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Sparse sums with bases of Chebyshev polynomials of the third and fourth kind

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Abstract: We derive a generalization for the reconstruction of M-sparse sums in Chebyshev bases of the third and fourth kind. This work is used for a polynomial with Chebyshev sparsity and samples on a Chebyshev grid of $[-1,1]$. Further, fundamental reconstruction algorithms can be a way for getting M-sparse expansions of Chebyshev polynomials of the third and fourth kind. The numerical results for these algorithms are designed to compare the time effects of doing them.

Key words: Sparse interpolation, Chebyshev polynomial, Prony method, eigenvalue problem, Toeplitz-plus-Hankel matrix, SVD, QR decomposition

1. Introduction

A linear combination of Chebyshev polynomials with M nonzero coefficients, where M is much smaller than the degree, is called a M-sparse polynomial in the corresponding Chebyshev basis. One of the applications is the recovery and the repair of the sparse signals from a small set of measurements [11, 12]. There are also some efficient reconstruction algorithms for this work. One of them is a random recovery method such as Legendre expansion with M nonzero coefficients; see [8, 11]. Moreover, there are some deterministic methods for the reconstruction

$$F(x) = \sum_{k=1}^{M} c_k e^{iw_k x}$$

with complex parameters c_k and w_k, $k = 1, \ldots, M$, and $-\pi < Imw_1 < \ldots < Imw_M < \pi$. We hope to reconstruct c_k and w_k from a given small amount of (possibly noisy) measurement values $F(x)$.

In [9], Potts and Tasche introduced some processes for reconstruction of sparse expansions in bases of Chebyshev polynomials of the first and second kind. We are motivated to generalize this reconstruction of M-sparse sums in Chebyshev bases of the third and fourth kind. There are some methods for these works, such as the Prony method [10], the matrix pencil method [4, 5], and the ESPRIT method [12]. Thus we want to generalize these processes for recovery of all parameters c_k, w_k, $k = 1, \ldots, M$ for sparse Chebyshev-3 and Chebyshev-4 interpolations.

Usually Chebyshev polynomials of third and fourth kind that are special cases of Jacobi polynomials are known less than first and second kind in the literature. However, these polynomials appear in various applications such as potential theory of differential equations, recurrence relations, decomposition of sequences, Rodrigues type formula, hypergeometric functions, and generating functions. Some of the explicit advantages of

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Chebyshev polynomials of third and fourth kind are shown in [1] to estimate some definite integrals and solving boundary value problems in [7].

This paper is organized as follows: in Section 2, we study the Prony method for sparse polynomials with bases of Chebyshev polynomials of the third and fourth kind; in Section 3, we discuss QR decomposition by matrix pencil factorization for sparse interpolations; Section 4 is dedicated to the ESPRIT method; and finally in Section 5, some numerical examples and comparisons for different algorithms are collected. The following part shows standard and emphasis notations for the reader. We denote the set of all positive integers with $\mathbb{N}$ and nonnegative integers with $\mathbb{N}_0$. The Kronecker symbol is δ_k.

$A_{M,N} \in \mathbb{R}^{M\times N}$ is a matrix, its transpose is $A^T_{M,N}$, and its Moore–Penrose pseudoinverse is $A^{\dagger}_{M,N}$. A square matrix $A_{M,M}$ is abbreviated to A_M. $\mathbf{I}_M$ is an identity matrix in $\mathbb{R}^{M\times M}$. $\mathbf{O}_{M,N}$ is a zero matrix in $\mathbb{R}^{M\times N}$. $\mathbf{A}_{M,M+1}(1:M, 2:M+1)$ is the submatrix of $\mathbf{A}_{M,M+1}$ by extracting rows 1 through M and columns 2 through $M+1$. $\mathbf{A}_{M,M+1}(1:M, M+1)$ is the submatrix of $\mathbf{A}_{M,M+1}$ obtained by only the last column of $\mathbf{A}_{M,M+1}$. Definitions are presented by the symbol $:=$. All algorithms are tested for different matrices by floating point arithmetic and with double precision in MATLAB.

2. Prony method for sparse polynomials with bases of Chebyshev polynomials of the third and fourth kind

We begin this section with Chebyshev polynomials of the first, second, third, and fourth kinds. Chebyshev polynomials will be denoted by T_n, U_n, V_n, and W_n, respectively [6, 13]:

$$T_0(x) = 1, \quad T_1(x) = x, \quad T_n(\cos\theta) = \cos n\theta, \tag{1}$$

$$U_0(t) = 1, \quad U_1(x) = 2x, \quad U_n(\cos\theta) = \frac{\sin(n+1)\theta}{\sin\theta},$$

$$V_0(t) = 1, \quad V_1(x) = 2x-1, \quad V_n(\cos\theta) = \frac{\cos\left(n+\frac{1}{2}\right)\theta}{\cos\frac{1}{2}\theta},$$

$$W_0(t) = 1, \quad W_1(x) = 2x+1, \quad W_n(\cos\theta) = \frac{\sin\left(n+\frac{1}{2}\right)\theta}{\sin\frac{1}{2}\theta},$$

with $n = 0, 1, 2, \ldots$.

All Chebyshev polynomials satisfy the three-term recurrence relation, for instance,

$$T_{i+1}(x) = 2xT_i(x) - T_{i-1}(x) \quad for \;\; i = 1, 2, \ldots. \tag{2}$$

Let M, N be positive integers that $M < N$;

$$f(x) = \sum_{k=0}^{d} a_k T_k(x),$$

is a polynomial of degree d such that $M \ll d$. If coefficients $a_k, \;\; k = 0, 1, \ldots, M$ are nonzero and the other d-M + 1 coefficients vanish, it is called M-sparse in the Chebyshev-1 basis and it is represented in the form

$$f(x) = \sum_{j=1}^{M} c_j T_{n_j}(x) \tag{3}$$

with $c_j := a_{n_j} \neq 0$ and $0 \leq n_1 < n_2 < \ldots < n_M = d$. As we know Chebyshev grids are much better than uniform grids for the recovery of polynomials [2]. In [9] is introduced Prony polynomial P of degree M with grids $x_j := T_{n_j}(u_N) = \cos \frac{n_j \pi}{2N-1}, \quad j = 1, \ldots, M$ where $u_N := \cos \frac{\pi}{2N-1}$ and

$$P(x) = 2^{M-1} \prod_{j=1}^{M} (x - \cos \frac{n_j \pi}{2N-1}). \tag{4}$$

Then $P(x)$ can be represented in the Chebyshev-1 basis by

$$P(x) = \sum_{k=0}^{M} p_k T_k(x), \quad p_M := 1. \tag{5}$$

In [9] is used in the nonequidistant Chebyshev grids by

$$u_{N,k} := T_k(u_N := \cos \frac{\pi}{2N-1}) = \cos \frac{k\pi}{2N-1}, \quad k = 0, \ldots, 2M-1$$

of the interval $[-1, 1]$ with sampled data

$$f_k := f(u_{N,k}) = f(\cos \frac{k\pi}{2N-1}), \quad k = 0, 1, \ldots, 2M-1. \tag{6}$$

Now we try to achieve the sparse interpolation on bases of Chebyshev polynomials of the third and fourth kind.

2.1. Sparse polynomial interpolation in Chebyshev-3 basis with Prony method

For $n \in \mathbb{N}_0$ and $x \in (-1, 1)$, the Chebyshev polynomial of the third kind is defined by

$$V_n(x) := \frac{\cos((n + 1/2) \arccos x)}{\cos(\frac{1}{2} \arccos x)}$$

see [3, 6]. As we know, these polynomials are orthogonal with respect to the weight $(1-x)^{-1/2}(1+x)^{1/2}$ in $(-1, 1)$. Now we consider a polynomial f of degree at most $2N-1$, which is M-sparse in the Chebyshev-3 basis, i.e.

$$f(x) = \sum_{j=1}^{M} c_j V_{n_j}(x) \tag{7}$$

with $M < N$ and $\mathbf{c} := \{c_j\}_{j=1}^{M} \neq \mathbf{0}$ and $0 \leq n_1 < n_2 < \ldots < n_M \leq 2N-1$. The integer M is called the Chebyshev-3 sparsity of the polynomial (7).

