Construction of biorthogonal wavelet packets on local fields of positive characteristic

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Abstract: Orthogonal wavelet packets lack symmetry, which is a much desired property in image and signal processing. The biorthogonal wavelet packets achieve symmetry where the orthogonality is replaced by biorthogonality. In the present paper, we construct biorthogonal wavelet packets on local fields of positive characteristic and investigate their properties by means of Fourier transforms. We also show how to obtain several new Riesz bases of the space $L^2(K)$ by constructing a series of subspaces of these wavelet packets. Finally, we provide algorithms for the decomposition and reconstruction using these biorthogonal wavelet packets.

Key words: Wavelet, multiresolution analysis, scaling function, wavelet packet, Riesz basis, local field, Fourier transform

1. Introduction

A field K equipped with a topology is called a *local field* if both the additive K^+ and multiplicative groups K^* of K are locally compact Abelian groups. The local fields are essentially of two types: zero and positive characteristic (excluding the connected local fields $\mathbb{R}$ and $\mathbb{C}$). Examples of local fields of characteristic zero include the p-adic field $\mathbb{Q}_p$ where as local fields of positive characteristic are the Cantor dyadic group and the Vilenkin p-groups. Even though the structures and metrics of local fields of zero and positive characteristics are similar, their wavelet and multiresolution analysis theory are quite different. In recent years, local fields have attracted the attention of several mathematicians, and have found innumerable applications not only in number theory but also in representation theory, division algebras, quadratic forms, and algebraic geometry. As a result, local fields are now consolidated as part of the standard repertoire of contemporary mathematics. For more about local fields and their applications, we refer to the monographs [15, 24].

In recent years there has been considerable interest in the problem of constructing wavelet bases on various groups, namely, Cantor dyadic groups [12], locally compact Abelian groups [9], p-adic fields [11], and Vilenkin groups [14]. Benedetto and Benedetto [3] developed a wavelet theory for local fields and related groups. They did not develop the multiresolution analysis (MRA) approach; their method is based on the theory of wavelet sets and only allows the construction of wavelet functions whose Fourier transforms are characteristic functions of some sets. The concept of multiresolution analysis on local fields of positive characteristic was introduced by Jiang et al. [10]. They pointed out a method for constructing orthogonal wavelets on local field K with a

constant generating sequence. Subsequently, tight wavelet frames on local fields of positive characteristic were constructed by Shah and Debnath [21] using extension principles. On the other hand, Behera and Jahan [1] have constructed biorthogonal wavelets on local fields of positive characteristic and showed that if φ and $\tilde{\varphi}$ are the dual scaling functions associated with dual MRAs on local fields of positive characteristic such that their translates are biorthogonal, then the corresponding wavelet families are also biorthogonal. More results in this direction can also be found in [8,18–20] and the references therein.

It is well known that the classical orthogonal wavelet bases have poor frequency localization. To overcome this disadvantage, Coifman et al. [7] constructed univariate orthogonal wavelet packets. Well-known Daubechies orthogonal wavelets are a special case of wavelet packets. Later, Chui and Li [5] generalized the concept of orthogonal wavelet packets to the case of nonorthogonal wavelet packets so that they can be applied to spline wavelets and so on. The introduction of biorthogonal wavelet packets is attributed to Cohen and Daubechies [6]. They have also shown that all the wavelet packets constructed in this way did not lead to Riesz bases for $L^2(\mathbb{R})$. Shen [23] generalized the notion of univariate orthogonal wavelet packets to the case of multivariate wavelet packets. Other notable generalizations are the orthogonal version of vector-valued wavelet packets [4], multiwavelet packets [13], and wavelet packets and framelet packets related to the Walsh polynomials [16,17,22].

Recently, Behera and Jahan [2] constructed orthogonal wavelet packets and wavelet frame packets on local field K of positive characteristic and showed how to construct an orthonormal basis from a Riesz basis. Orthogonal wavelet packets have many desired properties such as compact support, good frequency localization, and vanishing moments. However, there is no continuous symmetry, which is a much desired property in imaging compression and signal processing. To achieve symmetry, several generalizations of scalar orthogonal wavelet packets have been investigated in the literature. The biorthogonal wavelet packets achieve symmetry where the orthogonality is replaced by the biorthogonality. Therefore, the objective of this paper is to construct biorthogonal wavelet packets on local fields of positive characteristic and investigate their properties by means of Fourier transforms and construct several new Riesz bases of space $L^2(K)$. Finally, we establish some algorithms for decomposition and reconstruction using these biorthogonal wavelet packets.

This paper is organized as follows. In Section 2, we discuss some preliminary facts about local fields of positive characteristic and also some results required in the subsequent sections including the definition of an MRA on local fields. In Section 3, we examine some of the properties of the biorthogonal wavelet packets via Fourier transforms. In Section 4, we generate Riesz bases of $L^2(K)$ from these wavelet packets. Section 5 deals with the decomposition and reconstruction algorithms corresponding to these wavelet packets.

2. Preliminaries and multiresolution analysis on local fields

Let K be a field and a topological space. Then K is called a *local field* if both K^+ and K^* are locally compact Abelian groups, where K^+ and K^* denote the additive and multiplicative groups of K, respectively. If K is any field and is endowed with the discrete topology, then K is a local field. Further, if K is connected, then K is either $\mathbb{R}$ or $\mathbb{C}$. If K is not connected, then it is totally disconnected. Hence, by a local field, we mean a field K that is locally compact, nondiscrete, and totally disconnected. The p-adic fields are examples of local fields. More details are contained in [15,24]. In the rest of this paper, we use the symbols $\mathbb{N}, \mathbb{N}_0$, and $\mathbb{Z}$ to denote the sets of natural, nonnegative integers, and integers, respectively.

Let K be a local field. Let dx be the Haar measure on the locally compact Abelian group K^+. If $\alpha \in K$ and $\alpha \neq 0$, then $d(\alpha x)$ is also a Haar measure. Let $d(\alpha x) = |\alpha|dx$. We call $|\alpha|$ the *absolute value* of α.

Moreover, the map $x \to |x|$ has the following properties: (a) $|x| = 0$ if and only if $x = 0$; (b) $|xy| = |x||y|$ for all $x, y \in K$; and (c) $|x + y| \leq \max\{|x|, |y|\}$ for all $x, y \in K$. Property (c) is called the *ultrametric inequality*. The set $\mathfrak{D} = \{x \in K : |x| \leq 1\}$ is called the *ring of integers* in K. Define $\mathfrak{B} = \{x \in K : |x| < 1\}$. The set $\mathfrak{B}$ is called the *prime ideal* in K. The prime ideal in K is the unique maximal ideal in $\mathfrak{D}$ and hence as a result $\mathfrak{B}$ is both principal and prime. Since the local field K is totally disconnected, there exists an element of $\mathfrak{B}$ of maximal absolute value. Let $\mathfrak{p}$ be a fixed element of maximum absolute value in $\mathfrak{B}$. Such an element is called a *prime element* of K. Therefore, for such an ideal $\mathfrak{B}$ in $\mathfrak{D}$, we have $\mathfrak{B} = \langle \mathfrak{p} \rangle = \mathfrak{p}\mathfrak{D}$. As proved in [24], the set $\mathfrak{D}$ is compact and open. Hence, $\mathfrak{B}$ is compact and open. Therefore, the residue space $\mathfrak{D}/\mathfrak{B}$ is isomorphic to a finite field $GF(q)$, where $q = p^k$ for some prime p and $k \in \mathbb{N}$.

Let $\mathfrak{D}^* = \mathfrak{D} \setminus \mathfrak{B} = \{x \in K : |x| = 1\}$. Then it can be proved that $\mathfrak{D}^*$ is a group of units in K^* and if $x \neq 0$ then we may write $x = \mathfrak{p}^k x', x' \in \mathfrak{D}^*$. For a proof of this fact we refer to [15]. Moreover, each $\mathfrak{B}^k = \mathfrak{p}^k \mathfrak{D} = \{x \in K : |x| < q^{-k}\}$ is a compact subgroup of K^+ and usually known as the *fractional ideals* of K^+. Let $\mathcal{U} = \{a_i\}_{i=0}^{q-1}$ be any fixed full set of coset representatives of $\mathfrak{B}$ in $\mathfrak{D}$; then every element $x \in K$ can be expressed uniquely as $x = \sum_{\ell=k}^{\infty} c_\ell \mathfrak{p}^\ell$ with $c_\ell \in \mathcal{U}$. Let χ be a fixed character on K^+ that is trivial on $\mathfrak{D}$ but nontrivial on $\mathfrak{B}^{-1}$. Therefore, χ is constant on cosets of $\mathfrak{D}$ and so if $y \in \mathfrak{B}^k$, then $\chi_y(x) = \chi(yx), x \in K$. Suppose that χ_u is any character on K^+; then clearly the restriction $\chi_u|\mathfrak{D}$ is also a character on $\mathfrak{D}$. Therefore, if $\{u(n) : n \in \mathbb{N}_0\}$ is a complete list of distinct coset representative of $\mathfrak{D}$ in K^+, then, as proved in [24], the set $\{\chi_{u(n)} : n \in \mathbb{N}_0\}$ of distinct characters on $\mathfrak{D}$ is a complete orthonormal system on $\mathfrak{D}$.

