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Fixed point theory in WC–Banach algebras

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Abstract: In this paper, we will prove some fixed point theorems for the sum and the product of nonlinear weakly sequentially continuous operators acting on a WC–Banach algebra. Our results improve and correct some recent results given by Banas and Taoudi, and extend several earlier works using the condition $(\mathcal{P})$.

Key words: WC–Banach algebra, fixed point theorems, weak topology, measure of weak noncompactness

1. Introduction

In 1998, Dhage [10] proved the following fixed point theorem involving three operators in a Banach algebra by combining Schauder's fixed point theorem and Banach's contraction principle.

Theorem 1.1 *Let S be a nonempty, bounded, closed, and convex subset of a Banach algebra $\mathcal{X}$ and let A, B, $C : S \to S$ be three operators, such that:*

(i) A and C are Lipschitzian with Lipschitz constants α and β, respectively,

(ii) $\left(\frac{I-C}{A}\right)^{-1}$ exists on $B(S)$, I being the identity operator on $\mathcal{X}$, and the operator $\left(\frac{I-C}{A}\right) : \mathcal{X} \to \mathcal{X}$ is defined by $\left(\frac{I-C}{A}\right)x := \frac{x-Cx}{Ax}$,

(iii) B is completely continuous, and

(iv) $AxBy + Cx \in S,\ \forall\ x, y \in S$.

Then the operator equation $x = AxBx + Cx$ has, at least, a solution in S, whenever $\alpha M + \beta < 1$, where $M = \|B(S)\| = \sup\{\|Bx\| : \ x \in S\}$. ♢

It is well known that Theorem 1.1 has a wide range of applications to nonlinear integral equations of mixed type. Moreover, it has also been extensively used in differential equations. Several attempts have been made in the literature to prove the analogousness of Theorem 1.1 under weak topology. In [6], Ben Amar et al. established a weak variant of Theorem 1.1 in Banach algebras, satisfying the following sequential condition:

$$(\mathcal{P}) \qquad \begin{cases} \text{For any sequences } (x_n)_{n\in\mathbb{N}} \text{ and } (y_n)_{n\in\mathbb{N}} \text{ of } \mathcal{X} \text{ such that } x_n \rightharpoonup x \\ \text{and } y_n \rightharpoonup y, \text{ then } x_n y_n \rightharpoonup xy; \text{ where } \mathcal{X} \text{ is a Banach algebra.} \end{cases}$$

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It is interesting to note that their result requires both the weak sequential continuity and the weak compactness of the operators A, B, and C. Their proof is based on the Arino, Gautier, and Penot's fixed point theorem [3] and also on the weak sequential continuity of $\left(\frac{I-C}{A}\right)^{-1}B$.

In their paper [7], Ben Amar et al. have established some fixed point theorems in Banach algebras satisfying the condition $(\mathcal{P})$ under the weak topology. Their results were based on the class of weakly condensing, weak sequential continuity, and weakly compact.

Recently, Banas and Taoudi [4] gave a generalization of some results established in [6] for the weak topology case. The authors used the notion of a WC–Banach algebra (cf. Definition 3.1), which is weaker than the property $(\mathcal{P})$ and the concept of the De Blasi measure of weak noncompactness [9]. However, in the proofs of Theorems 3.1 and 3.3, Banas and Taoudi used the sequential condition $(\mathcal{P})$ in any Banach algebra, which is not always valid.

The main goal of the present section is to establish new variants of Theorem 1.1 for three operators acting on WC–Banach algebras, without using the sequential condition $(\mathcal{P})$. This result can be considered an extension of [7, 12, 15, 17].

2. Preliminaries

Throughout this section, $\mathcal{X}$ denotes a Banach space. For any $r > 0$, B_r denotes the closed ball in $\mathcal{X}$ centered at $0_{\mathcal{X}}$ with radius r and $\mathcal{D}(A)$ denotes the domain of an operator A. $\Omega_{\mathcal{X}}$ is the collection of all nonempty bounded subsets of $\mathcal{X}$ and $\mathcal{K}^w$ is the subset of $\Omega_{\mathcal{X}}$ consisting of all weakly compact subsets of $\mathcal{X}$. In the remainder, $\rightharpoonup$ denotes the weak convergence and $\to$ denotes the strong convergence in $\mathcal{X}$. Recall that the notion of the measure of weak noncompactness was introduced by De Blasi [9]; it is the map $\omega : \Omega_{\mathcal{X}} \to [0, +\infty)$ defined in the following way:

$$\omega(\mathcal{M}) = \inf\big\{r > 0 : \text{ there exists } K \in \mathcal{K}^w \text{ such that } \mathcal{M} \subseteq K + B_r\big\},$$

for all $\mathcal{M} \in \Omega_{\mathcal{X}}$. For convenience we recall some basic properties of $\omega(.)$ needed below [2, 9].