If we let $x = \cos t$, for all $t \in [0, \pi]$, thus

$$f(\cos t) \cos \frac{t}{2} = \sum_{j=1}^{M} c_j \cos((n_j + \frac{1}{2})t),$$

we use the grids $t = \frac{k\pi}{2N-1}, \quad k = 0, 1, \ldots, 2N-1$, and we set

$$\tilde{f}_k := f(\cos \frac{k\pi}{2N-1}) \cos \frac{k\pi}{2(2N-1)} = \sum_{j=1}^{M} c_j \cos \frac{(n_j + \frac{1}{2})k\pi}{2N-1} \tag{8}$$

with Prony polynomial:

$$\tilde{P}(x) = 2^{M-1} \prod_{j=1}^{M} (x - \cos \frac{(n_j + \frac{1}{2})\pi}{2N-1}) = \sum_{k=0}^{M} p_k T_k(x), \quad p_M := 1. \tag{9}$$

The coefficients $p_k, \quad k = 0, \ldots, M-1$, of the Prony polynomial can be determined by the following theorem:

Theorem 2.1 *With the above assumptions, the coefficients p_k of the Prony polynomial (9) satisfy:*

$$\sum_{j=0}^{M-1} (\tilde{f}_{j+k} + \tilde{f}_{|j-k|}) p_j = -(\tilde{f}_{M+k} + \tilde{f}_{|M-k|}). \tag{10}$$

Proof We have

$$\tilde{f}_{j+k} = \sum_{l=1}^{M} c_l \cos \frac{(n_l + \frac{1}{2})(j+k)\pi}{2N-1},$$

and

$$\tilde{f}_{|j-k|} = \sum_{l=1}^{M} c_l \cos \frac{(n_l + \frac{1}{2})|j-k|\pi}{2N-1},$$

we know $\cos(x+y) + \cos(x-y) = 2\cos x \cos y$; then

$$\tilde{f}_{j+k} + \tilde{f}_{|j-k|} = 2 \sum_{l=1}^{M} c_l \cos \frac{(n_l + \frac{1}{2})j\pi}{2N-1} \cos \frac{(n_l + \frac{1}{2})k\pi}{2N-1}. \tag{11}$$

We deduce that

$$\sum_{j=0}^{M} (\tilde{f}_{j+k} + \tilde{f}_{|j-k|}) p_j = 2 \sum_{l=1}^{M} c_l \cos \frac{(n_l + \frac{1}{2})k\pi}{2N-1} \sum_{j=0}^{M} p_j \cos \frac{(n_l + \frac{1}{2})j\pi}{2N-1}$$

$$= 2 \sum_{l=1}^{M} c_l \cos \frac{(n_l + \frac{1}{2})k\pi}{2N-1} \tilde{P}(\cos \frac{(n_l + \frac{1}{2})k\pi}{2N-1}) = 0;$$

hence we get (10). □

Let $\tilde{\mathbf{f}}(k) := (\tilde{f}_{j+k} + \tilde{f}_{|j-k|})_{j=0}^{M-1}, \quad k = 0, 1, ..., M$ and set the square Toeplitz–Hankel ($T + H$) matrix such that

$$\tilde{\mathbf{F}}_M(0) := (\tilde{f}_{j+k} + \tilde{f}_{|j-k|})_{j,k=0}^{M-1} = \begin{pmatrix} \tilde{\mathbf{f}}(0) & \tilde{\mathbf{f}}(1) \ldots \tilde{\mathbf{f}}(M-1) \end{pmatrix}$$

$$= \begin{pmatrix} 2\tilde{f}_0 & 2\tilde{f}_1 & \ldots & 2\tilde{f}_{M-1} \\ 2\tilde{f}_1 & \tilde{f}_2 + \tilde{f}_0 & \ldots & \tilde{f}_M + \tilde{f}_{M-2} \\ \vdots & \vdots & & \vdots \\ 2\tilde{f}_{M-1} & \tilde{f}_M + \tilde{f}_{M-2} & \ldots & \tilde{f}_{2M-2} + \tilde{f}_0 \end{pmatrix}. \tag{12}$$

Eq. (11) follows the factorization of the $T+H$ matrix such that

$$\tilde{\mathbf{F}}_M(0) = 2\mathbf{V}_M(\mathbf{x})(diag\mathbf{c})\mathbf{V}_M(\mathbf{x})^T, \tag{13}$$

and

$$\mathbf{V}_M(\mathbf{x}) := (T_k(x_j))_{k=0,j=1}^{M-1,M} = \left(\cos\frac{(n_j+\frac{1}{2})k\pi}{2N-1}\right)_{k=0,j=1}^{M-1,M}.$$

$\mathbf{V}_M(\mathbf{x})$ is a nonsingular Vandermonde-like matrix with $\mathbf{x} := (x_j)_{j=1}^{M}$. Introducing

$$Y(x) := \sum_{l=0}^{M-1} y_l \cos(lx),$$

this is a trigonometric polynomial of order at most $M-1$ with M distinct zeros $\frac{(n_j+\frac{1}{2})\pi}{2N-1} \in (0,\pi)$ $j=1,\ldots,M$. In Eq. (7), $diag\ \mathbf{c}$ is nonsingular. Further, Eq. (13) concludes $\tilde{\mathbf{F}}_M(0)$ is nonsingular.

The results also show that

$$\tilde{\mathbf{F}}_M(1) := (\tilde{f}_{j+k+1} + \tilde{f}_{|j-k-1|})_{j,k=0}^{M-1} = \left(\tilde{\mathbf{f}}(1)\ \ \tilde{\mathbf{f}}(2)\ldots\tilde{\mathbf{f}}(M)\right),$$

and

$$\tilde{\mathbf{F}}_{M,M+1} := \left(\tilde{\mathbf{F}}_M(0)\ \ \tilde{\mathbf{F}}_M(1)(1:M,M)\right) = \left(\tilde{\mathbf{f}}(0)\ \ \tilde{\mathbf{f}}(1)\ldots\tilde{\mathbf{f}}(M-1)\ \ \tilde{\mathbf{f}}(M)\right) \tag{14}$$

$$\tilde{\mathbf{F}}_M(s) := \tilde{\mathbf{F}}_{M,M+1}(1:M, 1+s:M+s)\ \ (s=0,1). \tag{15}$$

Let $\mathbf{E}_M := diag(\frac{1}{2},1,\ldots,1)^T \in \mathbb{R}^M$,

$$\mathbf{S}_M := (\delta_{j-k-1} + \delta_{j-k+1})_{j,k=0}^{M-1} = \begin{pmatrix} 0 & 1 & 0 & \ldots & 0 & 0 & 0 \\ 1 & 0 & 1 & \ldots & 0 & 0 & 0 \\ 0 & 1 & 0 & \ldots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & 0 & 1 & 0 \\ 0 & 0 & 0 & \ldots & 1 & 0 & 1 \\ 0 & 0 & 0 & \ldots & 0 & 1 & 0 \end{pmatrix},$$

$$\mathbf{P}_M := \mathbf{S}_M - (\mathbf{0}\ldots\mathbf{0}\ \ \mathbf{p}) = \begin{pmatrix} 0 & 1 & 0 & \ldots & 0 & 0 & -p_0 \\ 1 & 0 & 1 & \ldots & 0 & 0 & -p_1 \\ 0 & 1 & 0 & \ldots & 0 & 0 & -p_2 \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & 0 & 1 & 0 \\ 0 & 0 & 0 & \ldots & 1 & 0 & 1-p_{M-2} \\ 0 & 0 & 0 & \ldots & 0 & 1 & -p_{M-1} \end{pmatrix}$$

By Lemma 2.4 and Lemma 2.5 in [9], we conclude that

$$det(2x\mathbf{E}_M - \mathbf{S}_M) = T_M(x), \quad x \in \mathbb{R}$$

and

$$det(2x\mathbf{E}_M - \mathbf{P}_M) = 2^{M-1}det(x\mathbf{I}_M - \frac{1}{2}\mathbf{E}_M^{-1}\mathbf{P}_M) = \tilde{P}(x), \quad x \in \mathbb{R}. \tag{16}$$

Theorem 2.2 *Let f be a M-sparse polynomial of degree at most $2N-1$ in the Chebyshev-3 basis with M and N integers ($1 \leq M \leq N$) with $2M$ samples*

$$\tilde{f}_k := \sum_{j=1}^{M} c_j \cos\frac{(n_j+\frac{1}{2})k\pi}{2N-1} \quad k = 0, \ldots, 2M-1:$$

i- we have $\tilde{\mathbf{f}}(M) \in span\{\tilde{\mathbf{f}}(0) \;\; \tilde{\mathbf{f}}(1) \ldots \tilde{\mathbf{f}}(M-1)\}$.

ii- we reconstruct M coefficients $c_j \in \mathbb{R}$ $(j = 1, \ldots M)$ and the M nonnegative integers n_j $(j = 1, \ldots M)$, of $f(x) = \sum_{j=1}^{M} c_j V_{n_j}(x)$.

Proof The coefficients of the Prony polynomial can be determined via the linear system:

$$\tilde{\mathbf{F}}_M(0)\mathbf{p} = -\tilde{\mathbf{f}}(M), \tag{17}$$

where $\mathbf{p} := (p_k)_{k=0}^{M-1}$. The equation (17) is spanned by Theorem 2.1 that shows $\tilde{\mathbf{f}}(M) \in span\{\tilde{\mathbf{f}}(0) \;\; \tilde{\mathbf{f}}(1) \ldots \tilde{\mathbf{f}}(M-1)\}$. With the above assumptions, the coefficients p_k of the Prony polynomial (9) satisfy the equations

$$\sum_{j=0}^{M-1} (\tilde{f}_{j+k} + \tilde{f}_{|j-k|})p_j = -(\tilde{f}_{M+k} + \tilde{f}_{|M-k|}).$$

The zeros of the Prony polynomial are the eigenvalues of the companion matrix $\frac{1}{2}\mathbf{E}_M^{-1}\mathbf{P}_M$ in (16). Then we can compute not only M nonnegative integers n_j $(j = 1, \ldots M)$, but also we denote M coefficients $c_j \in \mathbb{R}$ $(j = 1, \ldots M)$ by solution of the square Vandermonde-like relation $\mathbf{V}_M(\mathbf{x})\mathbf{c} = (\tilde{f}_k)_{k=0}^{M-1}$. □

The following algorithm can be used for the interpolation of known Chebyshev-3 sparsity.

