The dual generalized Chernoff inequality for star-shaped curves

Deyan ZHANG[1,2,*], **Yunlong YANG**[2]
[1]School of Mathematical Sciences, Huaibei Normal University, Huaibei, P.R. China
[2]Department of Mathematics, Tongji University, Shanghai, P.R. China.

Abstract: In this paper, we first introduce the k-order radial function $\rho_k(\theta)$ for star-shaped curves in $\mathbb{R}^2$ and then prove a geometric inequality involving $\rho_k(\theta)$ and the area A enclosed by a star-shaped curve, which can be looked upon as the dual Chernoff–Ou–Pan inequality. As a by-product, we get a new proof of the classical dual isoperimetric inequality. We also prove that $\frac{C^2}{k^2} \leq A < \frac{\pi C^2}{k}$ for star-shaped curves with $\rho_k(\theta) = C(\text{const.})$. In particular, if the curve is equichordal, then $\frac{C^2}{4} \leq A < \frac{\pi C^2}{2}$.

Key words: Star curves, the dual Chernoff–Ou–Pan inequality, equichordal curves

1. Introduction

Let α be a convex curve in the Euclidean plane $\mathbb{R}^2$ with area A and width function $w(\theta)$. In 1969, Chernoff [1] proved an inequality that says

$$A \leq \frac{1}{2} \int_0^{\frac{\pi}{2}} w(\theta)w(\theta + \frac{\pi}{2}) \mathrm{d}\theta,$$

where the equality holds if and only if α is a circle. Recently, Ou and Pan in [8] introduced the higher-order width function $w_k(\theta)$ and got the *Chernoff–Ou–Pan inequality* (see [3]) as follows:

$$A \leq \frac{1}{k} \int_0^{\frac{\pi}{k}} w_k(\theta)w_k(\theta + \frac{\pi}{k}) \mathrm{d}\theta, \qquad (1.1)$$

where the equality holds if and only if α is a circle. $w_k(\theta)$ is defined by

$$w_k(\theta) = h(\theta) + \cdots + h(\theta + \frac{2(k-1)\pi}{k}),$$

and $h(\theta)$ is the support function of α.

Let a compact subset K of $\mathbb{R}^n$ be star-shaped with respect to the origin; for $u \in S^{n-1}$, its radial function $\rho_K(\cdot)$ is defined by

$$\rho_K(u) = \max\{\lambda > 0 : \lambda u \in K\}.$$

If $\rho_K(\cdot)$ is continuous and positive, then K is called a *star body* (about the origin).

In contrast to the theory of convex bodies, in the dual theory, convex bodies are replaced by star bodies, the support function of a convex body is replaced by the radial function of a star body, and many geometric inequalities for star bodies are obtained (see [4, 5, 6, 10, 11]).

In this paper, a simple closed curve γ is called a *star-shaped curve* (about the origin) if it is the boundary of a planar star body. Therefore, the radial function $\rho(\theta)$ of γ is positive, continuous. Our goal is to extend the Chernoff–Ou–Pan inequality for convex curves to its dual form for star-shaped curves. To do so, we introduce for star-shaped curves the function $\rho_k(\theta)$ for an integer $k \geq 2$. This function is defined in (2.4) below. Furthermore, let γ be a C^1 (i.e. its radial function $\rho(\theta)$ is C^1) star-shaped curve with area A; then we can obtain the dual Chernoff–Ou–Pan inequality

$$A \geq \frac{1}{k} \int_0^{\frac{\pi}{k}} \rho_k(\theta) \rho_k(\theta + \frac{\pi}{k}) \mathrm{d}\theta, \tag{1.2}$$

where the equality holds if and only if the radial function $\rho(\theta)$ of γ is of the form

$$\rho(\theta) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_{2nk}\cos 2nk\theta + b_{2nk}\sin 2nk\theta), \ n \in \mathbb{Z}^+.$$