The Fourier transform $\hat{f}$ of a function $f \in L^1(K) \cap L^2(K)$ is defined by

$$\hat{f}(\xi) = \int_K f(x)\overline{\chi_\xi(x)}dx. \tag{2.1}$$

It is noted that

$$\hat{f}(\xi) = \int_K f(x)\,\overline{\chi_\xi(x)}dx = \int_K f(x)\chi(-\xi x)dx.$$

Furthermore, the properties of the Fourier transform on local field K are very similar to those of on the real line. In particular, the Fourier transform is unitary on $L^2(K)$.

We now impose a natural order on the sequence $\{u(n)\}_{n=0}^{\infty}$. We have $\mathfrak{D}/\mathfrak{B} \cong GF(q)$ where $GF(q)$ is a c-dimensional vector space over the field $GF(p)$. We choose a set $\{1 = \zeta_0, \zeta_1, \zeta_2, \ldots, \zeta_{c-1}\} \subset \mathfrak{D}^*$ such that span $\{\zeta_j\}_{j=0}^{c-1} \cong GF(q)$. For $n \in \mathbb{N}_0$ satisfying

$$0 \leq n < q, \quad n = a_0 + a_1 p + \cdots + a_{c-1}p^{c-1}, \quad 0 \leq a_k < p, \quad \text{and } k = 0, 1, \ldots, c-1,$$

we define

$$u(n) = (a_0 + a_1\zeta_1 + \cdots + a_{c-1}\zeta_{c-1})\,\mathfrak{p}^{-1}. \tag{2.2}$$

Moreover, for $n = b_0 + b_1 q + b_2 q^2 + \cdots + b_s q^s$, $n \in \mathbb{N}_0$, $0 \leq b_k < q, k = 0, 1, 2, \ldots, s$, we set

$$u(n) = u(b_0) + u(b_1)\mathfrak{p}^{-1} + \cdots + u(b_s)\mathfrak{p}^{-s}. \tag{2.3}$$

This defines $u(n)$ for all $n \in \mathbb{N}_0$. In general, it is not true that $u(m + n) = u(m) + u(n)$. However, if $r, k \in \mathbb{N}_0$ and $0 \leq s < q^k$, then $u(rq^k + s) = u(r)\mathfrak{p}^{-k} + u(s)$. Further, it is also easy to verify that $u(n) = 0$

if and only if $n = 0$ and $\{u(\ell) + u(k) : k \in \mathbb{N}_0\} = \{u(k) : k \in \mathbb{N}_0\}$ for a fixed $\ell \in \mathbb{N}_0$. Hereafter we use the notation $\chi_n = \chi_{u(n)}$, $n \geq 0$.

Let the local field K be of characteristic $p > 0$ and $\zeta_0, \zeta_1, \zeta_2, \ldots, \zeta_{c-1}$ be as above. We define a character χ on K as follows:

$$\chi(\zeta_\mu \mathfrak{p}^{-j}) = \begin{cases} \exp(2\pi i/p), & \mu = 0 \text{ and } j = 1, \\ 1, & \mu = 1, \ldots, c - 1 \text{ or } j \neq 1. \end{cases} \tag{2.4}$$

Definition 2.1 *Let $\{x_n : n \in \mathbb{N}_0\}$ be a subset of a Hilbert space H. Then $\{x_n : n \in \mathbb{N}_0\}$ is said to form a Riesz basis for H if*

(a) $\overline{span}\{x_n : n \in \mathbb{N}_0\} = H$, and

(b) there exist constants A and B with $0 < A \leq B < \infty$ such that

$$A \sum_{n \in \mathbb{N}_0} |c_n|^2 \leq \left\| \sum_{n \in \mathbb{N}_0} c_n x_n \right\|^2 \leq B \sum_{n \in \mathbb{N}_0} |c_n|^2. \tag{2.5}$$

A generalization of the classical theory of multiresolution analysis on local fields of positive characteristic was considered by Jiang et al. [10]. Analogous to the Euclidean case, following is a definition of MRA on the local field K of positive characteristic.

Definition 2.2 *Let K be a local field of positive characteristic $p > 0$ and $\mathfrak{p}$ be a prime element of K. An MRA of $L^2(K)$ is a sequence of closed subspaces $\{V_j : j \in \mathbb{Z}\}$ of $L^2(K)$ satisfying the following properties:*

(a) $V_j \subset V_{j+1}$ for all $j \in \mathbb{Z}$;

(b) $\bigcup_{j \in \mathbb{Z}} V_j$ is dense in $L^2(K)$;

(c) $\bigcap_{j \in \mathbb{Z}} V_j = \{0\}$;

(d) $f(\cdot) \in V_j$ if and only if $f(\mathfrak{p}^{-1} \cdot) \in V_{j+1}$ for all $j \in \mathbb{Z}$;

(e) There is a function $\varphi \in V_0$, called the scaling function, such that $\{\varphi(\cdot - u(k)) : k \in \mathbb{N}_0\}$ forms an orthonormal basis for V_0.

Since $\varphi \in V_0 \subset V_1$ and $\{\varphi(\mathfrak{p}^{-1}x - u(k)) : k \in \mathbb{N}_0\}$ is a Riesz basis of V_1, there exists $\{a_k\} \in l^2(\mathbb{N}_0)$ such that

$$\varphi(x) = \sqrt{q} \sum_{k \in \mathbb{N}_0} a_k \, \varphi(\mathfrak{p}^{-1}x - u(k)). \tag{2.6}$$

On taking the Fourier transform, we have

$$\hat{\varphi}(x) = m_0(\mathfrak{p}\xi) \, \hat{\varphi}(\mathfrak{p}\xi), \tag{2.7}$$

where

$$m_0(\xi) = \frac{1}{\sqrt{q}} \sum_{k \in \mathbb{N}_0} a_k \, \overline{\chi_k(\xi)}.$$

Let W_j, $j \in \mathbb{Z}$ be the direct complementary subspace of V_j in V_{j+1}. Assume that there exist $q - 1$ functions $\{\psi_1, \psi_2, \ldots, \psi_{q-1}\}$ in $L^2(K)$ such that their translates and dilations form Riesz bases of W_j, i.e.

$$W_j = \overline{\text{span}} \left\{ q^{j/2} \psi_\ell \big(\mathfrak{p}^{-j} \cdot -u(k) \big) : k \in \mathbb{N}_0, \ 1 \leq \ell \leq q - 1 \right\}, \ j \in \mathbb{Z}. \tag{2.8}$$

Since $\psi_\ell \in W_0 \subset V_1$, $1 \leq \ell \leq q - 1$, there exists a sequence $\{a_k^\ell\} \in \ell^2(\mathbb{N}_0)$ such that

$$\psi_\ell(x) = \sqrt{q} \sum_{k \in \mathbb{N}_0} a_k^\ell \varphi \big(\mathfrak{p}^{-1} x - u(k) \big), \quad 1 \leq \ell \leq q - 1. \tag{2.9}$$

Eq. (2.9) can be written in the frequency domain as

$$\hat{\psi}_\ell(\xi) = \frac{1}{\sqrt{q}} \sum_{k \in \mathbb{N}_0} a_k^\ell \ \overline{\chi_k(\mathfrak{p}\xi)} \hat{\varphi}(\mathfrak{p}\xi)$$

$$[1em] = m_\ell(\mathfrak{p}\xi) \, \hat{\varphi}(\mathfrak{p}\xi), \tag{2.10}$$

where $m_\ell(\xi) = \dfrac{1}{\sqrt{q}} \displaystyle\sum_{k \in \mathbb{N}_0} a_k^\ell \ \overline{\chi_k(\xi)}, \ \ 1 \leq \ell \leq q - 1$.

Definition 2.3 Let $f, \tilde{f} \in L^2(K)$ be given. We say that they are biorthogonal if

$$\Big\langle f(\cdot), \tilde{f} \big(\cdot -u(k) \big) \Big\rangle = \delta_{0,k}, \tag{2.11}$$

where $\delta_{0,k}$ is the Kronecker's delta function.

