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# Solving an initial boundary value problem on the semiinfinite interval

**Ferihe ATALAN***, **Gusein Sh. GUSEINOV**
Department of Mathematics, Atılım University, Ankara, Turkey

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**Abstract:** We explore the sign properties of eigenvalues and the basis properties of eigenvectors for a special quadratic matrix polynomial and use the results obtained to solve the corresponding linear system of differential equations on the half line subject to an initial condition at $t = 0$ and a condition at $t = \infty$.

**Key words:** Quadratic eigenvalue problem, eigenvalues, eigenvectors

## 1. Introduction

For various types of infinite interval problems we refer to the book [1] by Agarwal and O'Regan.

In this paper, we deal with the existence and uniqueness and the explicit form of solution $u(t)$ to the problem

$$C\frac{d^2u(t)}{dt^2} = J\frac{du(t)}{dt} + Ru(t), \quad 0 \le t < \infty, \tag{1}$$

$$u(0) = f, \quad \lim_{t\to\infty} u(t) = 0, \tag{2}$$

where

$$u(t) = \begin{bmatrix} u_0(t) \\ u_1(t) \\ \vdots \\ u_{N-1}(t) \end{bmatrix}, \; f = \begin{bmatrix} f_0 \\ f_1 \\ \vdots \\ f_{N-1} \end{bmatrix},$$

$$C = \begin{bmatrix} c_0 & 0 & 0 & \cdots & 0 \\ 0 & c_1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & c_{N-1} \end{bmatrix}, \; R = \begin{bmatrix} r_0 & 0 & 0 & \cdots & 0 \\ 0 & r_1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & r_{N-1} \end{bmatrix},$$

$$J = \begin{bmatrix} b_0 & a_0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ a_0 & b_1 & a_1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & a_1 & b_2 & a_2 & 0 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & a_{N-3} & b_{N-2} & a_{N-2} \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & a_{N-2} & b_{N-1} - ha_{N-1} \end{bmatrix}.$$

*Correspondence: ferihe.atalan@atilim.edu.tr
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We also consider conditions of the form

$$\lim_{t\to\infty} u(t) = v$$

and

$$\lim_{t\to\infty} [u(t) - (v + tw)] = 0$$

instead of the second condition in (2), where $v$ and $w$ are constant vectors.

Let us seek a nontrivial solution of Eq. (1), which has the form

$$u(t) \quad = \quad e^{\lambda t} y, \tag{3}$$

where $\lambda$ is a complex number and $y = \{y_n\}_{n=0}^{N-1}$ is a nonzero constant vector in $\mathbb{C}^N$. Substituting (3) into (1), we get

$$(\lambda^2 C - \lambda J - R)y \quad = \quad 0. \tag{4}$$

A complex number $\lambda_0$ is said to be an eigenvalue of Eq. (4) if there exists a nonzero vector $y_0 \in \mathbb{C}^N$ satisfying Eq. (4) for $y = y_0$ and $\lambda = \lambda_0$. This vector $y_0$ is called an eigenvector of Eq. (4) corresponding to the eigenvalue $\lambda_0$.

Thus, the vector-function $u(t)$ in (3) is a nontrivial solution of Eq. (1) if and only if $\lambda$ is an eigenvalue and $y$ is a corresponding eigenvector of Eq. (4).

Note that Eq. (4) is equivalent to the boundary value problem:

$$\lambda a_{n-1} y_{n-1} + (\lambda b_n + r_n) y_n + \lambda a_n y_{n+1} \quad = \quad \lambda^2 c_n y_n, \tag{5}$$

$$n = 0, 1, \cdots, N-1,$$

$$y_{-1} = 0, \quad y_N + h y_{N-1} = 0. \tag{6}$$

Namely, if $\{y_n\}_{n=-1}^{N}$ is a solution of problem (5), (6), then the vector $y = \{y_n\}_{n=0}^{N-1}$ satisfies Eq. (4), and, conversely, if the vector $y = \{y_n\}_{n=0}^{N-1}$ is a solution of Eq. (4), then $\{y_n\}_{n=-1}^{N}$ with $y_{-1} = 0$ and $y_N = -h y_{N-1}$ is a solution of problem (5), (6).