Let D be the star body enclosed by γ. From the definition of the dual mixed volume in [4, 7], the dual mixed area of D and the unit disc B in $\mathbb{R}^2$, $\tilde{A}(D, B)$, can be expressed by

$$\tilde{A}(D, B) = \frac{1}{2} \int_0^{2\pi} \rho(\theta) \mathrm{d}\theta.$$

Lutwak in [4] got the *classical dual isoperimetric inequality*

$$\tilde{A}(D, B)^2 \leq \pi A(D), \tag{1.3}$$

where the equality holds if and only if D is a disc. However, we can calculate

$$\lim_{k \to \infty} \frac{1}{k} \int_0^{\frac{\pi}{k}} \rho_k(\theta) \rho_k(\theta + \frac{\pi}{k}) \mathrm{d}\theta = \frac{1}{\pi} \tilde{A}(D, B)^2, \tag{1.4}$$

Thus (1.2) and (1.4) give a new proof of the classical dual isoperimetric inequality (1.3).

Another purpose of this paper is to show that there are many curves with $\rho_k(\theta) = \text{const.}$, and to build the following inequality: If the star-shaped curve γ satisfies $\rho_k(\theta) = C(\text{const.})$, then

$$\frac{C^2}{k^2} \leq A < \frac{\pi C^2}{k}, \tag{1.5}$$

where the equality on the left-hand side holds if and only if γ is a circle of radius $\frac{1}{k}C$.

In particular, if γ is *equichordal with respect to the origin*, i.e. all chords of γ through the origin have the same length (for more information on equichordal curves one may consult [2, 9] and the literature therein), then the inequality (1.5) leads to

$$\frac{C^2}{4} \leq A < \frac{\pi C^2}{2},$$

where the equality on the left-hand side holds if and only if γ is a circle of radius $\frac{1}{2}C$.

2. Preliminaries

Let γ be a C^1 star-shaped curve (about the origin) in the Euclidean plane $\mathbb{R}^2$ and $\rho(\theta)$ be its radial function, $\theta \in [0, 2\pi]$. Here, we say that γ is C^1 if and only if $\rho(\theta)$ is C^1 with respect to θ. Clearly, γ can be parameterized in terms of θ and $\rho(\theta)$ as

$$\gamma(\theta) = \big(x(\theta), y(\theta)\big) = \big(\rho(\theta)\cos\theta, \rho(\theta)\sin\theta\big).$$

Denote by A the area γ bounds. Using Green's formula, one can get

$$A = \frac{1}{2} \int_0^{2\pi} \rho^2(\theta)\mathrm{d}\theta. \tag{2.1}$$

Since the radial function of a given star-shaped curve γ is always continuous, bounded, and 2π-periodic, it has a Fourier series of the form

$$\rho(\theta) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n\cos n\theta + b_n\sin n\theta), \tag{2.2}$$

where

$$a_0 = \frac{1}{\pi}\int_0^{2\pi}\rho(\theta)\mathrm{d}\theta,$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi}\rho(\theta)\cos n\theta\,\mathrm{d}\theta, \quad b_n = \frac{1}{\pi}\int_0^{2\pi}\rho(\theta)\sin n\theta\,\mathrm{d}\theta, \quad n \in \mathbb{Z}^+. \tag{2.3}$$

In this paper, we introduce for the star-shaped curve γ the function $\rho_k(\theta)$:

$$\rho_k(\theta) = \rho(\theta) + \rho\left(\theta + \frac{2\pi}{k}\right) + \cdots + \rho\left(\theta + \frac{2(k-1)\pi}{k}\right) \tag{2.4}$$

for an integer $k \geq 2$, which is called the *k-order radial function* of γ. For $k = 2$, $\rho_2(\theta)$ is the length of the chord passing through the origin in the direction $\vec{u} = (\cos\theta, \sin\theta)$. Thus, the function $\rho_k(\theta)$ is a natural generalization of the chord of γ passing through the origin.

3. Main results

For a given integer $k \geq 2$, we get the following dual Chernoff–Ou–Pan inequality.

Theorem 3.1 *Let γ be a C^1 star-shaped curve in $\mathbb{R}^2$ with area A it bounds. Then*

$$A \geq \frac{1}{k}\int_0^{\frac{\pi}{k}} \rho_k(\theta)\rho_k\left(\theta + \frac{\pi}{k}\right)\mathrm{d}\theta, \tag{3.1}$$

and the equality in (3.1) *holds if and only if the radial function of γ is of the form*

$$\rho(\theta) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_{2nk}\cos 2nk\theta + b_{2nk}\sin 2nk\theta). \tag{3.2}$$

The following lemma plays a crucial role in the proof of Theorem 3.1.

