup, and the center of a group $G$ will be denoted by $\Phi(G)$, $G'$, and $Z(G)$ respectively. We use $c(G)$ and $G_i$ to denote the nilpotent class and the $i$th term of the lower central series of $G$, respectively. The number of subgroups of order $p^m$, abelian subgroups of order $p^m$, and cyclic subgroups of order $p^m$ are denoted by $s_m(G)$, $a_m(G)$, and $c_m(G)$, respectively. For a subgroup $H$ in $G$, the centralizer of $H$ in $G$ is denoted by $C_G(H)$.

There is much interest in investigating the structure of a group whenever the number of some kind of subgroups is given. For example, finite $p$-groups with exactly one minimal nonabelian subgroup of given structure of order $p^3$ are classified by [5]. In [3], finite $p$-groups with exactly one minimal nonabelian subgroup of index $p$ are investigated. In this paper, we are interested in the finite $p$-groups with unique cyclic subgroup of given order.

In [1] and [2], the authors proved that

**Theorem 1** ([1]) *Suppose that a 2-group $G$ is neither cyclic nor of maximal class. If $n > 1$, then $c_n(G)$ is even.*

**Theorem 2** ([2]) *Let $G$ be a noncyclic $p$-group, $p > 2$, and $n > 0$. If $n > 1$, then $p$ divides $c_n(G)$.*

By the above two theorems, we see that finite $p$-groups with unique cyclic subgroup of order $p^m$ are cyclic groups or 2-groups of maximal class. In this paper, we give a direct elementary proof. Moreover, we proved the following theorem:

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**Theorem 3** *Let $G$ be a nonabelian $p$-group and $|G| > p^4$. Then $c_m(G) = 1$ if and only if $a_3(G) = 1$ if and only if $G$ is 2-group of maximal class, where $m \geq 3$.*

## 2. Preliminaries

We gathered all the results used in what follows.

**Lemma 4** ([4] Theorem 2.2.13) *Let $G$ be a finite $p$-group with unique subgroup of order $p$. Then*

(i) *$G$ is cyclic if $p > 2$;*

(ii) *$G$ is a cyclic group or generalized quaternion group when $p = 2$.*

**Lemma 5** ([4] Exercise 2.2.3) *Let $G$ be a nonabelian $p$-group. Then the number of abelian maximal subgroups of $G$ is 0, 1, or $1 + p$.*

**Lemma 6** ([4] Theorem 2.7.1) *Let $G$ be a $p$-group with $|G| = p^n$ and $1 < m < n$. If $s_m(G) = 1$, then $G$ is cyclic.*

**Lemma 7** ([4] Theorem 2.7.2) *Let $G$ be a $p$-group with $|G| = p^n$ and $1 < m \leq n$. If $s_m(G) = c_m(G)$, then*

*(1) $G$ is cyclic if $p^m \neq 4$;*

*(2) $G$ is a cyclic group or generalized quaternion group if $p^m = 4$.*

**Lemma 8** ([4] Theorem 2.5.2) *Let $G$ be a $p$-group of maximal class with $|G| = p^n$. Then*

*(1) $G_i$ is the unique normal subgroup of order $p^{n-i}$;*

*(2) If $p > 2$ and $n > 3$, then $G$ does not have a cyclic normal subgroup of order $p^2$.*

**Lemma 9** ([4] Theorem 2.5.5) *$G$ is a 2-group of maximal class if and only if $|G : G'| = 4$.*

**Lemma 10** ([4] Theorem 2.5.6) *Let $G$ be a nonabelian $p$-group with $p > 2$. If $G$ has an abelian maximal subgroup, then $G$ is of maximal class if and only if $|G : G'| = p^2$.*

**Lemma 11** ([4] Theorem 2.5.7) *Let $G$ be a nonabelian $p$-group. If $G$ has a subgroup $A$ of order $p^2$ such that $C_G(A) = A$, then $G$ is of maximal class.*

**Lemma 12** ([4] Theorem 7.1.6) *Let $G$ be an extra-special $p$-group. Then $|G| = p^{2m+1}$ for some integer $m$.*

## 3. Finite $p$-groups with $c_m(G) = 1$

Firstly, we have the following Lemmas.