We denote all the eigenvalues of Eq. (4) by $\lambda_1, \cdots, \lambda_m$ and the corresponding eigenvectors by $y^{(1)}, \cdots, y^{(m)}$. Since Eq. (1) is linear, the vector-function

$$u(t) \quad = \quad \sum_{j=1}^{m} \alpha_j e^{\lambda_j t} y^{(j)} \tag{7}$$

is a solution of Eq. (1), where $\alpha_1, \cdots, \alpha_m$ are arbitrary constants. Next we have to choose the constants $\alpha_1, \cdots, \alpha_m$ so that (7) satisfies the conditions in (2):

$$\sum_{j=1}^{m} \alpha_j y^{(j)} = f, \quad \lim_{t\to\infty} \sum_{j=1}^{m} \alpha_j e^{\lambda_j t} y^{(j)} = 0. \tag{8}$$

In this paper we show that such a choice of the constants $\alpha_j$ is possible and we indicate a way in which one can realize this choice.

Problem (1), (2) was earlier solved in [7] under the conditions

$$a_n, b_n, c_n, h \in \mathbb{R}, \quad a_n \neq 0, \quad c_n > 0 \qquad (0 \leq n \leq N-1), \tag{9}$$

$$r_n > 0 \qquad (0 \leq n \leq N-1). \tag{10}$$

In the present paper, we replace condition (10) by the condition

$$r_0 = 0, \quad r_n > 0 \qquad (1 \leq n \leq N-1), \tag{11}$$

allowing one of the $r_n$'s to be zero, and study the consequences of the condition $r_0 = 0$. It turns out that if $r_0 = 0$, then $\lambda = 0$ is an eigenvalue of Eq. (4), and, moreover, this eigenvalue is defective if $b_0 = 0$. For solving the problem (1), (2) it will also be important to investigate the sign properties of the nonzero eigenvalues.

To fix the terminology used in the paper let us recall some concepts related to the quadratic eigenvalue problems.

Let $N$ be a positive integer and $M$, $L$, and $K$ be $N \times N$ complex matrices. The quadratic matrix polynomial (quadratic matrix pencil)

$$Q(\lambda) = \lambda^2 M + \lambda L + K \tag{12}$$

is called *regular* when $\det Q(\lambda)$ is not identically zero for all values of $\lambda$, and *nonregular* otherwise. We assume that $Q(\lambda)$ is regular. By the *quadratic eigenvalue problem* (QEP) is meant the equation

$$Q(\lambda)y := (\lambda^2 M + \lambda L + K)y = 0.$$

The complex scalar $\lambda$ and the corresponding nonzero vector $y \in \mathbb{C}^N$ are respectively called the *eigenvalue* and the *eigenvector* of the quadratic pencil $Q(\lambda)$. The general theory of the linear second-order differential equation

$$M\frac{d^2u(t)}{dt^2} + L\frac{du(t)}{dt} + Ku(t) = 0, \quad 0 \leq t < \infty,$$

is based on the theory of matrix pencil (12); see [6, 9, 10].

Let $\lambda_0$ be an eigenvalue of the quadratic pencil $Q(\lambda)$. We say that the vectors $y^{(0)}, y^{(1)}, \cdots, y^{(m)}$ in $\mathbb{C}^N$ form a *Jordan chain* of length $m+1$ for $Q(\lambda)$ associated with the eigenvalue $\lambda_0$ if

$$\begin{aligned}
Q(\lambda_0)y^{(0)} &= 0, \\
Q(\lambda_0)y^{(1)} + Q'(\lambda_0)y^{(0)} &= 0, \\
Q(\lambda_0)y^{(2)} + Q'(\lambda_0)y^{(1)} + \frac{1}{2}Q''(\lambda_0)y^{(0)} &= 0, \\
&\vdots \\
Q(\lambda_0)y^{(m)} + Q'(\lambda_0)y^{(m-1)} + \frac{1}{2}Q''(\lambda_0)y^{(m-2)} &= 0,
\end{aligned}$$

where $Q'(\lambda_0) = 2\lambda_0 M + L$, $Q''(\lambda_0) = 2M$. The vector $y^{(0)}$ is an eigenvector and the subsequent vectors $y^{(1)}, \cdots, y^{(m)}$ are called the *generalized eigenvectors* (associated with the eigenvector $y^{(0)}$).