Lemma 3.2

$$\int_0^{\frac{\pi}{k}} \rho_k(\theta)\rho_k\left(\theta + \frac{\pi}{k}\right) \mathrm{d}\theta = \frac{1}{2} \sum_{i=0}^{k-1} \int_0^{2\pi} \rho(\theta)\rho(\theta + \frac{(2i+1)\pi}{k}) \mathrm{d}\theta. \tag{3.3}$$

Proof   It follows from (2.4) that

$$\rho_k(\theta)\rho_k\left(\theta + \frac{\pi}{k}\right)$$

$$= \sum_{m=0}^{2(k-1)} \rho\left(\theta + \frac{m\pi}{k}\right) \rho\left(\theta + \frac{(m+1)\pi}{k}\right) + \sum_{m=0}^{2(k-2)} \rho\left(\theta + \frac{m\pi}{k}\right) \rho\left(\theta + \frac{(m+3)\pi}{k}\right)$$

$$+ \cdots + \sum_{m=0}^{2i} \rho\left(\theta + \frac{m\pi}{k}\right) \rho\left(\theta + \frac{(m+2k-1-2i)\pi}{k}\right)$$

$$+ \cdots + \sum_{m=0}^{2} \rho\left(\theta + \frac{m\pi}{k}\right) \rho\left(\theta + \frac{(m+2k-3)\pi}{k}\right) + \rho(\theta)\rho\left(\theta + \frac{(2k-1)\pi}{k}\right).$$

Moreover,

$$\rho_k(\theta)\rho_k\left(\theta + \frac{\pi}{k}\right) = \sum_{i=0}^{k-1} \sum_{m=0}^{2i} \rho(\theta + \frac{m\pi}{k})\rho(\theta + \frac{(m+2k-1-2i)\pi}{k}). \tag{3.4}$$

A simple computation can give us

$$\sum_{m=0}^{2i} \int_0^{\frac{\pi}{k}} \rho(\theta + \frac{m\pi}{k})\rho(\theta + \frac{(m+2k-2i-1)\pi}{k}) \mathrm{d}\theta$$

$$= \int_0^{\frac{(2i+1)\pi}{k}} \rho(\theta)\rho(\theta + \frac{(2k-2i-1)\pi}{k}) \mathrm{d}\theta, \tag{3.5}$$

where $i = 0, 1, \cdots, k-1$. Let $j = (k-1) - i$; thus

$$\sum_{i=0}^{k-1} \int_0^{\frac{(2i+1)\pi}{k}} \rho(\theta)\rho(\theta + \frac{(2k-2i-1)\pi}{k}) \mathrm{d}\theta$$

$$= \sum_{j=0}^{k-1} \int_0^{\frac{(2k-2j-1)\pi}{k}} \rho(\theta)\rho(\theta + \frac{(2j+1)\pi}{k}) \mathrm{d}\theta. \tag{3.6}$$

From (3.4)–(3.6) and the fact that $\rho(\theta)$ is 2π-periodic, one can obtain

$$\int_0^{\frac{\pi}{k}} \rho_k(\theta)\rho_k(\theta + \frac{\pi}{k})\mathrm{d}\theta = \sum_{i=0}^{k-1} \int_0^{\frac{(2i+1)\pi}{k}} \rho(\theta)\rho(\theta + \frac{(2k-2i-1)\pi}{k})\mathrm{d}\theta$$

$$= \frac{1}{2}\sum_{i=0}^{k-1}\left(\int_0^{\frac{(2i+1)\pi}{k}} \rho(\theta)\rho(\theta + \frac{(2k-2i-1)\pi}{k})\mathrm{d}\theta + \int_0^{[2-\frac{(2i+1)}{k}]\pi} \rho(\theta)\rho(\theta + \frac{(2i+1)\pi}{k})\mathrm{d}\theta\right)$$