Lemma 2.1 *Let $\mathcal{M}_1$, $\mathcal{M}_2$ be two elements of $\Omega_{\mathcal{X}}$. Then the following conditions are satisfied:*

1. $\mathcal{M}_1 \subseteq \mathcal{M}_2$ *implies* $\omega(\mathcal{M}_1) \leq \omega(\mathcal{M}_2)$.
2. $\omega(\mathcal{M}_1) = 0$, *if and only if* $\overline{\mathcal{M}_1}^w \in \mathcal{K}^w$, *i.e.* $\overline{\mathcal{M}_1}^w$ *is the weak closure of* $\mathcal{M}_1$.
3. $\omega(\overline{\mathcal{M}_1}^w) = \omega(\mathcal{M}_1)$.
4. $\omega(\mathcal{M}_1 \cup \mathcal{M}_2) = \max\{\omega(\mathcal{M}_1), \omega(\mathcal{M}_2)\}$.
5. $\omega(\lambda\mathcal{M}_1) = |\lambda|\omega(\mathcal{M}_1)$ *for all* $\lambda \in \mathbb{R}$.
6. $\omega(co(\mathcal{M}_1)) = \omega(\mathcal{M}_1)$, *where* $co(\mathcal{M}_1)$ *denotes the convex hull of* $\mathcal{M}_1$.
7. $\omega(\mathcal{M}_1 + \mathcal{M}_2) \leq \omega(\mathcal{M}_1) + \omega(\mathcal{M}_2)$.

8. if $(\mathcal{M}_n)_{n\geq 1}$ *is a decreasing sequence of nonempty bounded and weakly closed subsets of* $\mathcal{X}$ *with* $\lim_{n\to\infty}\omega(\mathcal{M}_n) = 0$*, then* $\mathcal{M}_\infty := \cap_{n=1}^{\infty}\mathcal{M}_n$ *is nonempty and* $\omega(\mathcal{M}_\infty) = 0$*, that is,* $\mathcal{M}_\infty$ *is relatively weakly compact.* ♢

Definition 2.1 *An operator* $A : \mathcal{D}(A) \subseteq \mathcal{X} \to \mathcal{X}$ *is said to be weakly sequentially continuous on* $\mathcal{D}(A)$ *if for every sequence* $(x_n)_{n\in\mathbb{N}} \subseteq \mathcal{D}(A)$*,* $x_n \rightharpoonup x$ *implies* $Ax_n \rightharpoonup Ax$*.* ♢

Definition 2.2 *An operator* $A : \mathcal{D}(A) \subseteq \mathcal{X} \to \mathcal{X}$ *is said to be* ω*-contractive (or an* ω*-*α*-contraction) if it maps bounded sets into bounded sets, and there exists some* $\alpha \in [0, 1)$ *such that* $\omega(A\mathcal{S}) \leq \alpha\omega(\mathcal{S})$ *for all bounded subsets* $\mathcal{S} \subseteq \mathcal{D}(A)$*.*

An operator $A : \mathcal{D}(A) \subseteq \mathcal{X} \to \mathcal{X}$ *is said to be* ω*-condensing if it maps bounded sets into bounded sets, and* $\omega(A(\mathcal{S})) < \omega(\mathcal{S})$ *for all bounded sets* $\mathcal{S} \subseteq \mathcal{D}(A)$ *with* $\omega(\mathcal{S}) > 0$. ♢

Remark 2.1 *Obviously, every* ω*-*α*-contraction with* $0 \leq \alpha < 1$ *is* ω*-condensing.* ♢