Algorithm 2.3 *Prony method for known Chebyshev-3 sparsity*

Input: *Matrix $\tilde{\mathbf{F}}_M(0)$ and vector $\tilde{\mathbf{f}}(M)$ by*

$$\tilde{f}_k := \sum_{j=1}^{M} c_j \cos\frac{(n_j+\frac{1}{2})k\pi}{2N-1} \quad k = 0, \ldots, 2M-1.$$

Step 1: *Solve the following linear square problem $\tilde{\mathbf{F}}_M(0)\mathbf{p} = -\tilde{\mathbf{f}}(M)$.*

Step 2: *Find roots $-1 \leq x_M < \ldots < x_1 \leq 1$ of the Prony polynomial in (9).*

Step 3: *Compute*

$$n_j := \left[\frac{2N-1}{\pi}\arccos x_j - \frac{1}{2}\right], \quad j = 1, \ldots, M,$$

by rounding to the nearest integer.

Table 1. Numerical evaluation of indices by Algorithm 2.3.

j	n_j	c_j	$\tilde{n}_j$	$\tilde{c}_{n_j}$
1	60	10	60	10.000000000004942
2	120	20	120	19.999999999994991
3	1760	30	1760	30.000000000009710
4	1780	40	1780	39.999999999989704
5	2000	50	2000	50.000000000000647

Step 4: *Solve the square Vandermonde-like problem*

$$\mathbf{V}_M(\mathbf{x})\mathbf{c} = (\tilde{f}_k)_{k=0}^{M-1}.$$

Output: *We have* $c_j \in \mathbb{R}$ $(j = 1, \dots M)$ *and* M *nonnegative integers* n_j $(j = 1, \dots M)$.

Example 2.4 *We use Algorithm 2.3 with* $N = 2000$ *for the recovery of the sparse polynomial in the Chebyshev-3 basis*

$$f(x) = 10V_{60}(x) + 20V_{120}(x) + 30V_{1760}(x) + 40V_{1780}(x) + 50V_{2000}(x).$$

Table 1 shows the approximations for $\tilde{n}_j$ *and* $\tilde{c}_j$ *of the original parameters* n_j *and* c_j, $j = 1, 2, 3, 4, 5$ *and* $u_N = \cos\frac{\pi}{2N-1}$.

Let L, K, M, N be positive integers with $M \leq L \leq K \leq N$. We try to generalize the last results of this section to a rectangular $T + H$ matrix and rectangular Vandermonde-like matrices. We try to factorize the rectangular $T + H$ matrix and the modified Prony polynomial

$$Q(x) = 2^{L-1}\prod_{j=1}^{L}(x - \cos\frac{(n_j + \frac{1}{2})\pi}{2N-1}) = \sum_{k=0}^{L} q_k T_k(x), \quad q_L := 1, \quad x \in R. \tag{18}$$

The zeros of the modified Prony polynomial can be computed via solving a rectangular eigenvalues problem. We choose more sampling points by improving the numerical stability; then we introduce $\tilde{f}_k = f(u_{N,k}) \in R, \quad k = 0, 1, \dots, L + K - 1$ and $T + H$ matrix

$$\tilde{\mathbf{F}}_{K,L+1} := (\tilde{f}_{j+k} + \tilde{f}_{|j-k|})_{l,m=0}^{K-1,L} = \Big(\tilde{\mathbf{F}}_{K,L}(0) \;\; \tilde{\mathbf{F}}_{K,L}(1)(1:K,L)\Big), \tag{19}$$

$$\tilde{\mathbf{F}}_{K,L}(s) := (\tilde{f}_{j+k+s} + \tilde{f}_{|j-k-s|})_{l,m=0}^{K-1,L-1} \quad s = 0, 1. \tag{20}$$

$\tilde{\mathbf{F}}_{K,L}(1)$ is a shifted version of the matrix $\tilde{\mathbf{F}}_{K,L}(0)$, namely

$$\tilde{\mathbf{F}}_{K,L}(1) = \Big(\tilde{\mathbf{f}}(1) \;\; \tilde{\mathbf{f}}(2)\dots\tilde{\mathbf{f}}(L)\Big) = \Big(\tilde{\mathbf{F}}_{K,L}(0)(1:K, 1:L-1) \;\; \tilde{\mathbf{f}}(L)\Big) \tag{21}$$

$$\tilde{\mathbf{F}}_{K,L}(s) := \tilde{\mathbf{F}}_{K,L+1}(1:K, 1+s:L+s) \;\; (s = 0, 1). \tag{22}$$

Note that if $M = L = K$ we obtain $T+H$ matrix in (14), (15) and the vector $\mathbf{p} := (p_k)_{k=0}^{L-1}$ with $p_M := 1, p_{M+1} = \ldots = p_{L-1} := 0$.

Without loss of generality we can use the results of Lemma 3.1 in [9] for sparse polynomial interpolation in the Chebyshev-3 basis,

$$rank \;\; \tilde{\mathbf{F}}_{K,L+1} = rank \;\; \tilde{\mathbf{F}}_{K,L+1}(s) = M \quad s = 0,1$$

$$\tilde{\mathbf{F}}_{K,L}(0) = 2\mathbf{V}_{K,M}(\mathbf{x})(diag\ \mathbf{c})\mathbf{V}_{L,M}(\mathbf{x})^T, \tag{23}$$

where

$$\mathbf{V}_{K,M}(\mathbf{x}) := (T_{k-1}(x_j))_{k,j=1}^{K,M} = \left(\cos\frac{(n_j+\frac{1}{2})(k-1)\pi}{2N-1}\right)_{k,j=1}^{K,M},$$

is a rectangular Vandermonde-like matrix with $\mathbf{x} := (x_j)_{j=1}^M$ and

$$dim(null\tilde{\mathbf{F}}_{K,L+1}) = L - M + 1,$$

$$dim(null\tilde{\mathbf{F}}_{K,L+1}(s)) = L - M \quad s = 0,1.$$

Lemma 2.5 *Let L, K, M, N be positive integers with $M \le L \le K \le N$, $\tilde{f}_k = \tilde{f}(u_{N,k}) \quad k = 0,\ldots,L+K-1$ be sampled data of the sparse polynomial (7) of degree at most $2N-1$ and the coefficients $c_j \in R-\{0\}$; then the following results are equivalent:*

i-

$$Q(x) := \sum_{k=0}^{L} q_k T_k(x), \quad q_L := 1 \quad q_k \in R \tag{24}$$

is a polynomial with distinct roots $\{x_j\}_{j=1}^M$ such that $-1 \le x_M < \ldots < x_1 \le 1$.

ii- A solution of the following linear system is the vector $\mathbf{q} = (q_k)_{k=0}^{L-1}$ such that

$$\tilde{\mathbf{F}}_{K,L}(0)\mathbf{q} = -\tilde{\mathbf{f}}(L) \quad that \quad \tilde{\mathbf{f}}(L) := (\tilde{f}_{L+m} + \tilde{f}_{|L-m|})_{m=0}^{K-1}. \tag{25}$$

iii-The matrix $\mathbf{Q}_L := \mathbf{S}_L - (\mathbf{0}\ldots\mathbf{0} \;\; \mathbf{q}) \in R^{L\times L}$ has the property

$$\tilde{\mathbf{F}}_{K,L}(0)\mathbf{Q}_L = \tilde{\mathbf{F}}_{K,L}(1) + \Big(\mathbf{0} \;\; \tilde{\mathbf{f}}(0) \;\; \tilde{\mathbf{f}}(1)\ldots\tilde{\mathbf{f}}(L-2)\Big). \tag{26}$$

Therefore, the eigenvalues $\frac{1}{2}\mathbf{E}_L^{-1}\mathbf{Q}_L$ are the zeros of the polynomial $Q(x)$ in (24).

For proof refer to [9].

Now we formulate Lemma 2.5 as an algorithm for the modified Prony method for sparse Chebyshev-3 interpolation. Since the unknown coefficients $c_j, \;\; j = 1,...,M$ do not vanish, for convenient bound $\epsilon \;\; (0 < \epsilon \ll 1)$, we suppose $|c_j| > \epsilon$.

Algorithm 2.6 *Prony method for unknown Chebyshev-3 sparsity*

Input: *Matrix* $\tilde{\mathbf{F}}_{K,L}(0)$ *and vector* $\tilde{\mathbf{f}}(L)$ *by*

$$\tilde{f}_k := \sum_{j=1}^{M} c_j \cos\frac{(n_j + \frac{1}{2})k\pi}{2N-1} \quad k = 0, \ldots, L+K-1$$

that L, K, M, N *are positive integers with* $M \leq L \leq K \leq N$.