$$= \frac{1}{2}\sum_{i=0}^{k-1}\left(\int_{[2-\frac{(2i+1)}{k}]\pi}^{2\pi} \rho(\theta + \frac{(2i+1)\pi}{k})\rho(\theta)\mathrm{d}\theta + \int_0^{[2-\frac{(2i+1)}{k}]\pi} \rho(\theta)\rho(\theta + \frac{(2i+1)\pi}{k})\mathrm{d}\theta\right)$$

$$= \frac{1}{2}\sum_{i=0}^{k-1} \int_0^{2\pi} \rho(\theta)\rho(\theta + \frac{(2i+1)\pi}{k})\mathrm{d}\theta.$$

$\square$

Based on Lemma 3.2, we give two proofs of Theorem 3.1. The first uses the Schwarz inequality and the second uses the Parseval equality. However, the first proof is easier for establishing the inequality, while the second proof is easier for obtaining the equality condition.

First proof of Theorem 3.1 Lemma 3.2 and the Schwarz inequality yield

$$\frac{1}{k}\int_0^{\frac{\pi}{k}} \rho_k(\theta)\rho_k(\theta + \frac{\pi}{k})\mathrm{d}\theta$$

$$= \frac{1}{2k}\sum_{i=0}^{k-1} \int_0^{2\pi} \rho(\theta)\rho(\theta + \frac{(2i+1)\pi}{k})\mathrm{d}\theta$$

$$\leq \frac{1}{k}\sum_{i=0}^{k-1}\left\{\frac{1}{2}\int_0^{2\pi} \rho^2(\theta)\mathrm{d}\theta\right\}^{\frac{1}{2}}\left\{\frac{1}{2}\int_0^{2\pi} \rho^2(\theta + \frac{(2i+1)\pi}{k})\mathrm{d}\theta\right\}^{\frac{1}{2}}$$

$$= A,$$

where the equality holds if and only if for $i = 0, 1, \ldots, k-1$,

$$\rho(\theta) = r\rho(\theta + \frac{(2i+1)\pi}{k}),$$

where r is a constant. It follows that

$$\rho(\theta) = r\rho(\theta + \frac{(2i+1)\pi}{k}) = r^2\rho(\theta + \frac{2(2i+1)\pi}{k}) = \cdots = r^{2k}\rho(\theta),$$

which together with $\rho(\theta) > 0$ implies that $r = 1$, that is,

$$\rho(\theta) = \rho(\theta + \frac{(2i+1)\pi}{k}). \tag{3.7}$$

By a direct computation, it follows from (3.7) and (2.2) that for $i = 0, 1, \ldots, k-1$,

$$\sum_{n=1}^{\infty} a_n \sin \frac{n(2i+1)\pi}{2k} \sin \left(n\theta + \frac{n(2i+1)\pi}{2k} \right)$$

$$-\sum_{n=1}^{\infty} b_n \sin \frac{n(2i+1)\pi}{2k} \cos \left(n\theta + \frac{n(2i+1)\pi}{2k} \right) = 0.$$

Hence, $a_n \sin \frac{n(2i+1)\pi}{2k} = b_n \sin \frac{n(2i+1)\pi}{2k} = 0$ for $i = 0, 1, \ldots, k-1$, $n \in \mathbb{Z}^+$, which implies that $a_n = b_n = 0$ or $n = 2km$, $m \in \mathbb{Z}$, which leads us to the desired result. $\qquad \square$

Second proof of Theorem 3.1 It follows from (2.2) that

$$\rho\left(\theta + \frac{(2i+1)\pi}{k}\right)$$

$$= \frac{1}{2}a_0 + \sum_{n=1}^{\infty} \left(a_n \cos \left(n\theta + \frac{(2i+1)n\pi}{k} \right) + b_n \sin \left(n\theta + \frac{(2i+1)n\pi}{k} \right) \right)$$

$$= \frac{1}{2}a_0 + \sum_{n=1}^{\infty} \left(a_n \cos n\theta \cos \frac{(2i+1)n\pi}{k} - a_n \sin n\theta \sin \frac{(2i+1)n\pi}{k} \right)$$

$$+ \sum_{n=1}^{\infty} \left(b_n \sin n\theta \cos \frac{(2i+1)n\pi}{k} + b_n \cos n\theta \sin \frac{(2i+1)n\pi}{k} \right).$$