Let $A : \mathcal{D}(A) \subseteq \mathcal{X} \to \mathcal{X}$ be an operator. In what follows, we will use the following conditions:

$$(\mathcal{H}1) \qquad \begin{cases} \text{If } (x_n)_{n\in\mathbb{N}} \subseteq \mathcal{D}(A) \text{ is a weakly convergent sequence in } \mathcal{X}\text{, then} \\ (Ax_n)_{n\in\mathbb{N}} \text{ has a strongly convergent subsequence in } \mathcal{X}. \end{cases}$$

$$(\mathcal{H}2) \qquad \begin{cases} \text{If } (x_n)_{n\in\mathbb{N}} \subseteq \mathcal{D}(A) \text{ is a weakly convergent sequence in } \mathcal{X}\text{, then} \\ (Ax_n)_{n\in\mathbb{N}} \text{ has a weakly convergent subsequence in } \mathcal{X}. \end{cases}$$

The conditions $(\mathcal{H}1)$ and $(\mathcal{H}2)$ were already considered in the papers [1, 12, 14, 17].

Definition 2.3 *An operator* $A : \mathcal{D}(A) \subseteq \mathcal{X} \to \mathcal{X}$ *is called* $\mathcal{D}$*-Lipschitzian if there exists a continuous and nondecreasing function* $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ *such that*

$$\|Ax - Ay\| \leq \phi(\|x - y\|),$$

for all x*,* $y \in \mathcal{D}(A)$*, where* $\phi(0) = 0$*. The function* ϕ *is called a* $\mathcal{D}$*-function of* A*. Obviously every Lipschitzian mapping is* $\mathcal{D}$*-Lipschitzian, but the converse may not be true. Moreover, if* $\phi(r) < r$ *for* $r > 0$*, then the operator* A *is called a nonlinear contraction with a contraction function* ϕ*.* ♢

The following lemma is useful for the sequel.

Lemma 2.2 *[1] Let* $A : \mathcal{D}(A) \subseteq \mathcal{X} \to \mathcal{X}$ *be an operator* $\mathcal{D}$*-Lipschitzian with a* $\mathcal{D}$*-function* ϕ *on a Banach space* $\mathcal{X}$ *and satisfy* $(\mathcal{H}2)$*. Then for each bounded subset* $\mathcal{S}$ *of* $\mathcal{D}(A)$ *one has*

$$\omega(A(\mathcal{S})) \leq \phi(\omega(\mathcal{S})).$$

♢

3. Fixed point theorems in WC–Banach algebras

First, let us recall the following definition used by Banas and Taoudi [4].

Definition 3.1 *Let $\mathcal{X}$ be a Banach algebra. We say that $\mathcal{X}$ is a WC–Banach algebra if the product WW' of arbitrary weakly compact subsets W, W' of $\mathcal{X}$ is weakly compact.* ♢

Clearly, every finite dimensional Banach algebra is a WC–Banach algebra. Even if $\mathcal{X}$ is a WC–Banach algebra then the set $\mathcal{C}(\mathcal{K}, \mathcal{X})$ of all continuous functions from $\mathcal{K}$ to $\mathcal{X}$ is also a WC–Banach algebra, where $\mathcal{K}$ is a compact Hausdorff space. The proof is based on Dobrakov's Theorem [11].

In order to present the main fixed point results of this section, we certainly need the following helpful theorems.

Theorem 3.1 *[5] Let S be a nonempty, bounded, closed, and convex subset of a Banach space $\mathcal{Y}$ and let $A : S \to S$ be a weakly sequentially continuous mapping. If A is ω-condensing, then it has, at least, a fixed point in S.* ♢

Lemma 3.1 *[4] Let $\mathcal{M}$ and $\mathcal{M}'$ be two bounded subsets of a WC–Banach algebra $\mathcal{X}$. Then we have the following inequality*

$$\omega(\mathcal{M}\mathcal{M}') \leq \|\mathcal{M}'\|\omega(\mathcal{M}) + \|\mathcal{M}\|\omega(\mathcal{M}') + \omega(\mathcal{M})\omega(\mathcal{M}').$$

♢

Now we are ready to state our first fixed point theorem in WC–Banach algebra in order to provide the existence of solutions for the operator equation $x = AxBx + Cx$.