Step 1: *Solve the least squares problem*

$$\tilde{\mathbf{F}}_{K,L}(0)\mathbf{q} = -\tilde{\mathbf{f}}(L).$$

Step 2: *Find roots* $-1 \leq \tilde{x}_{\tilde{M}} < \ldots < \tilde{x}_2 < \tilde{x}_1 \leq 1$ *of the Prony polynomial in (24) such that* $\tilde{M} \geq M$.

Step 3: *Solve the least squares solution of the overdetermined linear Vandermonde-like system*

$$\mathbf{V}_{L+K,\tilde{M}}(\tilde{\mathbf{x}})(\tilde{\mathbf{c}}_j)_{j=1}^{\tilde{M}} = (\tilde{f}_k)_{k=0}^{L+K-1}.$$

with $\tilde{\mathbf{x}} := (\tilde{x}_j)_{j=1}^{\tilde{M}}$ *and* $\mathbf{V}_{L+K,\tilde{M}}(\tilde{\mathbf{x}})$ *in (23).*

Step 4: *Compute the remaining values of* x_j $(j = 1, \ldots, M)$ *by deleting all the* $\tilde{x}_l$ $(l \in \{1, \ldots, \tilde{M}\}$ *with* $|c_l| \leq \epsilon$).

Step 5: *Compute*

$$n_j := \left[\frac{2N-1}{\pi}\arccos x_j - \frac{1}{2}\right], \quad j = 1, \ldots, M \leq \tilde{M},$$

by rounding to the nearest integer.

Step 6: *Again solve the least squares problem of the overdetermined Vandermonde-like system*

$$\mathbf{V}_{L+K,M}(\mathbf{x})(\mathbf{c}_j)_{j=1}^{M} = (\tilde{f}_k)_{k=0}^{L+K-1}.$$

Output: *We have* $c_j \in \mathbb{R}$ $(j = 1, \ldots M)$ *and* M *nonnegative integers* n_j $(j = 1, \ldots M)$.

Example 2.7 *We use Algorithm 2.6 with* $N = 5000$, $K = 60$, $L = 15$, $M = 5$, *and* $\epsilon := 10^{-5}$ *for the recovery of the sparse polynomial in the Chebyshev-3 basis*

$$f(x) = -32V_{75}(x) + 45V_{129}(x) - 108.6V_{1763}(x) + 1057V_{1785}(x) - 5679.7V_{2067}(x).$$

Table 2 shows the approximations for $\tilde{n}_j$ *and* $\tilde{c}_j$ *of the original parameters* n_j *and* c_j, $j = 1, 2, 3, 4, 5$ *and* $u_N = \cos\frac{\pi}{2N-1}$.

2.2. Sparse polynomial interpolation in Chebyshev-4 basis with Prony method

For $n \in \mathbb{N}_0$ and $x \in (-1, 1)$, the Chebyshev polynomial of fourth kind is defined by

$$W_n(x) := \frac{\sin((n+1/2)\arccos x)}{\sin(\frac{1}{2}\arccos x)};$$

Table 2. Numerical evaluation of indices by Algorithm 2.6.

j	n_j	c_j	$\tilde{n}_j$	$\tilde{c}_{n_j}$
1	75	-32	75	$-0.031999999999999 \times 10^3$
2	129	+45	129	$+0.045000000000001 \times 10^3$
3	1763	-108.6	1763	$-0.108600000000001 \times 10^3$
4	1785	+1057	1785	$+1.056999999999999 \times 10^3$
5	2067	-5679.7	2067	$-5.679709999999997 \times 10^3$

see [3, 6]. As we know these polynomials are orthogonal with respect to the weight $(1-x)^{-1/2}(1+x)^{1/2}$ in $(-1,1)$. Now we consider a polynomial g of degree at most $2N-1$, which is M-sparse in the Chebyshev-4 basis, i.e.

$$g(x) = \sum_{j=1}^{M} c_j W_{n_j}(x) \tag{27}$$

$M < N$ and $0 \le n_1 < n_2 < \ldots < n_M \le 2N-1$ with $\mathbf{c} := \{c_j\}_{j=1}^{M} \neq \mathbf{0}$. The integer M is called the Chebyshev-4 sparsity of the polynomial (27).

We know $W_{2N-1} = U_{2N-1} + U_{2N-2} = T_0 + 2(T_1 + T_2 + \ldots + T_{2N-1})$; see [6]. Then $W_{2N-1} + W_{2N-2} = 2T_0 + 4(T_1 + T_2 + \ldots + T_{2N-2}) + 2T_{2N-1}$. Therefore, the 4-sparse polynomial $W_{2N-1} + W_{2N-2}$ in the Chebyshev-4 basis is not a sparse polynomial in the Chebyshev-1 basis. If we let $x = \cos t$, for all $t \in [0,\pi]$, thus

$$g(\cos t)\sin\frac{t}{2} = \sum_{j=1}^{M} c_j \sin((n_j + \frac{1}{2})t).$$

We use the grids $t = \frac{k\pi}{2N-1}$, $\ k = 0, 1, \ldots, 2N-1$, and set

$$\tilde{g_k} := g(\cos\frac{k\pi}{2N-1})\sin\frac{k\pi}{2(2N-1)} = \sum_{j=1}^{M} c_j \sin\frac{(n_j + \frac{1}{2})k\pi}{2N-1} \tag{28}$$

with Prony polynomial:

$$\tilde{P}(x) = 2^{M-1}\prod_{j=1}^{M}(x - \cos\frac{(n_j + \frac{1}{2})\pi}{2N-1}) = \sum_{k=0}^{M} p_k T_k(x), \quad p_M := 1. \tag{29}$$

Let $\tilde{g}_{-k} := -\tilde{g}_k \ \ k = 1, 2, \ldots, 2N-1$; the coefficients p_k, $\ k = 0, \ldots, M-1$, of the Prony polynomial can now be determined via the following theorem:

Theorem 2.8 *With the above assumptions, the coefficients p_k of the Prony polynomial (29) satisfy:*

$$\sum_{j=0}^{M-1} (\tilde{g}_{j+k} - \tilde{g}_{|j-k|})p_j = -(\tilde{g}_{M+k} - \tilde{g}_{|M-k|}). \tag{30}$$

Proof We have

$$\tilde{g}_{j+k} = \sum_{l=1}^{M} c_l \sin \frac{(n_l + \frac{1}{2})(j+k)\pi}{2N-1},$$

and

$$\tilde{g}_{|j-k|} = \sum_{l=1}^{M} c_l \sin \frac{(n_l + \frac{1}{2})|j-k|\pi}{2N-1},$$

we know $\sin(x+y) - \sin(x-y) = 2\sin x \cos y$; then

$$\tilde{g}_{j+k} - \tilde{g}_{|j-k|} = 2\sum_{l=1}^{M} c_l \sin \frac{(n_l + \frac{1}{2})j\pi}{2N-1} \cos \frac{(n_l + \frac{1}{2})k\pi}{2N-1}. \tag{31}$$

We deduce that

$$\sum_{j=0}^{M} (\tilde{g}_{j+k} - \tilde{g}_{|j-k|})p_j = 2\sum_{l=1}^{M} c_l \sin \frac{(n_l + \frac{1}{2})k\pi}{2N-1} \sum_{j=0}^{M} p_j \cos \frac{(n_l + \frac{1}{2})j\pi}{2N-1}$$

$$= 2\sum_{l=1}^{M} c_l \sin \frac{(n_l + \frac{1}{2})k\pi}{2N-1} \tilde{P}(\cos \frac{(n_l + \frac{1}{2})k\pi}{2N-1}) = 0;$$

hence we obtain (30). □

Let $\tilde{\mathbf{g}}(k) := (\tilde{g}_{j+k} - \tilde{g}_{|j-k|})_{k=1,j=0}^{M,M-1}$ and set the square $T+H$ matrix such that

$$\tilde{\mathbf{G}}_M(0) := (\tilde{g}_{j+k} - \tilde{g}_{|j-k|})_{k=1,j=0}^{M,M-1} = (\tilde{\mathbf{g}}(0) \;\; \tilde{\mathbf{g}}(1) \ldots \tilde{\mathbf{g}}(M-1))$$

$$= \begin{pmatrix} 2\tilde{g}_1 & \tilde{g}_2 - \tilde{g}_0 & \ldots & \tilde{g}_M - \tilde{g}_{M-2} \\ 2\tilde{g}_2 & \tilde{g}_3 + \tilde{g}_1 & \ldots & \tilde{g}_{M+1} - \tilde{g}_{M-3} \\ \vdots & \vdots & & \vdots \\ 2\tilde{g}_M & \tilde{g}_{M+1} + \tilde{g}_{M-1} & \ldots & \tilde{g}_{2M-1} + \tilde{g}_1 \end{pmatrix}. \tag{32}$$

Eq. (31) follows the factorization of the $T+H$ matrix such that

$$\tilde{\mathbf{G}}_M(0) = 2\mathbf{V}_M^s(\mathbf{x})(diag\mathbf{c})\mathbf{V}_M^c(\mathbf{x})^T \tag{33}$$

where

$$\mathbf{V}_M^c(\mathbf{x}) := \left(\cos \frac{(n_j + \frac{1}{2})k\pi}{2N-1}\right)_{k=0,j=1}^{M-1.M}$$

and

$$\mathbf{V}_M^s(\mathbf{x}) := \left(\sin \frac{(n_j + \frac{1}{2})k\pi}{2N-1}\right)_{k,j=1}^{M},$$

both Vandermonde-like matrices are nonsingular. Introducing

$$Y(x) := \sum_{l=0}^{M-1} y_l \cos(lx),$$

thus this is a trigonometric polynomial of order at most $M-1$ that has M distinct zeros $\frac{(n_j+\frac{1}{2})\pi}{2N-1} \in (0,\pi)$ $j = 1,\ldots,M$. This is possible only with $Y \equiv 0$.