If $\varphi(\cdot), \tilde{\varphi}(\cdot) \in L^2(K)$ are a pair of biorthogonal scaling functions, then we have

$$\Big\langle \varphi(\cdot), \hat{\varphi} \big(\cdot -u(k) \big) \Big\rangle = \delta_{0,k}, \quad k \in \mathbb{N}_0. \tag{2.12}$$

Further, we say that $\psi_\ell(\cdot), \tilde{\psi}_\ell(\cdot) \in L^2(K), 1 \leq \ell \leq q - 1$ are a pair of biorthogonal wavelets associated with a pair of biorthogonal scaling functions $\varphi(\cdot), \tilde{\varphi}(\cdot) \in L^2(K)$ if the set $\{\psi_\ell \big(\cdot -u(k) \big) : k \in \mathbb{N}_0, 1 \leq \ell \leq q - 1\}$ forms a Riesz basis of W_0, and

$$\Big\langle \varphi(\cdot), \tilde{\psi}_\ell \big(\cdot -u(k) \big) \Big\rangle = 0, \quad k \in \mathbb{N}_0, \ 1 \leq \ell \leq q - 1, \tag{2.13}$$

$$\Big\langle \tilde{\varphi}(\cdot), \psi_\ell \big(\cdot -u(k) \big) \Big\rangle = 0, \quad k \in \mathbb{N}_0, \ 1 \leq \ell \leq q - 1, \tag{2.14}$$

$$\Big\langle \psi_\ell(\cdot), \tilde{\psi}_{\ell'} \big(\cdot -u(k) \big) \Big\rangle = \delta_{\ell,\ell'} \, \delta_{0,k}, \quad k \in \mathbb{N}_0, \ 1 \leq \ell, \ell' \leq q - 1. \tag{2.15}$$

For $\ell = 1, 2, \ldots, q - 1$, we have

$$W_j^\ell = \overline{\text{span}} \left\{ q^{j/2} \psi_\ell \big(\mathfrak{p}^{-j} \cdot -u(k) \big) : k \in \mathbb{N}_0 \right\}, \quad j \in \mathbb{Z}. \tag{2.16}$$

Using the definition of W_j and identities (2.13)–(2.15), we have the following result:

Definition 2.4 *If $\psi_\ell(\cdot), \tilde{\psi}_\ell(\cdot) \in L^2(K)$, $1 \leq \ell \leq q-1$ are a pair of biorthogonal wavelets associated with a pair of biorthogonal scaling functions $\varphi(\cdot), \tilde{\varphi}(\cdot) \in L^2(K)$, then*

$$L^2(K) = \bigoplus_{j \in \mathbb{Z}} W_j = \bigoplus_{j \in \mathbb{Z}} \bigoplus_{\ell=1}^{q-1} W_j^\ell. \tag{2.17}$$

In the biorthogonal setting, the refinement equation and wavelet equation are very similar to Eqs. (2.6) and (2.9)

$$\tilde{\varphi}(x) = \sqrt{q} \sum_{k \in \mathbb{N}_0} \tilde{a}_k \, \tilde{\varphi}\big(\mathfrak{p}^{-1}x - u(k)\big), \tag{2.18}$$

and

$$\tilde{\psi}_\ell(x) = \sqrt{q} \sum_{k \in \mathbb{N}_0} \tilde{a}_k^\ell \, \tilde{\varphi}\big(\mathfrak{p}^{-1}x - u(k)\big), \quad 1 \leq \ell \leq q-1. \tag{2.19}$$

Taking the Fourier transform of Eqs. (2.18) and (2.19), we obtain

$$\hat{\tilde{\varphi}}(\xi) = \frac{1}{\sqrt{q}} \sum_{k \in \mathbb{N}_0} \tilde{a}_k \, \overline{\chi_k(\mathfrak{p}\xi)} \, \hat{\tilde{\varphi}}(\mathfrak{p}\xi)$$

$$= \tilde{m}_0(\mathfrak{p}\xi)\hat{\tilde{\varphi}}(\mathfrak{p}\xi), \tag{2.20}$$

where $\tilde{m}_0(\xi) = \dfrac{1}{\sqrt{q}} \sum_{k \in \mathbb{N}_0} \tilde{a}_k \, \overline{\chi_k(\xi)}$, and

$$\hat{\tilde{\psi}}_\ell(\xi) = \frac{1}{\sqrt{q}} \sum_{k \in \mathbb{N}_0} \tilde{a}_k^\ell \, \overline{\chi_k(\mathfrak{p}\xi)} \, \hat{\tilde{\varphi}}(\mathfrak{p}\xi)$$

$$= \tilde{m}_\ell(\mathfrak{p}\xi)\hat{\tilde{\varphi}}(\mathfrak{p}\xi), \tag{2.21}$$

where $\tilde{m}_\ell(\xi) = \dfrac{1}{\sqrt{q}} \sum_{k \in \mathbb{N}_0} \tilde{a}_k^\ell \, \overline{\chi_k(\xi)}$, $\quad 1 \leq \ell \leq q-1$.

It is proved in [1] that if $\varphi(\cdot), \tilde{\varphi}(\cdot) \in L^2(K)$ are a pair of biorthogonal scaling functions associated with the given MRA, then the system $\{\varphi\big(\cdot - u(k)\big) : k \in \mathbb{N}_0\}$ is biorthogonal to $\{\tilde{\varphi}\big(\cdot - u(k)\big) : k \in \mathbb{N}_0\}$ if and only if

$$\sum_{k \in \mathbb{N}_0} \hat{\varphi}\big(\xi + u(k)\big)\overline{\hat{\tilde{\varphi}}\big(\xi + u(k)\big)} = 1 \quad a.e. \tag{2.22}$$

3. Biorthogonal wavelet packets on local fields

For $n = 0, 1, \ldots,$ the *basic wavelet packets* associated with a scaling function $\varphi(\cdot)$ on a local field K of positive characteristic are defined recursively by

$$\omega_n(x) = \omega_{qr+s}(x) = \sqrt{q} \sum_{k \in \mathbb{N}_0} a_k^s \, \omega_r\big(\mathfrak{p}^{-1}x - u(k)\big), \ \ 0 \leq s \leq q-1 \tag{3.1}$$

where $r \in \mathbb{N}_0$ is the unique element such that $n = qr + s$, $0 \le s \le q - 1$ holds (see [2]).

Similar to the orthogonal wavelet packets, the *biorthogonal wavelet packets* associated with $\tilde{\varphi}(\cdot)$ are given by

$$\tilde{\omega}_n(x) = \tilde{\omega}_{qr+s}(x) = \sqrt{q} \sum_{k \in \mathbb{N}_0} \tilde{a}_k^s \, \tilde{\omega}_r \big(\mathfrak{p}^{-1}x - u(k)\big), \ \ 0 \le s \le q - 1. \tag{3.2}$$

Note that for $r = 0$ and $1 \le s \le q - 1$, we have

$$\omega_0(x) = \varphi(x), \quad \tilde{\omega}_0(x) = \tilde{\varphi}(x), \quad \omega_s(x) = \psi_s(x), \text{ and } \tilde{\omega}_s(x) = \tilde{\psi}_s(x).$$

Moreover, the Fourier transform of (3.1) and (3.2) gives

$$\hat{\omega}_{qr+s}(\xi) = m_s(\mathfrak{p}\xi) \, \hat{\omega}_r(\mathfrak{p}\xi), \tag{3.3}$$

and

$$\hat{\tilde{\omega}}_{qr+s}(\xi) = \tilde{m}_s(\mathfrak{p}\xi) \, \hat{\tilde{\omega}}_r(\mathfrak{p}\xi). \tag{3.4}$$

We are now in a position to discuss the biorthogonality properties for these wavelet packets by means of the Fourier transform.

Lemma 3.1 *Assume that* $\omega_s(x)$, $\tilde{\omega}_s(x) \in L^2(K)$ *are a pair of biorthogonal wavelets associated with a pair of biorthogonal scaling functions* $\omega_0(x)$, $\tilde{\omega}_0(x)$. *Then we have*

$$\sum_{\ell=0}^{q-1} m_r\big(\mathfrak{p}\xi + \mathfrak{p}u(\ell)\big)\overline{\tilde{m}_s\big(\mathfrak{p}\xi + \mathfrak{p}u(\ell)\big)} = \delta_{r,s}, \quad 0 \le r, s \le q - 1. \tag{3.5}$$