By (2.3) and the Parseval equality, we obtain

$$\int_0^{2\pi} \rho(\theta)\rho\left(\theta + \frac{(2i+1)\pi}{k}\right) d\theta = \frac{\pi}{2}a_0^2 + \pi \sum_{n=1}^{\infty} (a_n^2 + b_n^2) \cos \frac{(2i+1)n\pi}{k}. \qquad (3.8)$$

When $n \neq km$, $m \in \mathbb{Z}$,

$$\sum_{i=0}^{k-1} \cos \frac{(2i+1)n\pi}{k} = \frac{1}{\sin \frac{n\pi}{k}} \sum_{i=0}^{k-1} \cos \frac{(2i+1)n\pi}{k} \sin \frac{n\pi}{k}$$

$$= \frac{1}{\sin \frac{n\pi}{k}} \left(\cos \frac{n\pi}{k} \sin \frac{n\pi}{k} + \cos \frac{3n\pi}{k} \sin \frac{n\pi}{k} + \cdots + \cos \frac{(2k-1)n\pi}{k} \sin \frac{n\pi}{k} \right) \qquad (3.9)$$

$$= \frac{\sin 2n\pi}{2\sin(n\pi/k)} = 0.$$

From (3.8), (3.9), and (3.3) we can get

$$
\begin{aligned}
I =& \frac{1}{k} \int_0^{\frac{\pi}{k}} \rho_k(\theta) \rho_k(\theta + \frac{\pi}{k}) \mathrm{d}\theta \\
=& \frac{1}{2k} \sum_{i=0}^{k-1} \left(\frac{\pi}{2} a_0^2 + \pi \sum_{n=1}^{\infty} (a_n^2 + b_n^2) \cos \frac{(2i+1)n\pi}{k} \right) \\
=& \frac{\pi}{4} a_0^2 + \frac{\pi}{2} \sum_{n=1}^{\infty} (a_{nk}^2 + b_{nk}^2) \sum_{i=0}^{k-1} \frac{1}{k} \cos(2i+1)n\pi \\
=& \frac{\pi}{4} a_0^2 + \frac{\pi}{2} \sum_{n=1}^{\infty} (-1)^n (a_{nk}^2 + b_{nk}^2).
\end{aligned}
\tag{3.10}
$$

We also wish to express A in terms of the Fourier coefficients of $\rho(\theta)$. By the Parseval equality and (2.1), we get

$$
A = \frac{1}{4} \pi a_0^2 + \frac{1}{2} \pi \sum_{n=1}^{\infty} (a_n^2 + b_n^2),
\tag{3.11}
$$

which together with (3.10) gives us $I \leq A$ and the equality holds if and only if

$$
\rho(\theta) = \frac{1}{2} a_0 + \sum_{n=1}^{\infty} (a_{2nk} \cos 2nk\theta + b_{2nk} \sin 2nk\theta),
$$

and this completes the proof. $\square$

Corollary 3.3 *Let γ be a C^1 star-shaped curve in $\mathbb{R}^2$ with area A. If $\rho_k(\theta) = C$, where C is a constant; then*

$$
A = \frac{1}{k} \int_0^{\frac{\pi}{k}} \rho_k(\theta) \rho_k(\theta + \frac{\pi}{k}) \mathrm{d}\theta = \frac{C^2 \pi}{k^2}
$$

if and only if γ is a circle of radius $\frac{1}{k} C$ and centered at the origin.

Remark 1 *For $k = 2$, let γ be a star-shaped curve with $\rho(\theta) = 2 + \sin 4\theta$, then*

$$
\frac{1}{2} \int_0^{\frac{\pi}{2}} \rho_2(\theta) \rho_2(\theta + \frac{\pi}{2}) \mathrm{d}\theta = \frac{9}{2} \pi = A,
$$

but γ is not a circle. This shows that the condition $\rho_k(\theta) = C$ in Corollary 3.3 is necessary, which is different from the equality case of the Chernoff–Ou–Pan inequality [1, 8].