Theorem 2.9 *Let g be a M-sparse polynomial of degree at most $2N-1$ in the Chebyshev-4 basis with M and N integers ($1 \le M \le N$) and $2M$ samples*

$$\tilde{g}_k := \sum_{j=1}^{M} c_j \sin\frac{(n_j+\frac{1}{2})k\pi}{2N-1} \quad k = 0,\ldots,2M-1:$$

i- we show $\tilde{\mathbf{g}}(M) \in span\{\tilde{\mathbf{g}}(0) \;\; \tilde{\mathbf{g}}(1)\ldots\tilde{\mathbf{g}}(M-1)\}$.

ii- we reconstruct M coefficients $c_j \in \mathbb{R}$ $(j = 1,\ldots M)$ and M nonnegative integers n_j $(j = 1,\ldots M)$, of $g(x) = \sum_{j=1}^{M} c_j W_{n_j}(x)$.

Proof The coefficients of the Prony polynomial can be determined via the linear system:

$$\tilde{\mathbf{G}}_M(0)\mathbf{p} = -\tilde{\mathbf{g}}(M) \tag{34}$$

that $\mathbf{p} := (p_k)_{k=0}^{M-1}$. The equation (34) is spanned by Theorem 2.8 that shows $\tilde{\mathbf{g}}(M) \in span\{\tilde{\mathbf{g}}(0) \;\; \tilde{\mathbf{g}}(1)\ldots\tilde{\mathbf{g}}(M-1)\}$. With the above assumptions, the coefficients p_k of the Prony polynomial (29) satisfy the equations

$$\sum_{j=0}^{M-1}(\tilde{g}_{j+k} - \tilde{g}_{|j-k|})p_j = -(\tilde{g}_{M+k} - \tilde{g}_{|M-k|}).$$

The zeros of the Prony polynomial are the eigenvalues of the companion matrix $\frac{1}{2}\mathbf{E}_M^{-1}\mathbf{P}_M$ in (16). Then we can not only compute M nonnegative integers n_j $(j = 1,\ldots M)$ but also denote M coefficients $c_j \in \mathbb{R}$ $(j = 1,\ldots M)$ by solution of the square Vandermonde-like relation $\mathbf{V}_M^c(\mathbf{x})\mathbf{c} = (\tilde{g}_k)_{k=0}^{M-1}$. □

Analogously as Section 2.1 we obtain the algorithms for the known Chebyshev-4 sparsity polynomial.

Algorithm 2.10 *Prony method for known Chebyshev-4 sparsity*

Input: *Matrix $\tilde{\mathbf{G}}_M(0)$ and vector $\tilde{\mathbf{g}}(M)$ by*

$$\tilde{g}_k := \sum_{j=1}^{M} c_j \sin\frac{(n_j+\frac{1}{2})k\pi}{2N-1} \quad k = 0,\ldots,2M-1.$$

Step 1: *Solve the following linear square problem $\tilde{\mathbf{G}}_M(0)\mathbf{p} = -\tilde{\mathbf{g}}(M)$.*

Step 2: *Find roots $-1 \le x_M < \ldots < x_2 < x_1 \le 1$ of the Prony polynomial in (29).*

Table 3. Numerical evaluation of indices by Algorithm 2.10.

j	n_j	c_j	$\tilde{n}_j$	$\tilde{c}_{n_j}$
1	40	+10	40	+10.000000000055900
2	100	-20	100	-20.000000000038430
3	184	+30	184	+30.000000000012285
4	261	-40	261	-40.000000000002665
5	489	+50	489	+50.000000000000057

Step 3: *Compute*

$$n_j := \left[\frac{2N-1}{\pi}\arccos x_j - \frac{1}{2}\right], \quad j = 1, \ldots, M,$$

by rounding to the nearest integer.

Step 4: *Solve the square Vandermonde-like problem*

$$\mathbf{V}^s_M(\mathbf{x})\mathbf{c} = (g_k)_{k=0}^{M-1}.$$

Output: *We have* $c_j \in \mathbb{R}$ $(j = 1, \ldots M)$ *and* M *nonnegative integers* n_j $(j = 1, \ldots M)$.

Example 2.11 *We use Algorithm 2.10 with* $N = 500$ *for the recovery of the sparse polynomial in the Chebyshev-4 basis*

$$f(x) = 10W_{40}(x) - 20W_{100}(x) + 30W_{184}(x) - 40W_{261}(x) + 50W_{489}(x).$$

Table 3 shows the approximations for $\tilde{n}_j$ *and* $\tilde{c}_j$ *of the original parameters* n_j *and* c_j, $j = 1, 2, 3, 4, 5$ *and* $u_N = \cos\frac{\pi}{2N-1}$.

Let L, K, M, N be positive integers with $M \le L \le K \le N$. We try to generalize the last results of this section to a rectangular $T+H$ matrix and rectangular Vandermonde-like matrices. We try to factorize the rectangular $T+H$ matrix and the modified Prony polynomial

$$\tilde{Q}(x) = 2^{L-1}\prod_{j=1}^{L}(x - \cos\frac{(n_j+\frac{1}{2})\pi}{2N-1}) = \sum_{k=0}^{L} q_k T_k(x), \quad q_L := 1, \quad x \in R. \tag{35}$$

The zeros of the modified Prony polynomial can be computed via solving a rectangular eigenvalues problem. We choose more sampling points by improving the numerical stability; then we introduce $\tilde{g}_k = g(u_{N,k}) \in R, \quad k = 0, 1, \ldots, L+K-1$ and $T+H$ matrix

$$\tilde{\mathbf{G}}_{K,L+1} := (\tilde{g}_{j+k} + \tilde{g}_{|j-k|})_{l,m=0}^{K-1,L} = \Big(\tilde{\mathbf{G}}_{K,L}(0) \ \ \tilde{\mathbf{G}}_{K,L}(1)(1:K,L)\Big), \tag{36}$$

$$\tilde{\mathbf{G}}_{K,L}(s) := (\tilde{g}_{j+k+s} + \tilde{g}_{|j-k-s|})_{l,m=0}^{K-1,L-1} \quad s = 0, 1. \tag{37}$$

$\tilde{\mathbf{G}}_{K,L}(1)$ is a shifted version of the matrix $\tilde{\mathbf{G}}_{K,L}(0)$, namely

$$\tilde{\mathbf{G}}_{K,L}(1) = (\tilde{\mathbf{g}}(1) \ \ \tilde{\mathbf{g}}(2)\ldots\tilde{\mathbf{g}}(L)) = \Big(\tilde{\mathbf{G}}_{K,L}(0)(1:K, 1:L-1) \ \ \tilde{\mathbf{g}}(L)\Big) \tag{38}$$

$$\tilde{\mathbf{G}}_{K,L}(s) := \tilde{\mathbf{G}}_{K,L+1}(1:K, 1+s:M+s) \quad (s=0,1). \tag{39}$$

Without loss of generality we can use the results of the section 2.1 and Lemma (2.5) for sparse polynomial interpolation in the Chebyshev-4 basis.

Algorithm 2.12 *Prony method for unknown Chebyshev-4 sparsity*

Input: *matrix* $\tilde{\mathbf{G}}_{K,L}(0)$ *and vector* $\tilde{\mathbf{g}}(L)$ *by*

$$\tilde{g}_k := \sum_{j=1}^{M} c_j \sin\frac{(n_j+\frac{1}{2})k\pi}{2N-1} \quad k = 0,\ldots,L+K-1$$

that L, K, M, N *are positive integers with* $M \le L \le K \le N$.

Step 1: *Solve the least squares problem*

$$\tilde{\mathbf{G}}_{K,L}(0)\mathbf{q} = -\tilde{\mathbf{g}}(L).$$

Step 2: *Find roots* $-1 \le \tilde{x}_{\tilde{M}} < \ldots < \tilde{x}_2 < \tilde{x}_1 \le 1$ *of the Prony polynomial in (35) such that* $\tilde{M} \ge M$.