Proof For given $0 \le r, s \le q - 1$, we have

$$\begin{aligned}
\delta_{r,s} &= \sum_{k \in \mathbb{N}_0} \omega_r\big(\xi + u(k)\big)\overline{\tilde{\omega}_r\big(\xi + u(k)\big)} \\[2mm]
&= \sum_{k \in \mathbb{N}_0} m_r\big(\mathfrak{p}\xi + \mathfrak{p}u(k)\big)\hat{\omega}_0\big(\mathfrak{p}\xi + \mathfrak{p}u(k)\big)\overline{\hat{\tilde{\omega}}_0\big(\mathfrak{p}\xi + \mathfrak{p}u(k)\big)} \, \overline{\tilde{m}_s\big(\mathfrak{p}\xi + \mathfrak{p}u(k)\big)} \\[2mm]
&= \sum_{\ell=0}^{q-1}\sum_{k \in \mathbb{N}_0} m_r\big(\mathfrak{p}\xi + \mathfrak{p}u(qk+\ell)\big)\hat{\omega}_0\big(\mathfrak{p}\xi + \mathfrak{p}u(qk+\ell)\big)\overline{\hat{\tilde{\omega}}_0\big(\mathfrak{p}\xi + \mathfrak{p}u(qk+\ell)\big)} \\
&\hspace{7cm} \times \ \overline{\tilde{m}_s\big(\mathfrak{p}\xi + \mathfrak{p}u(qk+\ell)\big)} \\[2mm]
&= \sum_{\ell=0}^{q-1} m_r\big(\mathfrak{p}\xi + \mathfrak{p}u(\ell)\big)\overline{\tilde{m}_s\big(\mathfrak{p}\xi + \mathfrak{p}u(\ell)\big)} \\
&\hspace{3cm} \times \left\{ \sum_{k \in \mathbb{N}_0} \hat{\omega}_0\big(\mathfrak{p}\xi + \mathfrak{p}u(qk+\ell)\big)\overline{\hat{\tilde{\omega}}_0\big(\mathfrak{p}\xi + \mathfrak{p}u(qk+\ell)\big)} \right\} \\[2mm]
&= \sum_{\ell=0}^{q-1} m_r\big(\mathfrak{p}\xi + \mathfrak{p}u(\ell)\big)\overline{m_s\big(\mathfrak{p}\xi + \mathfrak{p}u(\ell)\big)}.
\end{aligned}$$

$\square$

Theorem 3.2 *If* $\{\omega_n(x) : n \in \mathbb{N}_0\}$ *and* $\{\tilde{\omega}_n(x) : n \in \mathbb{N}_0\}$ *are wavelet packets associated with a pair of biorthogonal scaling functions* $\omega_0(x)$ *and* $\tilde{\omega}_0(x)$*, respectively, then we have*

$$\left\langle \omega_n(\cdot), \tilde{\omega}_n(\,\cdot - u(k))\right\rangle = \delta_{0,k}, \quad k \in \mathbb{Z},\ n \in \mathbb{N}_0. \tag{3.6}$$

Proof   We will prove this result by using induction on n. It follows from (2.12) and (2.15) that the claim is true for $n = 0$ and $n = 1, 2, \ldots, q-1$. Assume (3.6) holds for $n < t$, where $t \in \mathbb{N}$. Then we prove the result (3.6) for $n = t$. Let $n = qr + s$, where $r \in \mathbb{N}_0$, $0 \le s \le q-1$, and $r < n$. Therefore, by the inductive assumption, we have

$$\left\langle \omega_r(\cdot), \tilde{\omega}_r(\,\cdot - u(k))\right\rangle = \delta_{0,k} \iff \sum_{k \in \mathbb{N}_0} \omega_r\big(\xi + u(k)\big)\, \overline{\tilde{\omega}_r\big(\xi + u(k)\big)} = 1.$$

Using Lemmas 2.5 and 3.1 and Eqs. (3.3) and (3.4), we obtain

$$\left\langle \omega_n(\cdot), \tilde{\omega}_n(\,\cdot - u(k))\right\rangle = \left\langle \hat{\omega}_n(\cdot), \hat{\tilde{\omega}}_n(\,\cdot - u(k))\right\rangle$$

$$= \int_K \hat{\omega}_{qr+s}(\xi)\, \overline{\hat{\tilde{\omega}}_{qr+s}(\xi)}\, \chi_k(\xi) d\xi$$

$$= \left\langle \hat{\omega}_n(\cdot), \hat{\tilde{\omega}}_n(\,\cdot - u(k))\right\rangle$$

$$= \int_K \hat{\omega}_{qr+s}(\xi)\, \overline{\hat{\tilde{\omega}}_{qr+s}(\xi)}\, \chi_k(\xi) d\xi$$

$$= \int_K m_s(\mathfrak{p}\xi)\hat{\omega}_r(\mathfrak{p}\xi)\, \overline{\tilde{m}_s(\mathfrak{p}\xi)}\, \overline{\hat{\tilde{\omega}}_r(\mathfrak{p}\xi)}\, \chi_k(\xi) d\xi$$

$$= \int_{\mathfrak{D}} \sum_{k \in \mathbb{N}_0} m_s\big(\mathfrak{p}\xi + \mathfrak{p}u(k)\big)\, \hat{\omega}_r\big(\mathfrak{p}\xi + \mathfrak{p}u(k)\big)$$
$$\times \overline{\tilde{m}_s\big(\mathfrak{p}\xi + \mathfrak{p}u(k)\big)}\, \overline{\hat{\tilde{\omega}}_r\big(\mathfrak{p}\xi + \mathfrak{p}u(k)\big)}\, \chi_k(\xi) d\xi$$

$$= \int_{\mathfrak{D}} \sum_{\ell=0}^{q-1} \sum_{k \in \mathbb{N}_0} m_s\big(\mathfrak{p}\xi + \mathfrak{p}u(qk+\ell)\big)\, \hat{\omega}_r\big(\mathfrak{p}\xi + \mathfrak{p}u(qk+\ell)\big)$$
$$\times \overline{\tilde{m}_s\big(\mathfrak{p}\xi + \mathfrak{p}u(qk+\ell)\big)}\, \overline{\hat{\tilde{\omega}}_r\big(\mathfrak{p}\xi + \mathfrak{p}u(qk+\ell)\big)}\, \chi_k(\xi) d\xi$$

$$= \int_{\mathfrak{D}} \sum_{\ell=0}^{q-1} m_s\big(\mathfrak{p}\xi + \mathfrak{p}u(\ell)\big)\, \overline{\tilde{m}_s\big(\mathfrak{p}\xi + \mathfrak{p}u(\ell)\big)}$$
$$\times \left\{ \sum_{k \in \mathbb{N}_0} \hat{\omega}_r\big(\mathfrak{p}\xi + \mathfrak{p}u(qk+\ell)\big) \overline{\hat{\tilde{\omega}}_r\big(\mathfrak{p}\xi + \mathfrak{p}u(qk+\ell)\big)}\, \chi_k(\xi) d\xi \right\}$$

$$= \int_{\mathfrak{D}} \sum_{\ell=0}^{q-1} m_s\big(\mathfrak{p}\xi + \mathfrak{p}u(\ell)\big)\, \overline{\tilde{m}_s\big(\mathfrak{p}\xi + \mathfrak{p}u(\ell)\big)}\, \chi_k(\xi) d\xi$$

$$= \delta_{0,k}.$$

$\square$

Theorem 3.3 *Suppose $\{\omega_n(x) : n \in \mathbb{N}_0\}$ and $\{\tilde{\omega}_n(x) : n \in \mathbb{N}_0\}$ are the biorthogonal wavelet packets associated with a pair of biorthogonal scaling functions $\omega_0(x)$ and $\tilde{\omega}_0(x)$, respectively. Then we have*

$$\left\langle \omega_{qr+s_1}(\cdot), \tilde{\omega}_{qr+s_2}\big(\cdot - u(k)\big) \right\rangle = \delta_{0,k} \delta_{s_1,s_2}, \ 0 \leq s_1, s_2 \leq q-1, \ r, k \in \mathbb{N}_0. \tag{3.7}$$

Proof By Lemma 2.5, we have

$$\left\langle \omega_{qr+s_1}(\cdot), \tilde{\omega}_{qr+s_2}\big(\cdot - u(k)\big) \right\rangle = \left\langle \hat{\omega}_{qr+s_1}(\cdot), \hat{\tilde{\omega}}_{qr+s_2}\big(\cdot - u(k)\big) \right\rangle$$

$$= \int_K \hat{\omega}_{qr+s_1}(\xi) \overline{\hat{\tilde{\omega}}_{qr+s_2}(\xi)} \, \chi_k(\xi) d\xi$$

$$= \int_K m_{s_1}(\mathfrak{p}\xi) \, \hat{\omega}_r(\mathfrak{p}\xi) \, \overline{\tilde{m}_{s_2}(\mathfrak{p}\xi)} \, \overline{\hat{\tilde{\omega}}_r(\mathfrak{p}\xi)} \, \chi_k(\xi) d\xi$$

$$= \int_{\mathfrak{D}} \sum_{k \in \mathbb{N}_0} m_{s_1}(\mathfrak{p}\xi + \mathfrak{p}u(k)) \, \hat{\omega}_r(\mathfrak{p}\xi + \mathfrak{p}u(k))$$
$$\times \overline{\tilde{m}_{s_2}(\mathfrak{p}\xi + \mathfrak{p}u(k))} \, \overline{\hat{\tilde{\omega}}_r(\mathfrak{p}\xi + \mathfrak{p}u(k))} \, \chi_k(\xi) d\xi$$

$$= \int_{\mathfrak{D}} \sum_{\ell=0}^{q-1} \sum_{k \in \mathbb{N}_0} m_{s_1}\big(\mathfrak{p}\xi + \mathfrak{p}u(qk + \ell)\big) \hat{\omega}_r\big(\mathfrak{p}\xi + \mathfrak{p}u(qk + \ell)\big)$$
$$\times \overline{\tilde{m}_{s_2}\big(\mathfrak{p}\xi + \mathfrak{p}u(qk + \ell)\big)} \, \overline{\hat{\tilde{\omega}}_r\big(\mathfrak{p}\xi + \mathfrak{p}u(qk + \ell)\big)} \chi_k(\xi) d\xi$$