Proof of Corollary 3.3 The assumption $\rho_k(\theta) = C$ implies that

$$
a_{nk} = b_{nk} = 0, \, n = 1, 2, \cdots.
\tag{3.12}
$$

In fact, (2.2) and (2.4) lead to

$$
\begin{aligned}
\rho_k(\theta) =& \frac{1}{2}ka_0 + \sum_{n=1}^{\infty}\left(a_n\cos n\theta + b_n\sin n\theta\right) \\
&+ \cdots + \sum_{n=1}^{\infty}\left(a_n\cos\left(n\theta + \frac{2n(k-1)\pi}{k}\right) + b_n\sin\left(n\theta + \frac{2n(k-1)\pi}{k}\right)\right) \\
=& \frac{1}{2}ka_0 + \sum_{n=1}^{\infty}\left(a_n\cos n\theta + b_n\sin n\theta\right)\left(1 + \cos\frac{2n\pi}{k} + \cdots + \cos\frac{2n(k-1)\pi}{k}\right) \\
&+ \sum_{n=1}^{\infty}\left(b_n\cos n\theta - a_n\sin n\theta\right)\left(\sin\frac{2n\pi}{k} + \cdots + \sin\frac{2n(k-1)\pi}{k}\right).
\end{aligned}
\tag{3.13}
$$

When $n = km$, $m \in \mathbb{Z}^+$,

$$
\begin{aligned}
1 + \cos\frac{2n\pi}{k} + \cdots + \cos\frac{2n(k-1)\pi}{k} &= k, \\
\sin\frac{2n\pi}{k} + \cdots + \sin\frac{2n(k-1)\pi}{k} &= 0.
\end{aligned}
\tag{3.14}
$$

When $n \neq km$, $m \in \mathbb{Z}^+$,

$$
\begin{aligned}
1 + \cos\frac{2n\pi}{k} + \cdots + \cos\frac{2n(k-1)\pi}{k} &= 0, \\
\sin\frac{2n\pi}{k} + \cdots + \sin\frac{2n(k-1)\pi}{k} &= 0.
\end{aligned}
\tag{3.15}
$$

From (3.13)–(3.15) we get

$$
\rho_k(\theta) = k\left(\frac{1}{2}a_0 + \sum_{n=1}^{\infty}\left(a_{nk}\cos nk\theta + b_{nk}\sin nk\theta\right)\right).
$$

Since $\rho_k(\theta) = C$, it is clear that

$$
a_{nk} = b_{nk} = 0, \ \ n = 1, 2, \cdots.
\tag{3.16}
$$

If γ is a circle, it is obvious that $I = A = \frac{\pi C^2}{k^2}$. Conversely, suppose that $I = A$; then the equality in Theorem 3.1 holds, which together with $\rho_k(\theta) = C$ gives us $\rho(\theta) = \frac{a_0}{2}$ and $\frac{a_0}{2} = \frac{C}{k}$. This completes the proof of Corollary 3.3. □

From the proof of Corollary 3.3, we can get

Corollary 3.4 *Let γ be a C^1 star-shaped curve in $\mathbb{R}^2$ with $\rho_k(\theta)$ equal to a constant C. Then the Fourier expansion of the radial function $\rho(\theta)$ of γ is of the form*

$$
\rho(\theta) = \frac{1}{2}a_0 + \sum_{\substack{n=1 \\ n \neq mk}}^{\infty}\left(a_n\cos n\theta + b_n\sin n\theta\right), \quad m \in \mathbb{Z}^+,
\tag{3.17}
$$

where $a_0 = \frac{2C}{k}$.