Step 3: *Solve the least squares solution of the overdetermined Vandermonde-like system*

$$\mathbf{V}^s_{M}(\tilde{\mathbf{x}})(\tilde{\mathbf{c}}_j)_{j=1}^{\tilde{M}} = (\tilde{g}_k)_{k=0}^{L+K-1},$$

with $\tilde{\mathbf{x}} := (\tilde{x}_j)_{j=1}^{\tilde{M}}$ *and* $\mathbf{V}^s_{L+K,\tilde{M}}(\tilde{\mathbf{x}}) := \left(\sin\frac{(n_j+\frac{1}{2})k\pi}{2N-1}\right)_{k=0,j=1}^{L+K-1,\tilde{M}}$.

Step 4: *Compute the remaining values of* x_j $(j=1,\ldots,M)$ *by deleting all* $\tilde{x}_l$ $(l \in \{1,\ldots,\tilde{M}\}$ *that* $|c_l| \le \epsilon$*).*

Step 5: *Compute* $n_j := \left[\frac{2N-1}{\pi}\arccos x_j - \frac{1}{2}\right]$, $j = 1,\ldots,M \le \tilde{M}$ *by rounding to the nearest integer.*

Step 6: *Again solve the least squares solution of the overdetermined Vandermonde-like system*

$$\mathbf{V}^s_{L+K,M}(\mathbf{x})(\mathbf{c}_j)_{j=1}^{\tilde{M}} = (\tilde{g}_k)_{k=0}^{L+K-1}.$$

Output: *We have* $c_j \in \mathbb{R}$ $(j=1,\ldots M)$ *and* M *nonnegative integers* n_j $(j=1,\ldots M)$.

Example 2.13 *We use Algorithm 2.12 with* $N=2000$, $K=50$, $L=50$, $M=4$, *and* $\epsilon := 10^{-0.5}$ *for the recovery of the sparse polynomial in the Chebyshev-3 basis*

$$f(x) = W_{60}(x) + 2W_{120}(x) + 3W_{1760}(x) + 4W_{1780}(x).$$

Table 4 shows the approximations for $\tilde{n}_j$ *and* $\tilde{c}_j$ *of the original parameters* n_j *and* c_j, $j = 1,2,3,4$ *and* $u_N = \cos\frac{\pi}{2N-1}$.

Table 4. Numerical evaluation of indices by Algorithm 2.12.

j	n_j	c_j	$\tilde{n}_j$	$\tilde{c}_{n_j}$
1	60	1	60	1.000000000000000
2	120	2	120	1.999999999999999
3	1760	3	176	2.999999999999998
4	1780	4	1780	3.999999999999998

3. QR decomposition by matrix pencil factorization for sparse Chebyshev-3 and Chebyshev-4 interpolations

In this section we show that the Prony method for sparse Chebyshev-3 and Chebyshev-4 interpolations can be improved to the matrix pencil method. The matrix pencil can be

$$2x\tilde{\mathbf{F}}_{K,L}(0)\mathbf{E}_L - \tilde{\mathbf{F}}_{K,L}(0)\mathbf{Q}_L$$

by eigenvalues $x = \{x_j\}_{j=1}^{M} \in [-1,1]$ and a right eigenvector $\mathbf{v} \in \mathbb{C}^L$ of x_j.

$$(2x_j\tilde{\mathbf{F}}_{K,L}(0)\mathbf{E}_L - \tilde{\mathbf{F}}_{K,L}(0)\mathbf{Q}_L)\mathbf{v} = \tilde{\mathbf{F}}_{K,L}(0)(2x_j\mathbf{E}_L - \mathbf{Q}_L)\mathbf{v}. \tag{40}$$

By Lemma 2.5, we denote $det(2x_j\mathbf{E}_L - \mathbf{Q}_L) = Q(x_j)$, that $Q(x_j) = 0$.

$$(2x_j\mathbf{E}_L - \mathbf{Q}_L)\mathbf{v} \Rightarrow \mathbf{Q}_L\mathbf{v} = 2\mathbf{v}x_j\mathbf{E}_L \Rightarrow \frac{1}{2}\mathbf{E}_L^{-1}\mathbf{Q}_L\mathbf{v} = \mathbf{v}x_j. \tag{41}$$

Now we factorize the rectangular $T+H$ matrix (19) by QR factorization and column pivoting

$$\tilde{\mathbf{F}}_{K,L+1}\mathbf{\Pi}_{L+1} = \mathbf{U}_K\mathbf{R}_{K,L+1}$$

such that $\mathbf{U}_K$ is an orthogonal matrix and $\mathbf{R}_{K,L+1}(1:M,1:M)$ is a nonsingular upper triangular matrix. As we denoted $\mathbf{\Pi}_{L+1}$ is a permutation matrix such that the diagonal entries of $\mathbf{R}_{K,L+1}(1:M,1:M)$ have nonincreasing absolute values. We conclude that the diagonal matrix contains their diagonal entries by $\mathbf{D}_M$. With

$$\mathbf{S}_{K,L+1} := \mathbf{R}_{K,L+1}\mathbf{\Pi}_{L+1}^T = \begin{pmatrix} \mathbf{S}_{K,L+1}(1:M,1:L+1) \\ \mathbf{0}_{K-M,L+1} \end{pmatrix},$$

we can conclude

$$\tilde{\mathbf{F}}_{K,L}(s) = \mathbf{U}_K\mathbf{S}_{K,L}(s) \quad s = 0,1,$$

with

$$\mathbf{S}_{K,L}(s) = \mathbf{S}_{K,L+1}(1:K, 1+s:L+s) \quad s = 0,1.$$

We factorize the following matrices

$$2\tilde{\mathbf{F}}_{K,L}(0)\mathbf{E}_L = 2\tilde{\mathbf{F}}_{K,L}(0)\left(\frac{1}{2},1,\ldots,1\right) = \tilde{\mathbf{F}}_{K,L}(0)\mathbf{I} + \tilde{\mathbf{F}}_{K,L}(0)\left(0,1,\ldots,1\right)$$

$$= \mathbf{U}_K\mathbf{S}_{K,L}(0) + \Big(\mathbf{0}, \ \ \tilde{\mathbf{f}}(1)\dots\tilde{\mathbf{f}}(L-1)\Big) = \mathbf{U}_K\mathbf{S}_{K,L}(0) + \Big(\mathbf{0}, \ \ \tilde{\mathbf{F}}_{K,L+1}(1:K,1:L-1)\Big)$$

$$= \mathbf{U}_K\mathbf{S}_{K,L}(0) + (\mathbf{0}, \ \ \mathbf{U}_K\mathbf{S}_{K,L}(1)(1:K,1:L-1)) = \mathbf{U}_K\mathbf{S}'_{K,L}(0), \tag{42}$$

and

$$\tilde{\mathbf{F}}_{K,L}(0)\mathbf{Q}_L = \tilde{\mathbf{F}}_{K,L}(1) + \Big(\mathbf{0} \ \ \tilde{\mathbf{f}}(0)\dots\tilde{\mathbf{f}}(L-2)\Big)$$

$$= \mathbf{U}_K\mathbf{S}_{K,L}(1) + (\mathbf{0}, \ \ \mathbf{U}_K\mathbf{S}_{K,L}(0)(1:K,1:L-1)) = \mathbf{U}_K\mathbf{S}'_{K,L}(1), \tag{43}$$

where

$$\mathbf{S}'_{K,L}(0) := \mathbf{S}_{K,L}(0) + (\mathbf{0} \ \ \mathbf{S}_{K,L}(1)(1:K,1:L-1))\,,$$

$$\mathbf{S}'_{K,L}(1) := \mathbf{S}_{K,L}(1) + (\mathbf{0} \ \ \mathbf{S}_{K,L}(0)(1:K,1:L-1))\,.$$

We know U_k is orthogonal; then the generalized eigenvalue problem of the matrix pencil $(2x\tilde{\mathbf{F}}_{K,L}(0)\mathbf{E}_L - \tilde{\mathbf{F}}_{K,L}(0)\mathbf{Q}_L)$ is equivalent to the generalized eigenvalue problem of the matrix pencil

$$x\mathbf{S}'_{K,L}(0) - \mathbf{S}'_{K,L}(1) \quad x \in \mathbb{R}.$$

We can simplify the following matrix pencil by

$$x\mathbf{T}_{M,L}(0) - \mathbf{T}_{M,L}(1) \quad x \in \mathbb{R} \tag{44}$$

where

$$\mathbf{T}_{M,L}(0) := \mathbf{S}'_{K,L}(1:M, 1+s:L+s) \quad s = 0,1. \tag{45}$$

We define

$$\mathbf{T}'_{M,L}(s) := \mathbf{D}_M^{-1}T_{M,L}(s), \tag{46}$$

where $\mathbf{D}_M$ is a diagonal preconditioner matrix. Then the generalized eigenvalue problem of the transposed matrix pencil

$$x\mathbf{T}'_{M,L}(0) - \mathbf{T}'_{M,L}(1)^T \tag{47}$$

has the same eigenvalues as the matrix pencil (44) except for the zero eigenvalues. Finally we obtain the eigenvalue problem of the M-by-M matrix

$$\left(\mathbf{T}'_{M,L}(0)^T\right)^{\dagger}\mathbf{T}'_{M,L}(1)^T$$

(namely the nodes $x_j \in [-1,1], \quad j = 1,\dots,M$).