$$= \int_{\mathfrak{D}} \sum_{\ell=0}^{q-1} m_{s_1}\big(\mathfrak{p}\xi + \mathfrak{p}u(\ell)\big) \overline{\tilde{m}_{s_2}\big(\mathfrak{p}\xi + \mathfrak{p}u(\ell)\big)}$$

$$\times \left\{ \sum_{k \in \mathbb{N}_0} \hat{\omega}_r\big(\mathfrak{p}\xi + \mathfrak{p}u(qk + \ell)\big) \, \overline{\hat{\tilde{\omega}}_r\big(\mathfrak{p}\xi + \mathfrak{p}u(qk + \ell)\big)} \, \chi_k(\xi) d\xi \right\}$$

$$= \int_{\mathfrak{D}} \sum_{\ell=0}^{q-1} m_{s_1}\big(\mathfrak{p}\xi + \mathfrak{p}u(\ell)\big) \overline{\tilde{m}_{s_2}\big(\mathfrak{p}\xi + \mathfrak{p}u(\ell)\big)} \, \chi_k(\xi) d\xi$$

$$= \delta_{0,k} \delta_{s_1,s_2}.$$

$\square$

Theorem 3.4 *Suppose $\{\omega_n(x) : n \in \mathbb{N}_0\}$ and $\{\tilde{\omega}_n(x) : n \in \mathbb{N}_0\}$ are wavelet packets with respect to a pair of biorthogonal scaling functions $\omega_0(x)$ and $\tilde{\omega}_0(x)$, respectively. Then we have*

$$\left\langle \omega_\ell(\cdot), \tilde{\omega}_n\big(\cdot - u(k)\big) \right\rangle = \delta_{\ell,n} \, \delta_{0,k}, \quad \ell, n, k \in \mathbb{N}_0. \tag{3.8}$$

Proof For $\ell = n$, the result (3.8) follows by Theorem 3.2. When $\ell \neq n$, and $0 \leq \ell, n \leq q-1$, the result (3.8) can be established from Theorem 3.3. Assume ℓ is not equal to n and at least one of ℓ, n does not belong to $\{1, 2, \ldots, q-1\}$; then we can write ℓ, n as $\ell = qr_1 + s_1$, $n = qu_1 + v_1$, $r_1, u_1 \in \mathbb{N}_0$, $s_1, v_1 \in \{0, 1, 2, \ldots, q-1\}$. $\square$

Case 1: If $r_1 = u_1$, then $s_1 \neq v_1$. Therefore, (3.8) follows by virtue of the properties (3.3)–(3.5) and Lemma 2.5 i.e.

$$
\begin{aligned}
\left\langle \omega_\ell(\cdot), \tilde{\omega}_n\big(\,\cdot -u(k)\big)\right\rangle &= \left\langle \omega_{qr_1+s_1}(\cdot), \tilde{\omega}_{qu_1+v_1}\big(\,\cdot -u(k)\big)\right\rangle \\[2mm]
&= \left\langle \hat{\omega}_{qr_1+s_1}(\cdot), \hat{\tilde{\omega}}_{qu_1+v_1}\big(\,\cdot -u(k)\big)\right\rangle \\[2mm]
&= \int_K \hat{\omega}_{qr_1+s_1}(\xi)\, \overline{\hat{\tilde{\omega}}_{qu_1+v_1}(\xi)}\, \chi_k(\xi)\, d\xi \\[2mm]
&= \int_K m_{s_1}(\mathfrak{p}\xi)\, \hat{\omega}_{r_1}(\mathfrak{p}\xi)\, \overline{\tilde{m}_{v_1}(\mathfrak{p}\xi)}\, \overline{\hat{\tilde{\omega}}_{u_1}(\mathfrak{p}\xi)}\, \chi_k(\xi)\, d\xi \\[2mm]
&= \int_{\mathfrak{D}} \sum_{k\in\mathbb{N}_0} m_{s_1}\big(\mathfrak{p}\xi + \mathfrak{p}u(k)\big)\, \hat{\omega}_{r_1}\big(\mathfrak{p}\xi + \mathfrak{p}u(k)\big) \\
&\qquad \times \overline{\tilde{m}_{v_1}\big(\mathfrak{p}\xi + \mathfrak{p}u(k)\big)}\, \overline{\hat{\tilde{\omega}}_{u_1}\big(\mathfrak{p}\xi + \mathfrak{p}u(k)\big)} \chi_k(\xi)\, d\xi \\[2mm]
&= \int_{\mathfrak{D}} \sum_{\ell=0}^{q-1} \sum_{k\in\mathbb{N}_0} m_{s_1}\big(\mathfrak{p}\xi + \mathfrak{p}u(qk+\ell)\big)\, \hat{\omega}_{r_1}\big(\mathfrak{p}\xi + \mathfrak{p}u(qk+\ell)\big) \\
&\qquad \times \overline{\tilde{m}_{v_1}\big(\mathfrak{p}\xi + \mathfrak{p}u(qk+\ell)\big)}\, \overline{\hat{\tilde{\omega}}_{u_1}\big(\mathfrak{p}\xi + \mathfrak{p}u(qk+\ell)\big)} \chi_k(\xi)\, d\xi \\[2mm]
&= \int_{\mathfrak{D}} \sum_{\ell=0}^{q-1} m_{s_1}\big(\mathfrak{p}\xi + \mathfrak{p}u(\ell) + \mathfrak{p}u(\ell)\big)\, \overline{\tilde{m}_{v_1}\big(\mathfrak{p}\xi + \mathfrak{p}u(\ell)\big)} \\
&\qquad \times \left\{ \sum_{k\in\mathbb{N}_0} \hat{\omega}_{r_1}\big(\mathfrak{p}\xi + \mathfrak{p}u(qk+\ell)\big)\, \overline{\hat{\tilde{\omega}}_{u_1}\big(\mathfrak{p}\xi + \mathfrak{p}u(qk+\ell)\big)} \chi_k(\xi)\, d\xi \right\} \\[2mm]
&= \int_{\mathfrak{D}} \sum_{\ell=0}^{q-1} m_{s_1}\big(\mathfrak{p}\xi + \mathfrak{p}u(\ell)\big)\, \overline{\tilde{m}_{v_1}\big(\mathfrak{p}\xi + \mathfrak{p}u(\ell)\big)}\, \chi_k(\xi)\, d\xi \\[2mm]
&= \delta_{0,k}.
\end{aligned}
$$

Case 2: If $r_1 \neq u_1$, then $r_1 = pr_2+s_2$, $u_1 = pu_2+v_2$, where $r_2, u_2 \in \mathbb{N}_0$, and $s_2, v_2 \in \{0, 1, \dots, q-1\}$. If $r_2 = u_2$, then $s_2 \neq v_2$. Similar to Case 1, (3.8) can be established. When $r_2 \neq u_2$, we order $r_2 = pr_3 + s_3$, $u_2 = pu_3 + v_3$, where $r_3, u_3 \in \mathbb{N}_0$, and $s_3, v_3 \in \{0, 1, \dots, q-1\}$. Thus, after taking finite steps (denoted by h), we obtain $r_h, u_h \in \mathbb{N}_0$ and $s_h, v_h \in \{0, 1, \dots, q-1\}$. If $r_h = u_h$, then $s_h \neq v_h$. Similar to Case 1, (3.8) can be established. When $r_h \neq u_h$, it follows from Eqs. (2.12)–(2.15) that

$$
\left\langle \omega_{r_h}(\cdot), \tilde{\omega}_{u_h}\big(\,\cdot -u(k)\big)\right\rangle = 0 \iff \sum_{k\in\mathbb{N}_0} \omega_{r_h}\big(\xi + u(k)\big)\, \overline{\tilde{\omega}_{u_h}\big(\xi + u(k)\big)} = 0, \quad \xi \in K.
$$