**Lemma 13** *Let $G$ be a metacyclic $p$-group with $p > 2$ and $H \leq G$. Then $H$ is abelian if $|H| \leq |G/G'|$ and $H$ is nonabelian if $|H| > |G/G'|$.*

**Proof** Suppose $G = \langle a\rangle\langle b\rangle$ and $G/\langle a\rangle \cong \langle\bar{b}\rangle$. For any subgroup $H$ of $G$, assume that $H/H\cap\langle a\rangle \cong H\langle a\rangle/\langle a\rangle = \langle\bar{b}^{p^j}\rangle$ and $H \cap \langle a\rangle = \langle a^{p^i}\rangle$. Thus, $|H| = |G|/p^{i+j}$ and $H = \langle a^{p^i}, b^{p^j}\rangle$. Since $p > 2$, we see that $[a^{p^i}, b^{p^j}] = 1$ if and only if $[a,b]^{p^{i+j}} = 1$. Hence, $H' = 1$ if and only if $p^{i+j} \geq |G'|$. By $|H| = |G|/p^{i+j}$, we see $p^{i+j} = |G|/|H|$. Then $H$ is abelian if and only if $|H| \leq |G/G'|$. □

**Lemma 14** *Let $G$ be a $p$-group with $|G| \geq p^4$. Then $G$ has an abelian normal subgroup of order $p^3$.*

**Proof** Take a subgroup $N$ of order $p^2$ that is normal in $G$. Since "$N/C$", $G/C_G(N) \lesssim \mathrm{Aut}(N)$. Thus, $|G/C_G(N)||p$, $|C_G(N)| \geq p^3$. By $C_G(N) \trianglelefteq G$, we see that there exists a normal subgroup $M$ such that $N \leq M \leq C_G(N)$ and $|M| = p^3$. Then $M = \langle N, c\rangle$, where $c \in C_G(N)$. Thus, $M$ is abelian. Therefore, $M$ is desired. □

**Proposition 15** *Let $G$ be a finite $p$-group and $n > 3$. If $c_n(G) = 1$, then $c_3(G) = 1$.*

**Proof** Suppose, by way of contradiction, that $C_{p^n} \cong \langle a\rangle \leq G$, $C_{p^3} \cong \langle x\rangle \leq G$, and $x \notin \langle a\rangle$. Since $\langle a^p\rangle char \langle a\rangle \trianglelefteq G$, $\langle a^p\rangle \trianglelefteq \langle a\rangle\langle x\rangle$. By $\langle a\rangle/\langle a^p\rangle \trianglelefteq \langle a, x\rangle/\langle a^p\rangle$ and $o(\langle a\rangle/\langle a^p\rangle) = p$, we see $[a, x] \in \langle a^p\rangle$. Thus, we may assume $[a, x] = a^{ip}$. If $p > 2$, then

$$(ax^{-1})^{p^3} = a^{p^3}[a, x]^{\binom{p^3}{2}}[a, x, x]^{\binom{p^3}{3}} ... x^{-p^3} = a^{vp^3}$$

with $(v, p) = 1$. Hence, $C_{p^n} \cong \langle ax^{-1}\rangle \neq \langle a\rangle$, a contradiction. Now, assume $p = 2$. If $x^2 \notin \langle a\rangle$, then it is easy to see $C_{2^n} \cong \langle ax^{-2}\rangle \neq \langle a\rangle$. When $2|i$, there exists $\langle ax^{-1}\rangle$ such that $C_{2^n} \cong \langle ax^{-1}\rangle \neq \langle a\rangle$. For $x^2 \in \langle a\rangle$ and $(2, i) = 1$, setting $x^2 = a^{j2^{n-2}}$ with $(j, 2) = 1$, $1 = [a^{j2^{n-2}}, x] = [a, x]^{j2^{n-2}} = a^{ij2^{n-1}}$. Thus, $a^{2^{n-1}} = 1$, a contradiction. The proof is complete. □