Theorem 3.2 *Let S be a nonempty, bounded, closed, and convex subset of a WC–Banach algebra $\mathcal{X}$ and let A, $C : \mathcal{X} \to \mathcal{X}$ and $B : S \to \mathcal{X}$ be three weakly sequentially continuous operators satisfying the following conditions:*

(i) $\left(\frac{I-C}{A}\right)^{-1}$ *exists on* $B(S)$,

(ii) A *satisfies* $(\mathcal{H}1)$, *and* $A(S)$ *is relatively weakly compact,*

(iii) B *is an* ω-β-*contraction,*

(iv) C *is an* ω-α-*contraction, and*

(v) $(x = AxBy + Cx,\ y \in S) \Longrightarrow x \in S$.

Then the operator equation $x = AxBx + Cx$ has, at least, a solution in S, whenever $\frac{\gamma\beta}{1-\alpha} < 1$, where $\gamma = \|A(S)\|$. ♢

Proof It is easy to check that the vector $x \in S$ is a solution for the equation $x = AxBx + Cx$, if and only if x is a fixed point for the operator $T := \left(\frac{I-C}{A}\right)^{-1}B$. From assumption (i), it follows that, for each $y \in S$, there is a unique $x_y \in \mathcal{X}$ such that

$$\left(\frac{I-C}{A}\right)x_y = By,$$

or, in an equivalent way,

$$Ax_yBy + Cx_y = x_y.$$

Since the hypothesis (v) holds, then $x_y \in S$. Hence, the map $T : S \to S$ is well defined. In order to achieve the proof, we will apply Theorem 3.1. Hence, we only have to prove that the operator $T : S \to S$ is weakly sequentially continuous and ω-condensing. Indeed, consider $(x_n)_{n\in\mathbb{N}}$ as a sequence in S that is weakly convergent to x. In this case, the set $\{x_n : n \in \mathbb{N}\}$ is relatively weakly compact, and since B is weakly sequentially continuous, then $\{Bx_n : n \in \mathbb{N}\}$ is also relatively weakly compact. Using the following equality

$$T = ATB + CT, \tag{3.1}$$

combined with the fact that $A(S)$ is relatively weakly compact and C is an ω-α-contraction, we obtain

$$\begin{aligned} \omega(\{Tx_n : n \in \mathbb{N}\}) &\leq \omega(\{A(Tx_n)Bx_n : n \in \mathbb{N}\}) + \omega(\{C(Tx_n) : n \in \mathbb{N}\}) \\ &\leq \alpha\omega(\{Tx_n : n \in \mathbb{N}\}) \\ &< \omega(\{Tx_n : n \in \mathbb{N}\}). \end{aligned}$$

Hence, $\{Tx_n : n \in \mathbb{N}\}$ is relatively weakly compact. Consequently, there exists a subsequence $(x_{n_i})_{i\in\mathbb{N}}$ of $(x_n)_{n\in\mathbb{N}}$ such that $Tx_{n_i} \rightharpoonup y$. Going back to Eq. (3.1); to the weak sequential continuity of A, B, and C; and to the fact that A verifies $(\mathcal{H}1)$ shows that there exists a subsequence $(x_{n_{i_j}})_{j\in\mathbb{N}}$ of $(x_{n_i})_{i\in\mathbb{N}}$ such that $Tx_{n_{i_j}} = A(Tx_{n_{i_j}})Bx_{n_{i_j}} + C(Tx_{n_{i_j}})$ and then $y = Tx$. Consequently, $Tx_{n_{i_j}} \rightharpoonup Tx$. Now we claim that $Tx_n \rightharpoonup Tx$. Suppose the contrary, then there exists a subsequence $(x_{n_i})_{i\in\mathbb{N}}$ of $(x_n)_{n\in\mathbb{N}}$ and a weak neighborhood V^w of Tx, such that $Tx_{n_i} \notin V^w$ for all $i \in \mathbb{N}$. Since $(x_{n_i})_{i\in\mathbb{N}}$ converges weakly to x then, arguing as before, we may extract a subsequence $(x_{n_{i_{j_k}}})_{k\in\mathbb{N}}$ of $(x_{n_i})_{i\in\mathbb{N}}$, such that $Tx_{n_{i_{j_k}}} \rightharpoonup Tx$, which is absurd, since $Tx_{n_{i_{j_k}}} \notin V^w$, for all $k \in \mathbb{N}$. As a result, T is weakly sequentially continuous.