Obviously, if γ is an equichordal curve, that is, $\rho_2(\theta) = $ const., then the Fourier series of $\rho(\theta)$ for γ has only odd terms and a constant term.
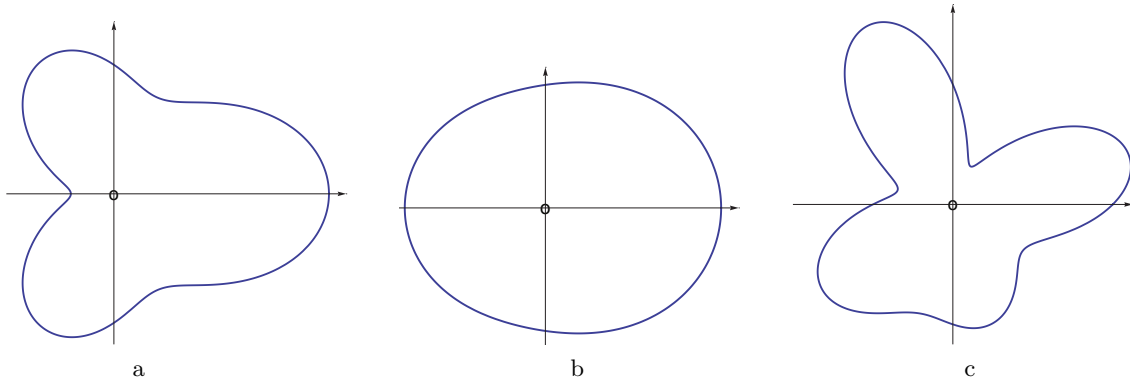
a   b   c

Figure. Examples.

4. Applications

Let γ be a star-shaped curve in $\mathbb{R}^2$ with area A. Let D denote the star body enclosed by γ and B denote the unit disc in $\mathbb{R}^2$. From the continuity of $\rho(\theta)$, it is easy to see that, for all $\theta_k \in [0, \frac{2\pi}{k}]$,

$$\lim_{k\to\infty} \frac{2\pi}{k} \rho_k(\theta_k) = \lim_{k\to\infty} \frac{2\pi}{k} \sum_{m=1}^{k} \rho\left(\theta_k + \frac{2m\pi}{k}\right) = \int_0^{2\pi} \rho(\theta)\mathrm{d}\theta.$$

Moreover, for any $k \in \mathbb{N}$, there exists a $\xi_k \in [0, \frac{\pi}{k}]$ such that

$$\frac{1}{k}\int_0^{\frac{\pi}{k}} \rho_k(\theta)\rho_k(\theta + \frac{\pi}{k})\mathrm{d}\theta = \frac{\pi}{k^2}\rho_k(\xi_k)\rho_k(\xi_k + \frac{\pi}{k}).$$

Since $\xi_k \in [0, \frac{\pi}{k}] \subset [0, \frac{2\pi}{k}]$, we have $\xi_k + \frac{\pi}{k} \in [0, \frac{2\pi}{k}]$. Thus, we obtain

$$\lim_{k\to\infty} \frac{1}{k}\int_0^{\frac{\pi}{k}} \rho_k(\theta)\rho_k(\theta + \frac{\pi}{k})\mathrm{d}\theta = \lim_{k\to\infty} \frac{1}{k^2}\rho_k(\xi_k)\rho_k(\xi_k + \frac{\pi}{k})$$

$$= \frac{1}{4\pi}\left(\int_0^{2\pi} \rho(\theta)\mathrm{d}\theta\right)^2 = \frac{1}{\pi}\tilde{A}^2(D, B),$$

which together with (3.1) gives us a new proof of the classical dual isoperimetric inequality (1.3).

Now, for a given integer $k \geq 2$, the star-shaped curve γ is called a *k-order equichordal curve with respect to the origin*, if $\rho_k(\theta)$ is a constant. Therefore, k-order equichordal curves can be regarded as a generalization of the equichordal curves.

By Corollary 3.4, it can be easily seen that there are many star-shaped curves with $\rho_k(\theta)$ equal to a constant C besides circles; here are three examples.

Examples (i) Take $\rho(\theta) = 3 + \cos\theta + \cos 3\theta$, then $\rho_2(\theta) = 6$, γ is an equichordal curve with respect to the origin; see Figure 1a.

(ii) Take $\rho(\theta) = \frac{1}{2}(8 + \cos\theta + \cos 2\theta)$, then $\rho_3(\theta) = 12$, γ is a 3-order equichordal curve with respect to the origin; see Figure 1b.