**Proposition 16** *Let $G$ be a nonabelian $p$-group with $|G| \geq p^4$. If $c_3(G) = 1$, then $a_3(G) = 1$.*

**Proof** Assume the contrary; there exists a subgroup $N$ such that $N \cong C_{p^2} \times C_p$ or $C_p \times C_p \times C_p$. Suppose the unique cyclic subgroup of order $p^3$ is $M = \langle a\rangle$. Now we divide our analysis into two cases: (1) $N \cong C_{p^2} \times C_p$ and (2) $N \cong C_p \times C_p \times C_p$.

Case 1: $C_{p^2} \times C_p \cong N = \langle x\rangle \times \langle y\rangle$.

Since $\langle a^p\rangle$ *char* $\langle a\rangle \trianglelefteq G$, we may assume that $[a, x] = a^{ip}$ and $[a, y] = a^{jp}$ for some integers $i$ and $j$.

(1.1) $x \in \langle a\rangle$. We see $y \notin \langle a\rangle$ from $y \notin \langle x\rangle$. If $p|j$, then $(ay^{-1})^p = a^p[a, y]^{\binom{p}{2}}y^{-p} = a^{pv}$ with $(v, p) = 1$. Thus, $C_{p^3} \cong \langle ay^{-1}\rangle \neq \langle a\rangle$. Therefore, $(p, j) = 1$. By $x \in \langle a\rangle$, we may assume $x = a^{rp}$ with $(r, p) = 1$. Then

$$1 = [x, y] = [a^{pr}, y] = [a, y]^{pr} = a^{p^2jr} \neq 1,$$

a contradiction.

(1.2) Let $x \notin \langle a\rangle$. Since $N = \langle x\rangle \times \langle y\rangle = \langle x^{-1}y\rangle \times \langle y\rangle$, we may assume that $x^{-1}y \notin \langle a\rangle$ by (1.1). If $x^p \notin \langle a\rangle$, then $C_{p^3} \cong \langle ay^{-p}\rangle \neq \langle a\rangle$. Therefore, $x^p = a^{kp^2}$ with $(k, p) = 1$. It is easy to see $(ij, p) = 1$ from $c_3(G) = 1$. If $p > 2$, then

$$1 = [a, x^p] = [a, x]^p[a, x, x]^{\binom{p}{2}} = a^{ip^2},$$

which contradicts $o(a) = p^3$. When $p = 2$,

$$(ax^{-1}y)^2 = a^2[a, xy]x^2 = a^2[a, y][a, x][a, x, y]x^2 = a^2a^{2(i+j)}a^{4ij}a^{4k} = a^{2v}$$

with $(v, 2) = 1$. Thus, $C_{p^3} \cong \langle ax^{-1}y\rangle \neq \langle a\rangle$, a contradiction.

Case 2: $C_p \times C_p \times C_p \cong N = \langle x \rangle \times \langle y \rangle \times \langle z \rangle$.

Since $|M \cap N| \leq p$, we may assume that $y, z \notin \langle a \rangle$ and $yz \notin \langle a \rangle$. By $\langle a^p \rangle \trianglelefteq MN$, $[a, y] = a^{jp}$ and $[a, z] = a^{kp}$. It is easy to see that $(jk, p) = 1$. If $p > 2$, then $C_{p^3} \cong \langle ay^{-1} \rangle \neq \langle a \rangle$. Therefore, $p = 2$. However, there exists $ayz$ such that $(ayz)^2 = a^2[a, yz] = a^2 a^{2(k+j)} a^{4kj}$. Hence, $\langle ayz \rangle \cong \langle a \rangle$, a contradiction. □

**Proposition 17** *Let $G$ be a $p$-group with $|G| \geq p^4$. Then $a_3(G) = 1$ and $c_3(G) = 1$ if and only if $G$ is a cyclic group or $2$-group of maximal class.*

**Proof** If $G$ is abelian, then $G$ is cyclic from Lemma 6. Now assume $G$ is nonabelian. If there exists a subgroup $A \cong C_p \times C_p$, then $C_G(A) = A$ from $a_3(G) = 1$ and $c_3(G) = 1$. Thus, we see $G$ is a $p$-group of maximal class by Lemma 11. If all the subgroups of order $p^2$ are cyclic, then it follows from Lemma 7 that $G$ is a generalized quaternion group, which is also of maximal class. Hence, $G$ is a 2-group of maximal class from Lemma 8.