Next we will prove that T is ω-condensing. For this purpose, let $\mathcal{M}$ be a subset of S with $\omega(\mathcal{M}) > 0$. Using Eq. (3.1), we infer that

$$\omega(T(\mathcal{M})) \leq \omega(A(T(\mathcal{M}))B(\mathcal{M}) + C(T(\mathcal{M}))).$$

The properties of ω in Lemmas 2.1 and 3.1, when combined with the assumptions (ii), (iii), and (iv) on A, B, and C allow us to show that

$$\begin{aligned} \omega(T(\mathcal{M})) &\leq \omega(A(T(\mathcal{M}))B(\mathcal{M})) + \omega(C(T(\mathcal{M}))) \\ &\leq \gamma\beta\omega(\mathcal{M}) + \alpha\omega(T(\mathcal{M})), \end{aligned}$$

and then

$$\omega(T(\mathcal{M})) \leq \frac{\gamma\beta}{1-\alpha}\omega(\mathcal{M}).$$

This inequality means, in particular, that is T is ω-condensing. □

Remark 3.1 *If we take $\alpha = \beta = 0$ in the above theorem, then we get the following Corollary 3.1, which represents the new version of the fixed point result obtained by Ben Amar et al. in [7] in a WC–Banach algebra case.* ♢

Corollary 3.1 *Let S be a nonempty, bounded, closed, and convex subset of a WC–Banach algebra $\mathcal{X}$ and let A, $C : \mathcal{X} \to \mathcal{X}$ and $B : S \to \mathcal{X}$ be three weakly sequentially continuous operators satisfying the following conditions:*

(i) $\left(\frac{I-C}{A}\right)^{-1}$ *exists on* $B(S)$,

(ii) A *satisfies* $(\mathcal{H}1)$,

(iii) $A(S)$, $B(S)$, *and* $C(S)$ *are relatively weakly compact, and*

(iv) $(x = AxBy + Cx,\ y \in S) \Longrightarrow x \in S$.

Then the operator equation $x = AxBx + Cx$ has, at least, a solution in S. ♢

Proof According to Theorem 3.2, it is sufficient to show that $T(S) := \left(\frac{I-C}{A}\right)^{-1}B(S)$ is relatively weakly compact.
Using both Lemmas 2.1 and 3.1 and knowing the weak compactness of $A(S)$, $B(S)$, and $C(S)$, we infer that

$$\omega(T(S)) \leq \omega(A(T(S))B(S)) + \omega(C(T(S))).$$

This shows that $\omega(T(S)) = 0$. Hence, $T(S)$ is relatively weakly compact. The use of Theorem 3.2 achieves the proof. □

In the following theorem, we will use the notion of $\mathcal{D}$-Lipschitzian operators.

Theorem 3.3 *Let S be a nonempty, bounded, closed, and convex subset of a WC–Banach algebra $\mathcal{X}$ and let A, $C : \mathcal{X} \to \mathcal{X}$ and $B : S \to \mathcal{X}$ be three weakly sequentially continuous operators satisfying the following conditions:*

(i) B *is an* ω-δ-*contraction,*

(ii) A *and* C *are* $\mathcal{D}$-*Lipschitzian with the* $\mathcal{D}$-*functions* ϕ_A *and* ϕ_C *respectively, where* $\phi_C(r) < (1-\delta Q)r$ *for* $r > 0$ *and* $Q = \|A(S)\|$,

(iii) A *is regular on* $\mathcal{X}$, *verifies* $(\mathcal{H}1)$, *and* $A(S)$ *is relatively weakly compact, and*

(iv) $(x = AxBy + Cx,\ y \in S) \Longrightarrow x \in S$.