Algorithm 3.1 *Chebyshev-3 basis with QR decomposition*

Input: *Matrix* $\tilde{\mathbf{F}}_{K,L+1}$ *by*

$$\tilde{f}_k := \sum_{j=1}^{M} c_j \cos\frac{(n_j+\frac{1}{2})k\pi}{2N-1} \quad k = 0,\dots,K+L-1.$$

Table 5. Numerical evaluation of indices by Algorithm 3.1.

j	n_j	c_j	$\tilde{n}_j$	$\tilde{c}_{n_j}$
1	10	+1	10	+0.999999999999712
2	20	-2	20	-2.000000000000363
3	30	+3	30	+2.999999999999822
4	40	-4	40	-4.000000000000144
5	50	+5	50	+4.999999999999629
6	60	-6	60	-6.000000000000366

Step 1: *Compute QR factorization of the rectangular* $T+H$ *matrix* $\tilde{\mathbf{F}}_{K,L+1}$.

Step 2: *Find roots* $-1 \le x_M < \ldots < x_2 < x_1 \le 1$ *of the M-by-M matrix* $\left(\mathbf{T}'_{M,L}(0)^T\right)^\dagger \mathbf{T}'_{M,L}(1)^T$.

Step 3: *Compute*

$$n_j := \left[\frac{2N-1}{\pi} \arccos x_j - \frac{1}{2}\right], \quad j = 1, \ldots, M,$$

by rounding to the nearest integer.

Step 4: *Solve the least squares problem of the overdetermined Vandermonde-like system*

$$\mathbf{V}_{L+K,M}(\mathbf{x})(c_j)_{j=1}^M = (\tilde{f}_k)_{k=0}^{L+K-1}$$

Output: *We have* $c_j \in \mathbb{R}$ $(j = 1, \ldots M)$ *and* M *nonnegative integers* n_j $(j = 1, \ldots M)$.

Example 3.2 *We use Algorithm 3.1 with* $N = 101$, $K = 6$, $L = 6$, $M = 6$ *for the recovery of the sparse polynomial in the Chebyshev-3 basis*

$$f(x) = V_{10}(x) - 2V_{20}(x) + 3V_{30}(x) - 4V_{40}(x) + 5V_{50}(x) - 6V_{60}(x).$$

Table 5 shows the approximations for $\tilde{n}_j$ *and* $\tilde{c}_j$ *of the original parameters* n_j *and* c_j, $j = 1, 2, 3, 4, 5$ *and* $u_N = \cos \frac{\pi}{2N-1}$.

We derive the sparse interpolation with the basis of Chebyshev polynomials of the fourth kind with QR decomposition. Here we applied analogous ideas as in Section 3 on the matrix (36). Then we can obtain an algorithm similar to Algorithm 3.1; it is called Algorithm $\tilde{3.1}$.

Example 3.3 *We use Algorithm* $\tilde{3.1}$ *with* $N = 1500$, $K = 200$, $L = 100$, $M = 3$ *for the recovery of the sparse polynomial in the Chebyshev-4 basis*

$$f(x) = 30.5W_{60}(x) - 40.89W_{120}(x) + 50.01W_{1000}(x).$$

Table 6 shows the approximations for $\tilde{n}_j$ *and* $\tilde{c}_j$ *of the original parameters* n_j *and* c_j, $j = 1, 2, 3, 4, 5$ *and* $u_N = \cos \frac{\pi}{2N-1}$.

Table 6. Numerical evaluation of indices by Algorithm $\tilde{3.1}$.

j	n_j	c_j	$\tilde{n}_j$	$\tilde{c}_{n_j}$
1	60	+30.5	60	+30.500000000000004
2	120	-40.89	120	-40.890000000000008
3	1000	+50.01	1000	+50.010000000000012

4. ESPRIT method for sparse Chebyshev-3 and Chebyshev-4 interpolations

In this section we obtain some results for sparse Chebyshev-3 and Chebyshev-4 interpolations by singular value decomposition (SVD) of the $T+H$ matrix (19):

$$\tilde{\mathbf{F}}_{K,L+1} = \mathbf{U}_k \mathbf{D}_{K,L+1} \mathbf{W}_{L+1},$$

where $\mathbf{U}_k$ and $\mathbf{W}_{L+1}$ are orthogonal matrices and $\mathbf{D}_{K,L+1}$ is a rectangular diagonal matrix. The diagonal entries of $\mathbf{D}_{K,L+1}$ are the singular values of (19) such that

$$\sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_M > \sigma_{M+1} = \ldots = \sigma_{L+1} = 0.$$

Thus we can denote the rank M of the Hankel matrix (19), which sets with the Chebyshev-3 sparsity of the polynomial (7). Therefore, we introduce the following matrices

$$\tilde{\mathbf{D}}_{K,M} := \mathbf{D}_{K,L+1}(1:K, 1:M) = \begin{pmatrix} diag(\sigma_j)_{j=1}^{M} \\ \mathbf{0}_{K-M,M} \end{pmatrix}$$

$$\mathbf{W}_{M,L}(s) = \mathbf{W}_{M,L+1}(1:M, 1+s:L+s), \quad s = 0,1.$$

$$\mathbf{W}_{M,L+1} = \mathbf{W}_{M,L+1}(1:M, 1:L+1),$$

By simplify the SVD of the Hankel matrix (19) we have

$$\tilde{\mathbf{F}}_{K,L+1} = \mathbf{U}_k \mathbf{D}_{K,M} \mathbf{W}_{M,L+1},$$

and

$$\tilde{\mathbf{F}}_{K,L}(s) = \mathbf{U}_k \mathbf{D}_{K,M} \mathbf{W}_{M,L}(s), \quad s = 0,1.$$

Hence we can factorize the following matrices by similar processes in (42) and (43).

$$2\tilde{\mathbf{F}}_{K,L}(0)\mathbf{E}_L = \tilde{\mathbf{F}}_{K,L}(0) + \Big(\mathbf{0} \;\; \tilde{\mathbf{f}}(1)\ldots\tilde{\mathbf{f}}(L-1)\Big) = \mathbf{U}_k \mathbf{D}_{K,M} \mathbf{W}'_{K,L}(0),$$

$$\tilde{\mathbf{F}}_{K,L}(0)\mathbf{Q}_L = \tilde{\mathbf{F}}_{K,L}(1) + \Big(\mathbf{0} \;\; \tilde{\mathbf{f}}(0)\ldots\tilde{\mathbf{f}}(L-2)\Big) = \mathbf{U}_k \mathbf{D}_{K,M} \mathbf{W}'_{K,L}(1),$$

where

$$\mathbf{W}'_{K,L}(0) := \mathbf{W}_{K,L}(0) + (\mathbf{0} \;\; \mathbf{W}_{K,L}(1)(1:K, 1:L-1)),$$

$$\mathbf{W}'_{K,L}(1) := \mathbf{W}_{K,L}(1) + (\mathbf{0} \;\; \mathbf{W}_{K,L}(0)(1:K, 1:L-1)).$$

We know U_k is orthogonal and so the generalized eigenvalue problem of the rectangular matrix pencil $(2x\tilde{\mathbf{F}}_{K,L}(0)\mathbf{E}_L - \tilde{\mathbf{F}}_{K,L}(0)\mathbf{Q}_L)$ is equivalent to the generalized eigenvalue problem of the matrix pencil

$$x\mathbf{D}_{K,M}\mathbf{W}'_{M,L}(0) - \mathbf{D}_{K,M}\mathbf{W}'_{M,L}(1).$$

Now we multiply the above transposed matrix pencil from the right side with

$$\begin{pmatrix} diag(\sigma_j^{-1})_{j=1}^{M} \\ \mathbf{0}_{K-M,M} \end{pmatrix},$$

and we obtain the generalized eigenvalue problem of the matrix pencil

$$x\mathbf{W}'_{M,L}(0)^T - \mathbf{W}_{M,L}(1)^T, \tag{48}$$

with the same eigenvalues except for the zero eigenvalues. Finally we obtain the eigenvalue problem of the M-by-M matrix $\left(\mathbf{W}'_{M,L}(0)^T\right)^{\dagger}\mathbf{W}'_{M,L}(1)^T$, similarly as nodes $x_j \in [-1,1]$ $(j = 1,\ldots,M)$. This completes the ESPRIT method for sparse Chebyshev-3 interpolation.

Algorithm 4.1 *Chebyshev-3 basis with ESPRIT method*

Input: *Matrix* $\tilde{\mathbf{F}}_{K,L+1}$ *by*

$$\tilde{f}_k := \sum_{j=1}^{M} c_j \cos\frac{(n_j + \frac{1}{2})k\pi}{2N-1} \quad k = 0,\ldots,K+L-1.$$

Step 1: *Compute the SVD factorization of the rectangular* $T+H$ *matrix* $\tilde{\mathbf{F}}_{K,L+1}$.

Step 2: *Find roots* $-1 \le x_M < \ldots < x_2 < x_1 \le 1$ *of the M-by-M matrix*

$$\left(\mathbf{W}'_{M,L}(0)^T\right)^{\dagger}\mathbf{W}'_{M,L}(1)^T.$$

Step 3: *Compute*

$$n_j := \left[\frac{2N-1}{\pi}\arccos x_j - \frac{1}{2}\right], \quad j = 1,\ldots,M,$$

by rounding to the nearest integer.

