

On le -semigroups

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Received: 24.03.2015

Accepted/Published Online: 20.07.2015

Final Version: 10.02.2016

Abstract: We characterize the idempotent ideal elements of the le -semigroups in terms of semisimple elements and we prove, among others, that the ideal elements of an le -semigroup S are prime (resp. weakly prime) if and only if they form a chain and S is intraregular (resp. semisimple). The corresponding results on semigroups (without order) can be also obtained as an application of the results of this paper. The study of poe -semigroups plays an essential role in the theory of fuzzy semigroups and the theory of hypersemigroups.

Key words: le -semigroup, left (right) ideal element, ideal element, prime, weakly prime, semiprime, intraregular

1. Introduction and prerequisites

A po -groupoid S is a groupoid under a multiplication “.” and at the same time an ordered set under an order “ \leq ” such that $a \leq b$ implies $ac \leq bc$ and $ca \leq cb$ for all $c \in S$. If the multiplication of S is associative, then S is called a po -semigroup. A \vee -groupoid is a groupoid that is also a semilattice under \vee such that $a(b \vee c) = ab \vee ac$ and $(a \vee b)c = ac \vee bc$ for all $a, b, c \in S$. A \vee -groupoid that is also a lattice is called an l -groupoid [1,3]. A po -groupoid, \vee -groupoid, or po -semigroup possessing a greatest element “ e ” (that is, $e \geq a$ for any $a \in S$) is called poe -groupoid, $\vee e$ -groupoid, or poe -semigroup, respectively. Let S be a po -groupoid. An element a of S is called *idempotent* if $a^2 = a$. An element a of S is called a *left ideal element* if $xa \leq a$ for all $x \in S$. It is called a *right ideal element* if $ax \leq a$ for all $x \in S$ [1]. It is called an *ideal element* if it is both a left and right ideal element. If S is a poe -groupoid, then a is a left (resp. right) ideal element of S if and only if $ea \leq a$ (resp. $ae \leq a$). A poe -semigroup S is called *intraregular* if $a \leq ea^2e$ for every $a \in S$ [4]. For a $\vee e$ -groupoid S , we denote by $l(a)$, $r(a)$ the left and the right ideal element of S , respectively, generated by a ($a \in S$), and we have $l(a) = ea \vee a$, $r(a) = ae \vee a$. We have $r(l(a)) = a \vee ea \vee ae \vee eae$, and this is the ideal element of S generated by a .

An *intraregular semigroup* is a semigroup S such that for every $a \in S$ there exist $x, y \in S$ such that $a = xa^2y$ [2]. Semigroups in which the ideals are prime, weakly prime have been considered by Szász, who showed that the ideals of a semigroup S are prime if and only if S is intraregular and the ideals of S form a chain, and that the ideals of S are weakly prime if and only if and only if they are idempotent and they form a chain. He also proved that an ideal of a semigroup is prime if and only if it is both semiprime and weakly prime, and that in commutative semigroups the prime and weakly prime ideals coincide [8]. Semigroups with

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2010 AMS Mathematics Subject Classification: 06F05.

idempotent ideals have been also studied by Szász. In [9] Szász showed that the semigroups with idempotent ideals and the semigroups with idempotent principal ideals are the same and these are the semigroups S in which $a \in SaSaS$ for every $a \in S$, that is, the semisimple semigroups [7, p. 58]. It was proved in [5] that similar results to the results of Szász mentioned above hold in ordered semigroups as well. Besides, a semigroup S endowed with the equality relation $\leq := \{(x, y) \mid x = y\}$ is a po -semigroup, and one can easily prove that the results on semigroups –without order– given by Szász can be also obtained as an application of the results in [5] (in the way indicated in Corollary 13 in [6]).