(iii) Take $\rho(\theta) = 3 + \cos 3\theta + \sin 4\theta$, then $\rho_5(\theta) = 15$, γ is a 5-order equichordal curve with respect to the origin; see Figure 1c.

Considering a class of star-shaped curves with $\rho_k(\theta) = \text{const.}$, we can obtain the following inequality:

Theorem 4.1 *Let γ be a C^1 star-shaped curve in $\mathbb{R}^2$ with $\rho_k(\theta)$ equal to a constant C; then*

$$\frac{C^2}{k^2} \le A < \frac{\pi C^2}{k}, \tag{4.1}$$

where A is the area it bounds, and the equality on the left-hand side holds if and only if γ is a circle of radius $\frac{1}{k}C$ and centered at the origin.

Proof By (2.1) and $\rho(\theta) > 0$, we have

$$
\begin{aligned}
A =& \frac{1}{2} \int_0^{2\pi} \rho^2(\theta) \mathrm{d}\theta \\
=& \frac{1}{2} \int_0^{\frac{2\pi}{k}} \left(\rho^2(\theta) + \rho^2(\theta + \frac{2\pi}{k}) + \cdots + \rho^2(\theta + \frac{2(k-1)\pi}{k}) \right) \mathrm{d}\theta \\
<& \frac{1}{2} \int_0^{\frac{2\pi}{k}} \left(\rho(\theta) + \rho(\theta + \frac{2\pi}{k}) + \cdots + \rho(\theta + \frac{2(k-1)\pi}{k}) \right)^2 \mathrm{d}\theta.
\end{aligned}
\tag{4.2}
$$

Combining (4.2) with $\rho_k(\theta) = C$ yields

$$A < \frac{\pi C^2}{k}.$$

From Theorem 3.1 and Corollary 3.3 we have $A \ge \frac{\pi C^2}{k^2}$ and the equality holds if and only if γ is a disc. □

In particular, let γ be a C^1 equichordal curve in $\mathbb{R}^2$ with area A. If $\rho(\theta) + \rho(\theta + \pi) = C$ (a constant), then

$$\frac{1}{4}\pi C^2 \le A < \frac{1}{2}\pi C^2, \tag{4.3}$$

and the equality on the left-hand side holds if and only if γ is a circle of radius $\frac{1}{2}C$ and centered at the origin.

It follows from (4.3) that circles have the minimal area among all the equichordal curves, but we do not know which has the greatest area. Naturally, we pose a dual Blaschke–Lebesgue problem:

Problem Among all curves with $\rho(\theta) + \rho(\theta + \pi)$ being equal to a constant C, which has the greatest area?

Acknowledgments

References

[1] Chernoff PR. An area-width inequality for convex curves. Amer Math Monthly 1969; 76: 34–35.

[2] Fujiwara M. Über die Mittelkurve zweier geschlossenen convexen kurven in Bezug auf einen Punkt. Töhoku Mathematical J 1916; 10: 99–103.

[3] Gao L, Wang YL. Stability properties of the generalizaed Chernoff inequality. Mathematical inequality and Applications 2012; 15: 281–287.

[4] Lutwak E. Dual mixed volumes. Pacific J Math 1975; 58: 531–538.

[5] Lutwak E. Intersection bodies and dual mixed volumes. Adv Math 1988; 71: 232–261.

[6] Lutwak E. Centroid bodies and dual mixed volumes. Proc Lond Math Soc 1990; 60: 365–391.

[7] Moszyńska M. Selected Topics in Convex Geometry. Boston, MA, USA: Birkhäuser, 2006.

[8] Ou K, Pan SL. Some remarks about closed convex curves. Pacific J Math 2010; 248: 393–401.

[9] Rychlik MR. A complete solution to the Equichordal Point Problem of Fujiwara, Blaschke, Rothe and Weizenböck. Inventiones Mathematicae 1997; 129: 141–212.

[10] Zhang G. Centered bodies and dual mixed volumes. Trans Amer Math Soc 1994; 345: 777–801.

[11] Zhao CJ, Leng GS. Inequalities for dual quermassintegrals of mixed intersection bodies. Proc Indian Acad Sci Math Sci 2005; 115: 79–91.