Since $G$ is a 2-group of maximal class, it is easy to check that $c_3(G) = 1$ and $a_3(G) = 1$. □

**Proposition 18** *Let $G$ be a $p$-group with $|G| \geq p^4$. Then $a_3(G) = 1$ and $c_3(G) = 0$ if and only if $G$ is of maximal class of order $p^4$ with $p > 2$.*

**Proof** If $G$ is of maximal class of order $p^4$, then $d(G) = 2$. Therefore, the number of maximal subgroups is $1 + p$. By Lemma 14, we see that $a_3(G) \geq 1$. It follows that $a_3(G) = 1$ or $1 + p$ from Lemma 5. If $a_3(G) = 1 + p$, then $G$ is minimal nonabelian and $c(G) = 2$, a contradiction. Therefore, $a_3(G) = 1$ and then $c_3(G) = 0$ or 1. If $c_3(G) = 1$, then there exists a cyclic normal subgroup of order $p^2$, which contradicts Lemma 8. Thus, $c_3(G) = 0$.

Conversely, we see that $G$ is nonabelian by Lemma 6. First we prove that the groups of order $p^4$ satisfying $a_3(G) = 1$ and $c_3(G) = 0$ are $p$-groups of maximal class with $p > 2$. In this case $|Z(G)| = p$. If not, $|Z(G)| = p^2$. We see $G/Z(G) \cong C_p \times C_p$ from $G$ is nonabelian. Then the number of abelian subgroups of order $p^3$ containing $Z(G)$ is $1 + p$, a contradiction. By Lemmas 9 and 10, we need to prove $|G'| = p^2$. Assume that $|G'| = p$. If $d(G) = 2$, then $G$ is a minimal nonabelian $p$-group. Hence, $|Z(G)| = p^2$, which is impossible. Therefore, $d(G) = 3$ and $G' = \Phi(G)$; therefore, $G$ is an extra-special $p$-group. Again, we have a contradiction, Lemma 12, because $|G| = p^4$. Thus, $|G'| = p^2$ and $G$ is of maximal class. Now, if $G$ is a 2-group of maximal class, then the abelian subgroup of order $p^3$ is cyclic. Therefore, $p > 2$.

Next, noting that the property is inherited by subgroups, we only need to prove that any group of order $p^5$ $(p > 2)$ does not satisfy $a_3(G) = 1$ and $c_3(G) = 0$. If there exists a group $G$ of order $p^5$ that satisfies the property, then for each maximal subgroup $M$ of $G$, $M$ has an abelian subgroup of order $p^3$ by Lemma 14. Thus, $M$ satisfies $a_3(G) = 1$ and $c_3(G) = 0$. Thus, $M$ is of class 3 by the above paragraph. Therefore, $c(G) = 3$ or 4.

Case (i) $c(G) = 3$. If $Z(G) \geq p^2$, then there exists $A$ such that $|A| = p^2$ and $A \leq Z(G)$. By the hypothesis, $G/A$ has the unique subgroup of order $p$, and $p > 2$. Thus, $G/A$ is cyclic since Lemma 4. Therefore, $G$ is abelian, a contradiction, and so $|Z(G)| = p$, $|G_3| = p$. By Lemma 14, we see $d(G) = 2$. Write $\bar{G} = G/G_3$. Then $|(\bar{G})'| = p$ or $p^2$. If $|(\bar{G})'| = p^2$, then $\bar{G}$ is of maximal class by Lemma 10, which contradicts