In the present paper, we deal with ordered semigroups possessing a greatest element: the poe -semigroups. In this type of semigroups the ideal elements (instead of ideals) play the essential role. It is interesting to know whether similar results to the results given in [5] hold in poe -semigroups using ideal elements instead of ideals. The present paper shows how similar the theory of po -semigroups based on ideals is to the theory of poe -semigroups based on ideal elements. The two theories are parallel to each other. We prove, among others, that the ideal elements of an le -semigroup S are idempotent if and only if the principal ideal elements of S are so, and this is equivalent to saying that $a \wedge b = ab$ for any ideal elements a, b of S . It might be noted that if an le -semigroup S is regular (that is, $a \leq aea$ for every $a \in S$), then for any right ideal element a and any left ideal element b of S we have $a \wedge b = ab$. “Conversely”, if $a \wedge b \leq ab$ for every right ideal element a and any left ideal element b of S , then S is regular. Thus, the relation $a \wedge b = ab$, and equivalently the relation $a \wedge b \leq ab$, characterizes the regular le -semigroups, while the condition $a \wedge b \leq ba$ for any right ideal element a and any left ideal element b of S characterizes the intraregular le -semigroups. A po -semigroup S is called *semisimple* if $a \leq eaeae$ for every $a \in S$. We prove that the ideal elements of an le -semigroup S are idempotent if and only if S is semisimple, and they are weakly prime if and only if they form a chain and S is semisimple. Finally, the ideal elements of an le -semigroup S are prime if and only if they form a chain and S is intraregular. If S is a semigroup, then the set $\mathcal{P}(S)$ of all subsets of S with the multiplication on $\mathcal{P}(S)$ induced by the multiplication on S and the inclusion relation is an le -semigroup. A set A is an ideal of S if and only if it is an ideal element on $\mathcal{P}(S)$. As a consequence the results given by Szász in [8,9] can be also proved as an application of the results of the present paper. Besides, the study of poe (le)-semigroups seems to be of particular interest as it gives information about fuzzy semigroups and hypersemigroups as well.

2. Main results

Definition 1 [4] An element a of a poe -groupoid S is called a *left* (resp. *right*) *ideal element* of S if $ea \leq a$ (resp. $ae \leq a$). It is called an *ideal element* of S if it is both a left and right ideal element of S . A poe -semigroup S is called *intraregular* (resp. *regular*) if

$$a \leq ea^2e \text{ (resp. } a \leq aea) \text{ for every } a \in S.$$

Definition 2 A poe -semigroup S is called *semisimple* if

$$a \leq eaeae \text{ for every } a \in S.$$

Remark 3 If S is a poe -groupoid, a a left ideal element, and b a right ideal element of S , then ab is an ideal element of S . Indeed, since $ea \leq a$ and $be \leq b$, we have $e(ab) = (ea)b \leq ab$ and $(ab)e = a(be) \leq ab$. Thus, if a, b are ideal elements of S , then ab is an ideal element of S as well. If S is a poe -groupoid and at the same time semilattice under “ \wedge ” and a, b ideal elements of S , then the element $a \wedge b$ is an ideal element of S as

well. Indeed, $(a \wedge b)e \leq ae \wedge be \leq a \wedge b$ and $e(a \wedge b) \leq ea \wedge eb \leq a \wedge b$. It might be also noted that if S is a $\vee e$ -groupoid and a, b ideal elements of S , then the element $a \vee b$ is so. Indeed, $(a \vee b)e = ae \vee be \leq a \vee b$ and $e(a \vee b) = ea \vee eb \leq a \vee b$.

Remark 4 Let S be a $\vee e$ -semigroup. Then we have the following:

- (1) $a \leq r(l(a))$ for every $a \in S$;
- (2) if $a \leq b$, then $l(a) \leq l(b)$ and $r(a) \leq r(b)$;
if $a = b$, then $l(a) = l(b)$ and $r(a) = r(b)$;
- (3) $l(l(a)) = l(a)$ and $r(r(a)) = r(a)$ for every $a \in S$;
- (4) $r(l(a)) = l(r(a))$ for every $a \in S$;
- (5) the element eae is an ideal element of S for every $a \in S$;
- (6) if t is an ideal element of S , then $r(l(t)) = t$; hence,

$$r(l(r(l(a)))) = r(l(a)) \text{ for every } a \in S;$$

- (7) if t is an ideal element of S , then t^2 is an ideal element of S as well.

Definition 5 Let S be a po -groupoid. An element t of S is called *prime* if for any a, b in S such that $ab \leq t$, we have $a \leq t$ or $b \leq t$ [1, p. 329]. It is called *weakly prime* if for any ideal elements a, b of S such that $ab \leq t$, we have $a \leq t$ or $b \leq t$. Finally, t is called *semiprime* [4] if $a \in S$ such that $a^2 \leq t$ implies $a \leq t$.

If an ideal element of a po -groupoid is prime, then it is weakly prime and semiprime.