Step 4: *Solve the least squares problem of the overdetermined Vandermonde-like system*

$$\mathbf{V}_{L+K,M}(\mathbf{x})(c_j)_{j=1}^{\tilde{M}} = (f_k)_{k=0}^{L+K-1}$$

Output: *We have* $c_j \in \mathbb{R}$ $(j = 1,\ldots M)$ *and* M *nonnegative integers* n_j $(j = 1,\ldots M)$.

Table 7. Numerical evaluation of indices by Algorithm 4.1.

j	n_j	c_j	$\tilde{n}_j$	$\tilde{c}_{n_j}$
1	500	-100	500	$-1.000000050833920 \times 10^2$
2	1500	+200	1500	$+2.000000089247412 \times 10^2$
3	2000	-300	2000	$-2.999999991339938 \times 10^2$
4	3000	+400	3000	$+3.999999949162848 \times 10^2$

Table 8. Numerical evaluation of indices by Algorithm $\tilde{4.1}$.

j	n_j	c_j	$\tilde{n}_j$	$\tilde{c}_{n_j}$
1	100	-1.01	100	-1.010000000000001
2	200	+2.02	200	+ 2.020000000000000
3	400	-3.03	400	-3.029999999999999
4	605	+4.04	605	+ 4.040000000000000
5	781	-5.05	781	-5.050000000000000

Example 4.2 *We use Algorithm 4.1 with* $N = 3000,\quad K = 100,\quad L = 80,\quad M = 4$ *for the recovery of the sparse polynomial in the Chebyshev-3 basis*

$$f(x) = -100V_{500}(x) + 200V_{1500}(x) - 300V_{2000}(x) + 400V_{3000}(x).$$

Table 7 shows the approximations for $\tilde{n}_j$ *and* $\tilde{c}_j$ *of the original parameters* n_j *and* c_j*,* $j = 1, 2, 3, 4, 5$ *and* $u_N = \cos\frac{\pi}{2N-1}$*.*

We derive the sparse interpolation with bases of the Chebyshev polynomials of the fourth kind with the ESPRIT method. Here we applied analogous ideas as in Section 4 on the matrix (36).

Then we can obtain an algorithm similar to Algorithm 4.1; it is called Algorithm $\tilde{4.1}$.

Example 4.3 *We use Algorithm* $\tilde{4.1}$ *with* $N = 10000,\quad K = L = M = 5$ *for the recovery of the sparse polynomial in the Chebyshev-4 basis*

$$f(x) = -1.01W_{1000}(x) + 2.02W_{2000}(x) - 3.03W_{4000}(x) + 4.04W_{6050}(x) - 5.05W_{9810}(x).$$

Table 8 shows the approximations for $\tilde{n}_j$ *and* $\tilde{c}_j$ *of the original parameters* n_j *and* c_j*,* $j = 1, 2, 3, 4, 5$ *and* $u_N = \cos\frac{\pi}{2N-1}$*.*

5. Numerical examples and comparisons

In this section, the discussed algorithms are tested for different matrices by using IEEE standard floating point arithmetic with double precision. We have implemented our algorithms in MATLAB. M-sparse polynomials are given in the forms (7) and (27) with Chebyshev polynomials V_{n_j} and W_{n_j} of degree n_j and real coefficients $c_j \neq 0 \quad j = 1, \dots, M$.

Figure 1 and Figure 2 in their parts (a, b, c, d) are designed to compare the time effects of doing them.

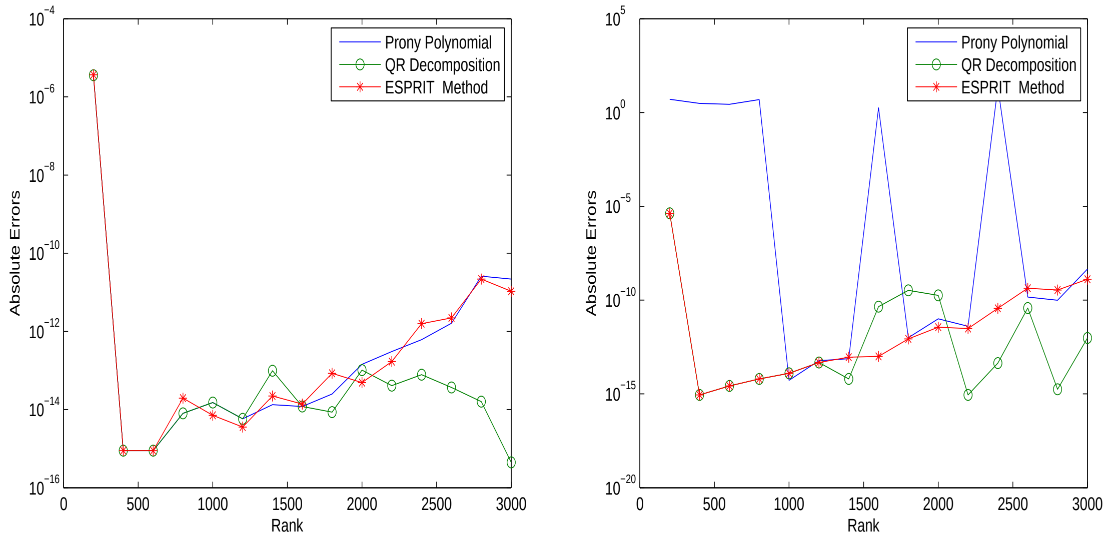

Figure 1. Comparison of the absolute error for Chebyshev-3 basis [left (a)] and Chebyshev-4 basis [right (b)].

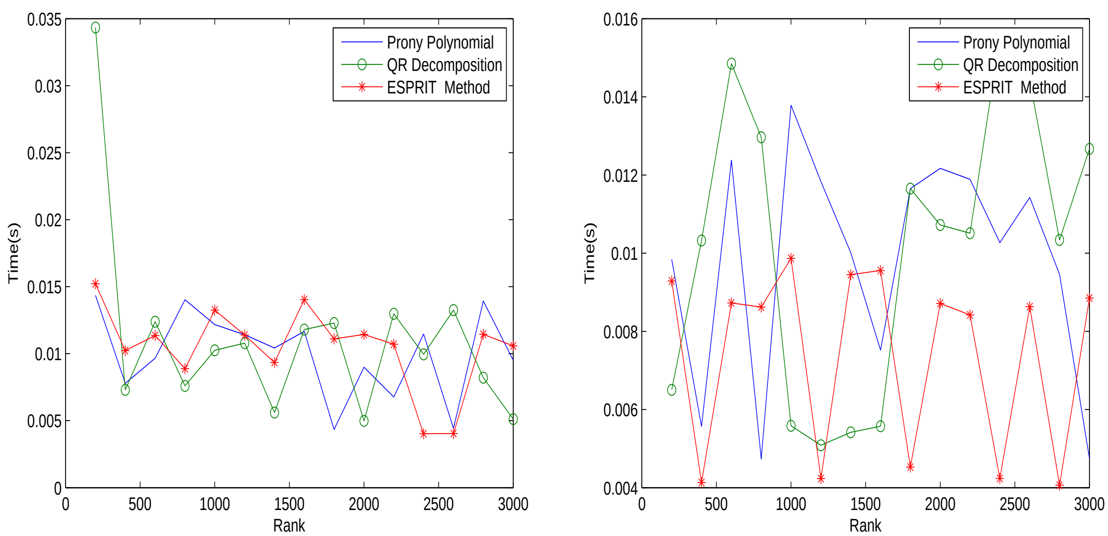

Figure 2. Comparison of the time for Chebyshev-3 basis [left (c)] and Chebyshev-4 basis [right (d)].

Example 5.1 *We start the following algorithms for different matrices with rank* $N = (200 : 200 : 3000)$ *with* $K = 100$, $L = 100$, $M = 5$ *and* $\mathbf{c} = (1, 2, 3, 4, 5)$, $n_j = (6, 12, 176, 178)$ *and* $u_N := \cos \frac{\pi}{2N-1}$. *We can see the success of the three different strategies for* M*-sparse expansions of Chebyshev polynomials of the third and fourth kind. We compared these methods by absolute error and time consuming.*

Absolute error of the coefficients is computed by

$$e(\mathbf{c}) := \max_{j=1,\ldots,5} |c_j - \tilde{c}_j| \quad \mathbf{c} := (c_j)_{j=1}^{5},$$

where $\tilde{c}_j$ are the coefficients calculated by our algorithms. Figure 1 presents absolute error for Chebyshev-3 basis [left (a)] and Chebyshev-4 basis [right (b)] and Figure 2 presents time consuming for Chebyshev-3 basis [left (c)] and Chebyshev-4 basis [right (d)] by different algorithms and matrices. In all examples the numerical stability of Algorithms 3.1 and $\tilde{3.1}$ in section 3 and Algorithms 4.1 and $\tilde{4.1}$ in section 4 can be improved by using more sampling values.

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