Then the operator equation $x = AxBx + Cx$ has, at least, a solution in S, whenever $L\phi_A(r) + \phi_C(r) < r$ for $r > 0$, where $L = \|B(S)\|$. ♢

Proof Let y be fixed in S and let us define the mapping

$$\begin{cases} \varphi_y : \mathcal{X} \longrightarrow \mathcal{X}, \\ \qquad x \longrightarrow \varphi_y(x) = AxBy + Cx. \end{cases}$$

Let $x_1, x_2 \in \mathcal{X}$. The use of assumption (i) leads to

$$\begin{aligned} \|\varphi_y(x_1) - \varphi_y(x_2)\| &\leq \|Ax_1B_y - Ax_2B_y\| + \|Cx_1 - Cx_2\| \\ &\leq \|Ax_1 - Ax_2\|\|By\| + \|Cx_1 - Cx_2\| \\ &\leq L\phi_A(\|x_1 - x_2\|) + \phi_C(\|x_1 - x_2\|). \end{aligned}$$

Now an application of Boyd and Wong's fixed point theorem [8] shows the existence of a unique point $x_y \in \mathcal{X}$, such that

$$\varphi_y(x_y) = x_y.$$

Hence, the operator $T := \left(\frac{I-C}{A}\right)^{-1} B : S \to \mathcal{X}$ is well defined. Moreover, the use of assumption (iv) allows us to have $T(S) \subset S$. Using arguments similar to those used in the proof of Theorem 3.2, we can deduce that the operator is weakly sequentially continuous. By using Theorem 3.1, it is sufficient to check that T is ω-condensing. In order to achieve this, let $\mathcal{M}$ be a subset of S with $\omega(\mathcal{M}) > 0$. Using Eq. (3.1), we have

$$T(\mathcal{M}) \subset A(T(\mathcal{M}))B(\mathcal{M}) + C(T(\mathcal{M})).$$

Making use of Lemmas 2.1, 2.2, and 3.1, together with the assumptions on A, B, and C, enables us to have

$$\begin{aligned} \omega(T(\mathcal{M})) &\leq \omega(A(T(\mathcal{M}))B(\mathcal{M})) + \omega(C(T(\mathcal{M}))) \\ &\leq \delta Q\, \omega(\mathcal{M}) + \phi_C(\omega(T(\mathcal{M}))). \end{aligned} \tag{3.2}$$

Now, if $\delta = 0$, inequality (3.2) becomes $\omega(T(\mathcal{M})) \leq \phi_C(\omega(T(\mathcal{M})))$, which implies that $\omega(T(\mathcal{M})) = 0$. Otherwise, by using the inequality

$$\phi_C(r) < (1 - \delta Q)r \ \text{ for } \ r > 0,$$

we have

$$\omega(T(\mathcal{M})) < \omega(\mathcal{M}).$$

In both cases, T is shown to be ω-condensing. The use of Theorem 3.1 achieves the proof. □

Remark 3.2

We should consider the case where $A = 1_{\mathcal{X}}$, in which $1_{\mathcal{X}}$ represents the unit element of the WC–Banach algebra $\mathcal{X}$. We have the following particular cases that constitute the versions of Krasnoselskii's-type fixed point theorems (see [13, 16, 18]).

(i) If we only take $\delta = 0$ in the above theorem, then we obtain the following Corollary 3.2, which extends one of the results obtained in [19].

(ii) If we only take the function $\phi_C(r) = \zeta r$, where $\zeta \in [0, 1-\delta)$ in the above theorem, then we obtain the following Corollary 3.3 which extends one of the results obtained in [12] . $\diamondsuit$

Corollary 3.2 *Let S be a nonempty, bounded, closed, and convex subset of a Banach algebra $\mathcal{X}$, and let $C : \mathcal{X} \to \mathcal{X}$ and $B : S \to \mathcal{X}$ be two weakly sequentially continuous operators satisfying the following conditions:*

(i) *C is a nonlinear contraction,*

(ii) *$B(S)$ is relatively weakly compact, and*

(iii) $(x = By + Cx,\ y \in S) \Longrightarrow x \in S$.

Then $B + C$ has, at least, a fixed point in S. $\diamondsuit$

Corollary 3.3 *Let S be a nonempty, bounded, closed, and convex subset of a Banach algebra $\mathcal{X}$, and let $C : \mathcal{X} \to \mathcal{X}$ and $B : S \to \mathcal{X}$ be two weakly sequentially continuous operators satisfying the following conditions:*

(i) *C is a strict contraction with constant $\zeta \in [0, 1-\delta)$,*

(ii) *B is an ω-δ-contraction, and*

(iii) $(x = By + Cx,\ y \in S) \Longrightarrow x \in S$.

Then $B + C$ has, at least, a fixed point in S. $\diamondsuit$

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