Proposition 6 Let S be a poe -semigroup and t an ideal element of S . If t is weakly prime, then for all ideal elements a, b of S for which $ab \wedge ba$ exists $ab \wedge ba \leq t$ implies $a \leq t$ or $b \leq t$. “Conversely”, suppose that for all ideal elements a, b of S the element $ab \wedge ba$ exists in S and $ab \wedge ba \leq t$ implies $a \leq t$ or $b \leq t$. Then t is weakly prime.

Proof \implies . Let a, b be ideal elements of S such that $ab \wedge ba$ exists and $ab \wedge ba \leq t$. Since ab and ba are ideal elements of S , we have $(ab)(ba) \leq (ab)e \leq ab$ and $(ab)(ba) \leq e(ba) \leq ba$, and then $(ab)(ba) \leq ab \wedge ba \leq t$. Since t is weakly prime and ab, ba ideal elements of S , we have $ab \leq t$ or $ba \leq t$. Again, since t is weakly prime and a, b are ideal elements of S , we have $a \leq t$ or $b \leq t$.

\impliedby . This holds in po -groupoids in general. In fact, let a, b be ideal elements of S such that $ab \leq t$. By hypothesis, the element $ab \wedge ba$ exists. Thus, we have $ab \wedge ba \leq ab \leq t$. Then, by hypothesis, we have $a \leq t$ or $b \leq t$, so t is weakly prime. \square

Proposition 7 An ideal element of a $\vee e$ -semigroup S is prime if and only if it is both semiprime and weakly prime; in commutative $\vee e$ -semigroups the prime and weakly prime ideal elements coincide.

Proof Let t be an ideal element of S both semiprime and weakly prime and $a, b \in S$ such that $ab \leq t$. Then $(bea)^2 \leq t$, $bea \leq t$, $(ebe)(eae) \leq t$, and so $ebe \leq t$ or $eae \leq t$. If $eae \leq t$, then

$$\left(r(l(a))\right)^2 r(l(a)) \leq eae \leq t.$$

If $\left(r(l(a))\right)^2 \leq t$, then $r(l(a)) \leq t$. If $r(l(a)) \leq t$, then $a \leq t$. If $ebe \leq t$, in the same way we get $b \leq t$, so t is prime.

Now let S be commutative, t a weakly prime ideal element of S , and $a, b \in S$, $ab \leq t$. Since $r(l(a))r(l(b)) \leq r(l(ab)) \leq r(l(t)) = t$, we have $a \leq r(l(a)) \leq t$ or $b \leq r(l(b)) \leq t$. \square

Proposition 8 *Let S be a poe-groupoid. If the ideal elements of S are idempotent then, for any ideal elements a, b of S such that $a \wedge b$ exists, we have $a \wedge b = ab$. “Conversely”, if S is a po-groupoid such that $a \wedge b = ab$ for all ideal elements a, b of S , then the ideal elements of S are idempotent.*

Proof Suppose the ideal elements of S are idempotent. Let a, b be ideal elements of S . Since $ab \leq ae \leq a$ and $ab \leq eb \leq b$, we have $ab \leq a \wedge b$. Besides, $a \wedge b$ is an ideal element of S . By hypothesis, we have $a \wedge b = (a \wedge b)^2 = (a \wedge b)(a \wedge b) \leq ab$, and thus $a \wedge b = ab$. For the converse statement, let a be an ideal element of S . By hypothesis, we have $a = a \wedge a = a^2$, so a is idempotent. \square

Proposition 9 *Let S be an le-semigroup. The following are equivalent:*

- (1) *The ideal elements of S are idempotent.*
- (2) *If a, b are ideal elements of S , then $a \wedge b = ab$.*
- (3) *$r(l(a)) \wedge r(l(b)) = r(l(a))r(l(b))$ for any $a, b \in S$.*
- (4) *$r(l(a)) = \left(r(l(a))\right)^2$ for every $a \in S$.*
- (5) *S is semisimple.*

Proof (1) \implies (2). By Proposition 8.

(2) \implies (3). Since $r(l(a))$ and $r(l(b))$ are ideal elements of S .

(3) \implies (4). It is clear.