Moreover, we have

$$
\begin{aligned}
\left\langle \omega_r(\cdot), \tilde{\omega}_u\big(\,\cdot -u(k)\big)\right\rangle &= \left\langle \hat{\omega}_r(\cdot), \hat{\tilde{\omega}}_u\big(\,\cdot -u(k)\big)\right\rangle \\[2mm]
&= \left\langle \hat{\omega}_{qr_1+s_1}(\cdot), \hat{\tilde{\omega}}_{qu_1+v_1}\big(\,\cdot -u(k)\big)\right\rangle
\end{aligned}
$$

$$
= \int_K \hat{\omega}_{qr_1+s_1}(\xi) \overline{\hat{\tilde{\omega}}_{qu_1+v_1}(\xi)} \chi_k(\xi) d\xi
$$

$$
= \int_K m_{s_1}(\mathfrak{p}\xi) \, m_{s_2}(\mathfrak{p}^2\xi) \, \hat{\omega}_{r_2}(\mathfrak{p}^2\xi) \, \overline{\tilde{m}_{v_1}(\mathfrak{p}\xi)} \, \overline{\tilde{m}_{v_2}(\mathfrak{p}^2\xi)} \, \overline{\hat{\tilde{\omega}}_{u_2}(\mathfrak{p}^2\xi)} \, \chi_k(\xi) d\xi
$$

$$
\vdots
$$

$$
= \int_K \left\{ \prod_{\ell=1}^{h} m_{s_\ell}(\mathfrak{p}^\ell\xi) \right\} \hat{\omega}_{r_h}(\mathfrak{p}^h\xi) \left\{ \prod_{\ell=1}^{h} \overline{\tilde{m}_{v_\ell}(\mathfrak{p}^\ell\xi)} \right\} \overline{\hat{\tilde{\omega}}_{u_h}(\mathfrak{p}^h\xi)} \chi_k(\xi) d\xi
$$

$$
= \int_{\mathfrak{D}} \sum_{k\in\mathbb{N}_0} \left\{ \prod_{\ell=1}^{h} m_{s_\ell}\left(\mathfrak{p}^\ell(\xi+u(k))\right) \right\} \left\{ \hat{\omega}_{r_h}\left(\mathfrak{p}^h(\xi+u(k))\right) \overline{\hat{\omega}_{u_h}\left(\mathfrak{p}^h(\xi+u(k))\right)} \right\}
$$

$$
\times \left\{ \prod_{\ell=1}^{h} m_{v_\ell}\left(\mathfrak{p}^\ell(\xi+u(k))\right) \right\} \chi_k(\xi) d\xi
$$

$$
= 0.
$$

4. Construction of Riesz bases from wavelet packets

In this section, we will decompose the subspaces V_j, $\tilde{V}_j$, W_j and $\tilde{W}_j$ by constructing subspaces of wavelet packets. We also present a direct decomposition for $L^2(K)$.

For any $n \in \mathbb{N}_0$, define

$$
E_n = \left\{ f(x) : f(x) = \sum_{k\in\mathbb{N}_0} a_k \omega_n(x - u(k)), \ \{a_k\}_{k\in\mathbb{N}_0} \in l^2(\mathbb{N}_0) \right\}, \tag{4.1}
$$

$$
\tilde{E}_n = \left\{ \tilde{f}(x) : \tilde{f}(x) = \sum_{k\in\mathbb{N}_0} \tilde{a}_k \tilde{\omega}_n(x - u(k)), \ \{\tilde{a}_k\}_{k\in\mathbb{N}_0} \in l^2(\mathbb{N}_0) \right\}. \tag{4.2}
$$

Clearly $E_0 = V_0$ and $E_s = W_0^s$, for any $1 \le s \le q - 1$. Assume that $\left\{ m_s\big(p\xi + \mathfrak{p}u(k)\big) \right\}_{s,k=0}^{q-1}$ is a unitary matrix.

Lemma 4.1 *For $n \in \mathbb{N}_0$, the space ΔE_n can be decomposed into the direct sum of $E_{qn+s}, 1 \le s \le q-1$, i.e.*

$$
\Delta E_n = \bigoplus_{s=0}^{q-1} E_{qn+s}, \tag{4.3}
$$

where Δ is the dilation operator such that $\Delta f(x) = f(\mathfrak{p}^{-1}x)$, for any $f \in L^2(K)$.

Proof For $n \in \mathbb{N}_0$, we claim that

$$
\Delta E_n = \left\{ f(x) : f(x) = \sum_{s=0}^{q-1} \sum_{k\in\mathbb{N}_0} a_k^s \, \omega_{qn+s}(x - u(k)), \ \{a_k^s\}_{k\in\mathbb{N}_0} \in l^2(\mathbb{N}_0) \right\}. \tag{4.4}
$$

As for any $0 \le s \le q-1$, by (3.1) and (4.1), $\omega_{qn+s}\big(x - u(k)\big) \in \Delta E_n$. Assume that $f(x) \in \Delta E_n$; then there exists a sequence $\{b_k\}_{k \in \mathbb{N}_0} \in l^2(\mathbb{N}_0)$ such that

$$f(x) = \sum_{k \in \mathbb{N}_0} b_k\, \omega_n\big(\mathfrak{p}^{-1}x - u(k)\big). \tag{4.5}$$

Similarly, for each $s = 0, 1, \ldots, q-1$, there exist a sequence $\{a_k^s\}_{k \in \mathbb{N}_0}$ in $l^2(\mathbb{N}_0)$ such that

$$f(x) = \sum_{s=1}^{q-1} \sum_{k \in \mathbb{N}_0} a_k^s\, \omega_n\big(\mathfrak{p}^{-1}x - u(k)\big), \tag{4.6}$$

provided $f(x) \in \Delta E_n$.

Taking the Fourier transform on both sides of (4.5) and (4.6), respectively, and by using (3.3), we obtain

$$\hat{f}(\xi) = h(\mathfrak{p}\xi)\hat{\omega}_n(\mathfrak{p}\xi) = \sum_{s=1}^{q-1} g_s(\xi) m_s(\mathfrak{p}\xi)\hat{\omega}_n(\mathfrak{p}\xi), \tag{4.7}$$

where $h(\xi) = \sum\limits_{k \in \mathbb{N}_0} b_k\overline{\chi_k(\xi)}$, $g_s(\xi) = \sum\limits_{k \in \mathbb{N}_0} a_k^s\overline{\chi_k(\xi)}$.

The above equality (4.7) follows if the following holds:

$$h(\mathfrak{p}\xi) = \sum_{s=1}^{q-1} g_s(\xi)\, m_s(\mathfrak{p}\xi). \tag{4.8}$$

For any $\{b_k\}_{k \in \mathbb{N}_0} \in l^2(\mathbb{N}_0)$, we will prove that there exists a sequence $\{a_k^s\}_{k \in \mathbb{N}_0} \in l^2(\mathbb{N}_0)$ such that (4.8) is satisfied. Moreover, Eq. (4.8) is equal to the following identity:

$$h\big(\mathfrak{p}\xi + \mathfrak{p}u(k)\big) = \sum_{s=1}^{q-1} g_s(\xi)\, m_s\big(\mathfrak{p}\xi + \mathfrak{p}u(k)\big). \tag{4.9}$$

The solvability of (4.9) for every sequence $\{b_k\}_{k \in \mathbb{N}_0} \in l^2(\mathbb{N}_0)$ follows from the fact that the matrix $\Big\{m_s\big(p\xi + \mathfrak{p}u(k)\big)\Big\}_{s,k=0}^{q-1}$ is unitary. Hence, Eq. (4.4) follows. Further, applying Theorem 3.3, it follows that

$$\big\{\omega_{qn+s}\big(\mathfrak{p}^{-1}x - u(k)\big)\ n \in \mathbb{N}_0,\ \ 0 \le s \le q-1,\ \ k \in \mathbb{N}_0\big\}$$

is a Riesz basis of ΔE_n.

Similar to (4.3), we can establish the following results:

$$\tilde{E}_0 = \tilde{V}_0,\ \ \tilde{E}_s = \tilde{W}_0^s,\ \ \ 1 \le s \le q-1,$$

$$\Delta\tilde{E}_n = \bigoplus_{s=0}^{q-1} \tilde{U}_{qn+s},\ \ \ 1 \le s \le q-1. \tag{4.10}$$

For $\ell \in \mathbb{N}$, define $\tilde{\vartheta}_\ell = \left\{ x : x = \sum_{j=0}^{\ell} a_j q^j, a_j = 0, 1, 2, \ldots, q - 1 \right\}$, and $\vartheta_\ell = \tilde{\vartheta}_\ell - \tilde{\vartheta}_{\ell-1}$. Next, we will establish the direct decomposition of space $L^2(K)$. □

Theorem 4.2 *The family of functions* $\left\{ \omega_n\big(x - u(k)\big) : n \in \vartheta_\ell, k \in \mathbb{N}_0 \right\}$ *constitutes the Riesz basis of* $\Delta^\ell W_0$. *In particular,* $\left\{ \omega_n\big(x - u(k)\big) : n, k \in \mathbb{N}_0 \right\}$ *constitutes the Riesz basis of* $L^2(K)$.