$c(G) = 3$. Thus, $|(\bar{G})'| = p$. If $\bar{G}$ is metacyclic, then $G$ is metacyclic. We may get a contradiction from Lemma 13 and $a_3(G) = 1$. Therefore,

$$\bar{G} \cong M_p(2,1,1) = \langle \bar{a}, \bar{b}, \bar{c} | \bar{a}^{p^2} = \bar{b}^p = \bar{c}^p = 1, [\bar{a}, \bar{b}] = \bar{c} \rangle.$$

Assume $G_3 = \langle x \rangle \cong C_p$. Since $c_3(G) = 0$, $a^{p^2} = 1$. Thus, $\langle a, x \rangle \cong C_{p^2} \times C_p$. By $Z(G) = G_3$, we see $a^p \notin Z(G)$. Since $[a^p, c] = [a, c]^p [a, c, a]^{\binom{p}{2}} = 1$, $[a^p, b] = c^p \neq 1$. Thus, $\langle c, a^p \rangle \cong C_{p^2} \times C_p$. However, $\langle c, a^p \rangle \neq \langle a, x \rangle$, a contradiction.

Case (ii) $c(G) = 4$. We see $G_3 \cong C_p \times C_p$ from Lemma 8. Assume that $G_4 = \langle z \rangle$ and $G_3 = \langle z \rangle \times \langle y \rangle$. It is easy to see that $G/G_3 = \langle \bar{a}, \bar{b}, \bar{c} | \bar{a}^p = \bar{b}^p = \bar{c}^p = 1, [\bar{a}, \bar{b}] = \bar{c} \rangle$, and $G = \langle a, b \rangle$. $G' = \langle c, z, y \rangle$ is the unique abelian subgroup of order $p^3$. Since $[\langle a, b \rangle, \langle z, y \rangle] = G_4$, we have $[a, y] = z^i$, $[b, y] = z^j$ and at least one of $i$ and $j$ cannot be divided exactly by $p$. Then $\langle a^{-j} b^i, y, z \rangle$ is another abelian subgroup of order $p^3$ of $G$, a contradiction. The proof is complete. □

By the above propositions, we easily get the following theorem.

**Theorem 19** *Let $G$ be a nonabelian $p$-group with $|G| > p^4$. Then the following conclusions are equivalent:*

*(1) $c_m(G) = 1$ where $m \geq 3$*

*(2) $a_3(G) = 1$*

*(3) $G$ is a 2-group of maximal class.*

**Proof** If (1), then (2) by Propositions 15 and 16. When (2) holds, we see (3) by Propositions 17 and 18. If $G$ is a 2-group of maximal class, then $G$ is isomorphic to one of the following three types of groups by Theorem 2.5.3 in [4]:

(a) $\langle a, b | a^{2^{n-1}} = b^2 = 1, a^b = a^{-1} \rangle$, $n \geq 3$;

(b) $\langle a, b | a^{2^{n-1}} = 1, b^2 = a^{2^{n-2}}, a^b = a^{-1} \rangle, n \geq 3$;

(c) $\langle a, b | a^{2^{n-1}} = b^2 = 1, a^b = a^{-1+2^{n-2}} \rangle, n \geq 4$.

By calculation, we see that cyclic subgroups of order $\geq 2^3$ are in $\langle a \rangle$. Therefore, $c_m(G) = 1$ where $m \geq 3$. □

**Theorem 20** *Let $G$ be a finite $p$-group. Then $c_2(G) = 1$ if and only if $G$ is a cyclic group or dihedral group.*

**Proof** If $G$ is abelian, then $G$ is cyclic. Assume that $G$ is nonabelian and $M$ is the unique cyclic subgroup of order $p^2$. If $C_G(M) = M$, then $G$ is of maximal class by Lemma 11. For $C_G(M) > M$, we see $C_G(M)$ is cyclic from Lemma 6. Since any cyclic subgroup of order $p^3$ contains $M$ and lies in $C_G(M)$, we have $a_3(G) = 1$. Therefore, $G$ is of maximal class. By Lemma 8, $G$ is a 2-group. It is easy to check that only the dihedral group in 2-groups of maximal class satisfies $c_2(G) = 1$. Conversely, the conclusion is obvious. □

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