(4) \implies (5). Let $a \in S$. By hypothesis, we have

$$a \leq r(l(a)) = \left(r(l(a))\right)^2 = \left(r(l(a))\right)^4 \left(r(l(a))\right).$$

Since

$$\begin{aligned} \left(r(l(a))\right)^3 &= (a \vee ea \vee ae \vee eae)^3 = (a \vee ea \vee ae \vee eae)^2 (a \vee ea \vee ae \vee eae) \\ &\leq (ea \vee eae)(a \vee ea \vee ae \vee eae) \leq eae, \end{aligned}$$

we have $\left(r(l(a))\right)^4 \leq eae(a \vee ea \vee ae \vee eae) = eaea \vee eaeae$.

Hence, we obtain

$$\begin{aligned} a &\leq (eaea \vee eaeae)(a \vee ea \vee ae \vee eae) = eaea^2 \vee eaeaea \vee eaea^2e \vee eaeaeae \\ &\leq eaeae, \end{aligned}$$

and S is semisimple.

(5) \implies (1). This holds in *poe*-semigroups in general. Indeed, let a be an ideal element of S . By hypothesis, we have $a \leq (eae)ae \leq aa = a^2 \leq ea \leq a$, so $a^2 = a$. \square

Proposition 10 *Let S be a *poe*-groupoid. If the ideal elements of S are weakly prime, then they are idempotent and they form a chain.*

Proof Let a be an ideal element of S . Since $a^2 \leq a^2$ and a^2 is an ideal element of S , by hypothesis, we have $a \leq a^2 \leq ae \leq a$, so $a^2 = a$, and a is idempotent. Let now a, b be ideal elements of S . Since ab is an ideal element of S and $ab \leq ab$, by hypothesis, we have $a \leq ab \leq eb \leq b$ or $b \leq ab \leq ae \leq a$, so the ideal elements of S form a chain. \square

Proposition 11 *Let S be a *po*-groupoid. Suppose that for any two ideal elements a, b of S , the infimum $a \wedge b$ exists and $a \wedge b = ab$. If the ideal elements of S form a chain, then they are weakly prime.*

Proof Let a, b, t be ideal elements of S such that $ab \leq t$. By hypothesis, we have $a \leq b$ or $b \leq a$. If $a \leq b$, then $a = a \wedge b = ab \leq t$. If $b \leq a$, then $b = a \wedge b = ab \leq t$. \square

By Propositions 9, 10, and 11, we have the following theorem:

Theorem 12 *Let S be an *le*-semigroup. The ideal elements of S are weakly prime if and only if they form a chain and one of the five equivalent conditions of Proposition 9 is satisfied.*

Proposition 13 *A *poe*-semigroup S is intraregular if and only if the ideal elements of S are semiprime.*

Proof \implies . Let t be an ideal element of S and $a \in S$ such that $a^2 \leq t$. Since S is intraregular, we have $a \leq ea^2e \leq ete \leq t$, and thus t is semiprime.

\impliedby . Let $a \in S$. Since ea^2e is an ideal element of S , by hypothesis, it is semiprime. Since $a^4 \leq ea^2e$, we have $a^2 \leq ea^2e$, and $a \leq ea^2e$, so S is intraregular. \square

Proposition 14 *Let S be a *poe*-semigroup. If the ideal elements of S are prime, then they form a chain and S is intraregular.*

Proof If the ideal elements of S are prime, then they are weakly prime and semiprime. Since the ideal elements of S are weakly prime, by Proposition 10, they form a chain. Since they are semiprime, by Proposition 13, S is intraregular. \square

Proposition 15 *Let S be an intraregular *le*-semigroup. If the ideal elements of S form a chain, then they are prime.*

Proof Since S is intraregular, by Proposition 13, the ideal elements of S are semiprime. Since the ideal elements of S are semiprime, we have the following:

(A) $r(l(x)) = exe$ for every $x \in S$. Indeed:

Let $x \in S$. Since exe is an ideal element of S and $x^4 \leq exe$, we have $x^2 \leq exe$, and $x \leq exe$. Since $r(l(x))$ is the ideal element of S generated by x , we have $r(l(x)) \leq exe$. Besides, $exe \leq x \vee ex \vee xe \vee exe = r(l(x))$, so $r(l(x)) = exe$.