Proof From Eq. (4.3), we have

$$\Delta E_0 = \bigoplus_{s=0}^{q-1} E_s \quad \text{i.e.,} \quad \Delta E_0 = E_0 \bigoplus_{s=1}^{q-1} E_s.$$

Since $E_0 = V_0$ and $W_0 = \bigoplus_{s=1}^{q-1} W_0^s = \bigoplus_{s=1}^{q-1} E_s$; therefore, $\Delta E_0 = V_0 \bigoplus W_0$. It can be inductively inferred from (4.3) that

$$\Delta^\ell E_0 = \Delta^{\ell-1} E_0 \bigoplus_{n \in \vartheta_\ell} E_n, \quad \ell \in \mathbb{N}. \tag{4.11}$$

Since $V_{j+1} = V_j \bigoplus W_j$, $j \in \mathbb{Z}$; hence, $\Delta^\ell E_0 = \Delta^{\ell-1} E_0 \bigoplus \Delta^{\ell-1} W_0$, $\ell \in \mathbb{N}$. Now it follows from (4.3) and Proposition 2.4 that $\Delta^\ell W_0 = \bigoplus_{n \in \vartheta_\ell} E_n$, and

$$L^2(K) = V_0 \bigoplus \left(\bigoplus_{\ell \geq 0} \Delta^\ell W_0 \right) = E_0 \bigoplus \left(\bigoplus_{\ell \geq 0} \left(\bigoplus_{n \in \vartheta_\ell} E_n \right) \right) = \bigoplus_{n \in \mathbb{N}_0} E_n. \tag{4.12}$$

In view of Theorem 3.3, the family of functions $\left\{ \omega_n\big(x - u(k)\big) : n \in \vartheta_\ell, k \in \mathbb{N}_0 \right\}$ is a Riesz basis of $\Delta^\ell W_0$. Thus, according to (4.12), the family $\left\{ \omega_n\big(x - u(k)\big) : n, k \in \mathbb{N}_0 \right\}$ forms a Riesz basis of $L^2(K)$. □

Corollary 4.3 *For every* $\ell \in \mathbb{N}$, *the family of functions* $\left\{ \tilde{\omega}_n\big(x - u(k)\big) : n \in \vartheta_\ell, k \in \mathbb{N}_0 \right\}$ *forms a Riesz basis of* $\tilde{\Delta}^l W_0$.

Corollary 4.4 *For every* $\ell \in \mathbb{N}$, *the family of functions* $\left\{ \omega_n\big(\mathfrak{p}^{-j} x - u(k)\big) : n, k \in \mathbb{N}_0 \right\}$ *forms a Riesz basis of* $L^2(K)$.

5. Decomposition and reconstruction algorithms

We begin this section with the decomposition formulae for the biorthogonal wavelet packets on local fields of positive characteristic followed by an algorithm.

Theorem 5.1 *Let* $\{\omega_n : n \in \mathbb{N}_0\}$ *and* $\{\tilde{\omega}_n : n \in \mathbb{N}_0\}$ *be the biorthogonal wavelet packets defined by (3.1) and (3.2), respectively. Then for all* $k \in \mathbb{N}_0$, *we have the following decomposition formulae:*

$$\omega_n\big(\mathfrak{p}^{-1} x - u(k)\big) = \frac{1}{\sqrt{q}} \sum_{\nu=1}^{q-1} \sum_{\mu \in \mathbb{N}_0} \tilde{a}_{k-q\mu}^\nu \, \omega_{qn+\nu}\big(x - u(\mu)\big), \tag{5.1}$$

and

$$\tilde{\omega}_n\left(\mathfrak{p}^{-1}x - u(k)\right) = \frac{1}{\sqrt{q}} \sum_{\nu=1}^{q-1} \sum_{\mu \in \mathbb{N}_0} a_{k-q\mu}^{\nu} \tilde{\omega}_{qn+\nu}\left(x - u(\mu)\right). \tag{5.2}$$

Proof We will prove only (5.1). The second formula (5.2), being the dual of (5.1), will follow. Using Eq. (3.1), we have

$$\frac{1}{\sqrt{q}} \sum_{\nu=0}^{q-1} \sum_{\mu \in \mathbb{N}_0} \tilde{a}_{k-q\mu}^{\nu} \omega_{qn+\nu}\left(x - u(\mu)\right)$$

$$= \frac{1}{\sqrt{q}} \sum_{\nu=0}^{q-1} \sum_{\mu \in \mathbb{N}_0} \tilde{a}_{k-q\mu}^{\nu} \, q^{1/2} \sum_{r \in \mathbb{N}_0} a_r^{\nu} \omega_n\left(\mathfrak{p}^{-1}\left(x - u(\mu)\right) - u(r)\right)$$

$$= \sum_{\nu=0}^{q-1} \sum_{\mu \in \mathbb{N}_0} \tilde{a}_{k-q\mu}^{\nu} \sum_{r \in \mathbb{N}_0} a_r^{\nu} \omega_n\left(\mathfrak{p}^{-1}x - u(q\mu - r)\right)$$

$$= \sum_{\nu=0}^{q-1} \sum_{t \in \mathbb{N}_0} \omega_n\left(\mathfrak{p}^{-1}x - u(t)\right) \sum_{\mu \in \mathbb{N}_0} \tilde{a}_{k-q\mu}^{\nu} a_{t-q\mu}^{\nu}$$

$$= \omega_n\left(\mathfrak{p}^{-1}x - u(k)\right).$$

This completes the proof of the Theorem. □

Given a level J and consider

$$f \approx f_J = \sum_{k \in \mathbb{N}_0} c_k^J \omega_0\left(\mathfrak{p}^{-J}x - u(k)\right),$$

where $\{c_k^J\} \in l^2(\mathbb{N}_0)$. Using the fact

$$V_J = W_{J-1} \oplus V_{J-1} = \cdots = W_{J-1} \oplus W_{J-2} \oplus \cdots W_{J-M} \oplus V_{J-M},$$

one obtains

$$f_J = g_{J-1} + g_{J-2} + \cdots + g_{J-M} + f_{J-M},$$

where $f_{J-M} \in V_{J-M}$ and $g_j \in W_j$, $j = J - M, \ldots, J - 1$.

Furthermore, by using Theorem 5.1, $g_j \in W_j, j = J - M, \ldots, J - 1$ can be further decomposed. To do this, let

$$f_j(x) = \sum_{k \in \mathbb{N}_0} c_k^j \, \omega_0\left(\mathfrak{p}^{-j}x - u(k)\right), \tag{5.3}$$

and

$$g_j(x) = \sum_{\nu=1}^{q-1} \sum_{k \in \mathbb{N}_0} d_k^{\nu,j} \, \omega_\nu\left(\mathfrak{p}^{-j}x - u(k)\right), \tag{5.4}$$

where $\{c_k^j\}_{k \in \mathbb{N}_0}, \{d_k^{\nu,j}\}_{k \in \mathbb{N}_0} \in l^2(\mathbb{N}_0)$.

Implementation of Eq. (5.1) for $n = 0$ gives the decomposition of $f_j(x)$ as

$$
\begin{aligned}
f_j(x) &= \sum_{k \in \mathbb{N}_0} c_k^j \omega_0\big(\mathfrak{p}^{-j}x - u(k)\big) \\[2mm]
&= \frac{1}{\sqrt{q}} \sum_{k \in \mathbb{N}_0} c_k^j \sum_{\nu=0}^{q-1} \sum_{\mu \in \mathbb{N}_0} \tilde{a}_{k-q\mu}^\nu \omega_{qn+\nu}\big(\mathfrak{p}^{-j}x - u(\mu)\big) \\[2mm]
&= \frac{1}{\sqrt{q}} \sum_{k \in \mathbb{N}_0} \sum_{\mu \in \mathbb{N}_0} c_\mu^j \sum_{\nu=0}^{q-1} \tilde{a}_{\mu-qk}^\nu \omega_\nu\big(\mathfrak{p}^{-j+1}x - u(k)\big) \\[2mm]
&= \frac{1}{\sqrt{q}} \sum_{k \in \mathbb{N}_0} \left(\sum_{\mu \in \mathbb{N}_0} c_\mu^j \tilde{a}_{\mu-qk}^0 \right) \omega_0\big(\mathfrak{p}^{-j+1}x - u(k)\big) \\[2mm]
&\quad + \frac{1}{\sqrt{q}} \sum_{k \in \mathbb{N}_0} \sum_{\nu=1}^{q-1} \left(\sum_{\mu \in \mathbb{N}_0} c_\mu^j \tilde{a}_{\mu-qk}^\nu \right) \omega_\nu\big(\mathfrak{p}^{-j+1}x - u(k)\big) \\[2mm]
&= \sum_{k \in \mathbb{N}_0} c_k^{j-1} \varphi\big(\mathfrak{p}^{-j+1}x - u(k)\big) + \sum_{\nu=1}^{q-1} \sum_{k \in \mathbb{N}_0} d_k^{i,j-1} \omega_\nu\big(\mathfrak{p}^{-j+1}x - u(k)\big) \\[2mm]
&= f_{j-1}(x) + g_{j-1}(x),
\end{aligned}
$$

where

$$
c_k^{j-1} = \frac{1}{\sqrt{q}} \sum_{\mu \in \mathbb{N}_0} c_\mu^j \, \tilde{a}_{\mu-qk}^0, \quad d_k^{i,j-1} = \frac{1}{\sqrt{q}} \sum_{\mu \in \mathbb{N}_0} c_\mu^j \, \tilde{a}_{\mu-qk}^\nu, \tag{5.5}
$$

$$
k \in \mathbb{N}_0, j = J, J-1, \ldots, J-M+1.
$$

For all $r \in \mathbb{N}_0$, we have

$$
g_j \in W_j = \Delta^j W_1 = \Delta^{j-r} \Delta^r W_1 = \Delta^{j-r} \bigoplus_{\nu=q^r}^{q^{r+1}-1} E_\nu.
$$

Using Theorem 5.1 for $n = 1, 2, \ldots, q^{r+1} - 1$ yields

$$
\begin{aligned}
g_j(x) &= \sum_{\nu=1}^{q-1} \sum_{k \in \mathbb{N}_0} d_k^{i,j} \omega_\nu\big(\mathfrak{p}^{-j}x - u(k)\big) \\[2mm]
&= \frac{1}{\sqrt{q}} \sum_{k \in \mathbb{N}_0} \sum_{\nu=1}^{q-1} d_k^{i,j} \sum_{\mu \in \mathbb{N}_0} \sum_{s=o}^{q-1} \tilde{a}_{k-q\mu}^s \omega_{q\nu+\mu}\big(\mathfrak{p}^{-j+1}x - u(\mu)\big) \\[2mm]
&= \frac{1}{\sqrt{q}} \sum_{k \in \mathbb{N}_0} \sum_{\nu=1}^{q^2-1} \left(\sum_{\mu \in \mathbb{N}_0} d_\mu^{\lfloor \nu/q \rfloor, j} \, \tilde{a}_{\mu-qk}^{\left(\nu - q \lfloor \nu/q \rfloor \right)} \right) \omega_\nu\big(\mathfrak{p}^{-j+1}x - u(k)\big)
\end{aligned}
$$

$$= \sum_{k\in\mathbb{N}_0}\sum_{\nu=1}^{q^2-1} d_k^{\nu,j,1}\omega_\nu\big(\mathfrak{p}^{-j+1}x - u(k)\big)$$

$$\vdots$$

$$= \sum_{k\in\mathbb{N}_0}\sum_{\nu=q^r}^{q^{r+1}-1} d_k^{\nu,j,r}\omega_\nu\big(\mathfrak{p}^{-j+r}x - u(k)\big),$$

where

$$d_k^{\nu,j,i} = \frac{1}{\sqrt{q}}\sum_{\mu\in\mathbb{N}_0} d_\mu^{\lfloor\nu/q\rfloor,j,i-1}\,\tilde{a}_{\mu-qk}^{\big(\nu-q\lfloor\nu/q\rfloor\big)}, \quad d_k^{\nu,j,0} = d_k^{\nu,j}. \tag{5.6}$$

$$i = 1,2,\ldots,r,\ \nu = q^i, q^i+1,\ldots,q^{i+1}-1.$$

Therefore, for $r\in\mathbb{N}_0$, f_J can be decomposed as:

$$
\begin{aligned}
f_J &= f_{J-M} + \sum_{j=J-M}^{J-1} g_j \\
&= \sum_{k\in\mathbb{N}_0} c_k^{J-M}\omega_0\big(\mathfrak{p}^{J-M}x - u(k)\big) + \sum_{j=J-M}^{J-1}\sum_{\nu=1}^{q-1}\sum_{k\in\mathbb{N}_0} d_k^{\nu,j}\omega_\nu\big(\mathfrak{p}^{-j}x - u(k)\big) \\
&= \sum_{k\in\mathbb{N}_0} c_k^{J-M}\varphi\big(\mathfrak{p}^{J-M}x - u(k)\big) + \sum_{j=J-M}^{J-1}\sum_{\nu=q^r}^{q^{r+1}-1}\sum_{k\in\mathbb{N}_0} d_k^{\nu,j,r}\omega_\nu\big(\mathfrak{p}^{r-j}x - u(k)\big),
\end{aligned}
$$

where the coefficients are given by Eqs. (5.5) and (5.6).

On the other hand, by using Eq. (3.1), we can reconstruct $g_j(\cdot)$ as follows:

$$
\begin{aligned}
g_j(x) &= \sum_{\nu=q^r}^{q^{r+1}-1}\sum_{k\in\mathbb{N}_0} d_k^{\nu,j,r}\omega_\nu\big(\mathfrak{p}^{r-j}x - u(k)\big) \\[2mm]
&= \sum_{\nu=q^r}^{q^{r+1}-1}\sum_{k\in\mathbb{N}_0} d_k^{\nu,j,r}\sum_{\mu\in\mathbb{N}_0} a_\mu^{\big(\nu-q\lfloor\nu/q\rfloor\big)}\omega_{\lfloor\nu/q\rfloor}\big(\mathfrak{p}^{r-j-1}x - u(qk-\mu)\big) \\[2mm]
&= \sum_{\nu=q^{r-1}}^{q^r-1}\sum_{k\in\mathbb{N}_0} d_k^{\nu,j,r-1}\omega_\nu(\mathfrak{p}^{r-j-1}x - u(k)) \\[2mm]
&\ \vdots \\[2mm]
&= \sum_{\nu=1}^{q-1}\sum_{k\in\mathbb{N}_0} d_k^{\nu,j}\omega_\nu(\mathfrak{p}^{-j}x - u(k)),
\end{aligned}
$$

where

$$d_k^{\nu,j,i-1} = \sum_{s=0}^{q-1}\sum_{\mu\in\mathbb{N}_0} d_\mu^{q\nu+s,j,i}\,a_{k-q\mu}^s, \quad d_k^{\nu,j} = d_k^{\nu,j,0}. \tag{5.7}$$

$$i = 1, 2, \ldots, r, \ \nu = q^{i-1}, q^{i-1} + 1, \ldots, q^i - 1.$$

Thus, after obtaining the coefficients $d_k^{\nu,j}$, $\nu = 1, 2, \ldots, q - 1$, $j = J - M, \ldots, J - 1$, $k \in \mathbb{N}_0$, we use Theorem 5.1 and (2.6) to construct f_j as follows:

$$
\begin{aligned}
f_j &= f_{j-1} + g_{j-1} \\[2mm]
&= \sum_{k \in \mathbb{N}_0} c_k^{j-1} \omega_0 \big(\mathfrak{p}^{-j-1} x - u(k)\big) + \sum_{\nu=1}^{q-1} \sum_{k \in \mathbb{N}_0} d_k^{\nu,j-1} \omega_\nu \big(\mathfrak{p}^{-j+1} x - u(k)\big) \\[2mm]
&= \sum_{k \in \mathbb{N}_0} c_k^{j-1} \sum_{\mu \in \mathbb{N}_0} a_\mu^0 \omega_0 \big(\mathfrak{p}^{-j} x - u(qk - \mu)\big) + \sum_{\nu=1}^{q-1} \sum_{k \in \mathbb{N}_0} d_k^{\nu,j-1} \sum_{\mu \in \mathbb{N}_0} a_\mu^\nu \omega_0 \big(\mathfrak{p}^{-j} x - u(qk - \mu)\big) \\[2mm]
&= \sum_{k \in \mathbb{N}_0} c_k^{j-1} \sum_{\mu \in \mathbb{N}_0} a_{\mu-qk}^0 \omega_0 \big(\mathfrak{p}^{-j} x - u(\mu)\big) + \sum_{\nu=1}^{q-1} \sum_{k \in \mathbb{N}_0} d_k^{\nu,j-1} \sum_{\mu \in \mathbb{N}_0} a_{\mu-qk}^\nu \omega_0 \big(\mathfrak{p}^{-j} x - u(\mu)\big) \\[2mm]
&= \sum_{k \in \mathbb{N}_0} \left(\sum_{\mu \in \mathbb{N}_0} c_k^{j-1} a_{\mu-qk}^0 + \sum_{\nu=1}^{q-1} \sum_{\mu \in \mathbb{N}_0} d_\mu^{\nu,j-1} a_{\mu-qk}^\nu \right) \omega_0 \big(\mathfrak{p}^{-j} x - u(k)\big) \\[2mm]
&= \sum_{k \in \mathbb{N}_0} c_k^j \varphi \big(\mathfrak{p}^{-j} x - u(k)\big),
\end{aligned}
$$

where

$$c_k^j = \sum_{\mu \in \mathbb{N}_0} c_k^{j-1} a_{\mu-qk}^0 + \sum_{\nu=1}^{q-1} \sum_{\mu \in \mathbb{N}_0} d_\mu^{\nu,j-1} a_{\mu-qk}^\nu, \ j = J - M + 1, J - M + 2, \ldots, J, \ k \in \mathbb{N}_0. \tag{5.8}$$

Therefore, with the given sequences $\big\{ c_k^{J-M} : k \in \mathbb{N}_0 \big\}$ and $\big\{ d_k^{\nu,J-M} : k \in \mathbb{N}_0 \big\}, \nu = 1, \ldots, q - 1$, and applying (5.8), one can reconstruct

$$f \approx f_J = \sum_{k \in \mathbb{N}_0} c_k^J \omega_0 \big(\mathfrak{p}^{-J} x - u(k)\big) \in V_J.$$

Acknowledgments

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