(B) $r(l(xy)) = r(l(x)) \wedge r(l(y))$ for every $x, y \in S$. Indeed:

Let $x, y \in S$. Since $xy \leq r(l(x))e \leq r(l(x))$, we have $r(l(xy)) \leq r(l(x))$. Since $xy \leq er(l(y)) \leq r(l(y))$, we have $r(l(xy)) \leq r(l(y))$. Then we have $r(l(xy)) \leq r(l(x)) \wedge r(l(y))$. On the other hand, by (A), we get

$$r(l(x)) \wedge r(l(y)) \leq r(l(x)) = exe \text{ and } r(l(x)) \wedge r(l(y)) \leq r(l(y)) = eye,$$

so we have

$$\left(r(l(x)) \wedge r(l(y)) \right)^2 \leq eyeexe.$$

We have $(ye^2x)(ye^2x) \leq exye = r(l(xy))$ (by (A)), where $r(l(xy))$ is an ideal element of S . Since $r(l(xy))$ is semiprime, we have $ye^2x \leq r(l(xy))$. Then

$$\left(r(l(x)) \wedge r(l(y)) \right)^2 \leq er(l(xy))e \leq r(l(xy)).$$

Again, since $r(l(xy))$ is a semiprime, we get $r(l(x)) \wedge r(l(y)) \leq r(l(xy))$. Then we have $r(l(x)) \wedge r(l(y)) = r(l(xy))$, and condition (B) is satisfied.

Now let t be an ideal element of S and $a, b \in S$ such that $ab \leq t$. By hypothesis, we have $r(l(a)) \leq r(l(b))$ or $r(l(b)) \leq r(l(a))$. If $r(l(a)) \leq r(l(b))$ then, by (B),

$$a \leq r(l(a)) = r(l(a)) \wedge r(l(b)) = r(l(ab)) \leq r(l(t)) = t.$$

If $r(l(b)) \leq r(l(a))$, then $b \leq r(l(b)) = r(l(a)) \wedge r(l(b)) = r(l(ab)) \leq r(l(t)) = t$. Thus, t is prime. \square

By Propositions 14 and 15 we have the following:

Theorem 16 *Let S be an le-semigroup. The ideal elements of S are prime if and only if they form a chain and S is intraregular.*

Remark 17 Are there *poe*-semigroups in which the ideal elements are idempotent? In regular *poe*-semigroups, the right and the left ideal elements are idempotent, so the ideal elements are idempotent. In intraregular *poe*-semigroups, the ideal elements are idempotent. A *poe*-semigroup is called *left* (resp. *right*) *regular* if $a \leq ea^2$ (resp. $a \leq a^2e$) for every $a \in S$ [4]. In left (resp. right) regular *poe*-semigroups, the left (resp. right) ideal elements are idempotent. Besides, the left (resp. right) regular *poe*-semigroups are intraregular.

We note that a semigroup S is intraregular if and only if $A \subseteq SA^2S$ for every $A \subseteq S$. Indeed: \implies . Let $A \subseteq S$ and $a \in A$. Since S is intraregular, there exist $x, y \in S$ such that $a = xa^2y \in SA^2S$. \impliedby . Let $a \in S$. Since $\{a\} \subseteq S$, by hypothesis, we have $\{a\} \subseteq S\{a\}^2S = S\{a^2\}S$, and then there exist $x, y \in S$ such that $a = xa^2y$, and S is intraregular.

We also note that if S is a semigroup, then the set $\mathcal{P}(S)$ of all subsets of S with the multiplication on $\mathcal{P}(S)$ induced by the multiplication on S and the inclusion relation " \subseteq " is an *le*-semigroup (S being the greatest element of $\mathcal{P}(S)$).

The results on semigroups given in [8,9] can be also proved as an application of the results of this paper. Let us apply Theorem 16 to get the corresponding result of the semigroup, and the rest of the results of this paper can be also applied in the same way.

Corollary 18 [8] *Let S be a semigroup. The ideals of S are prime if and only if they form a chain and S is intraregular.*

Proof \implies . Since the ideals of S are prime, the ideal elements of $\mathcal{P}(S)$ are prime. By Theorem 16, the ideal elements of $\mathcal{P}(S)$ form a chain, and $\mathcal{P}(S)$ is intraregular that is $A \subseteq SA^2S$ for every $A \in \mathcal{P}(S)$ (cf. Definition 1). This means that the ideals of S form a chain and S is intraregular.

\impliedby . Suppose the ideals of S form a chain and S is intraregular. Then the ideal elements of $\mathcal{P}(S)$ form a chain and the *le*-semigroup $\mathcal{P}(S)$ is intraregular. By Theorem 16, the ideal elements of $\mathcal{P}(S)$ are prime. Then the ideals of S are prime. \square

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