The eigenvalue $\lambda_0$ is called *simple* if there is only one linearly independent eigenvector $y^{(0)}$ corresponding to $\lambda_0$ and there is no generalized eigenvector associated with $y^{(0)}$. The eigenvalue $\lambda_0$ is called *semisimple* if there is no generalized eigenvectors associated with the eigenvectors corresponding to $\lambda_0$ (there may exist more than one linearly independent eigenvector corresponding to $\lambda_0$). A *defective* eigenvalue is an eigenvalue that is not semisimple (therefore, for a defective eigenvalue, there are one or more generalized eigenvectors).

The quadratic pencil (12) is called self-adjoint if $M$, $L$, and $K$ are self-adjoint matrices. The eigenvalues of a self-adjoint pencil $Q(\lambda)$ are real or arise in complex conjugate pairs.

A self-adjoint pencil (12) is said to be *weakly hyperbolic* (*hyperbolic*) if $M > 0$ and all roots of the polynomial $(Q(\lambda)x, x)$ are real (real and distinct) for any $x \neq 0$, where $(.\,,.)$ stands for the standard inner product in $\mathbb{C}^N$. Since the roots of $(Q(\lambda)x, x)$ are given by

$$\lambda \quad = \quad \frac{-(Lx,x) \pm \sqrt{(Lx,x)^2 - 4(Mx,x)(Kx,x)}}{2(Mx,x)},$$

we see that $Q(\lambda)$ is weakly hyperbolic if

$$(Lx,x)^2 \geq 4(Mx,x)(Kx,x)$$

for all $x \in \mathbb{C}^N$, and hyperbolic if

$$(Lx,x)^2 > 4(Mx,x)(Kx,x)$$

for all nonzero $x \in \mathbb{C}^N$.

The hyperbolic and weakly hyperbolic QEPs have been thoroughly analyzed [4, 5, 10]. In particular, Duffin showed that the eigenvalues of hyperbolic QEPs are not only real but also semisimple (i.e. there is no generalized eigenvector associated with the eigenvectors). It turns out that the length of any Jordan chain for a weakly hyperbolic pencil $Q(\lambda)$ associated with the any eigenvalue does not exceed 2 (see [10]).

We see that our quadratic pencil in (4) is hyperbolic under conditions (9) and (10), and weakly hyperbolic under conditions (9) and (11). A distinguishing feature of our quadratic pencil in (4) from general weakly hyperbolic quadratic pencils is that, due to the special structure of the coefficient matrices $C$, $J$, and $R$, the eigenvalue problem (4) is equivalent to the three-term recursion relation (second-order linear difference equation) (5) with the boundary conditions (6). This allows, using techniques from the theory of three-term linear difference equations [3], to develop a thorough analysis of eigenvalue problem (4) to get more specific results.

## 2. The form of general solution

In [2], we proved that under conditions (9) and (11) the eigenvalues are all real and $\lambda = 0$ is an eigenvalue of Eq. (4). The number of the nonzero eigenvalues depends on whether $b_0$ (the first element of the matrix $J$) is zero or not:

(i) If $b_0 \neq 0$, then Eq. (4) has, besides the zero eigenvalue, precisely $2N-1$ distinct nonzero real eigenvalues.

(ii) If $b_0 = 0$, then Eq. (4) has, besides the zero eigenvalue, precisely $2N-2$ distinct nonzero real eigenvalues so that in this case the zero eigenvalue is defective (there is a generalized eigenvector associated with the eigenvector corresponding to the zero eigenvalue).

Moreover, in [2], we proved that if $b_0 \neq 0$, the vectors

$$\Theta = [\theta, 0], \quad \Phi_j = [\varphi^{(j)}, \lambda_j \varphi^{(j)}], \qquad j = 1, \cdots, 2N-1,$$

form a basis of $\mathbb{C}^N \times \mathbb{C}^N$, where

$$\theta = (1, 0, \cdots, 0)^T$$

($T$ denotes the transpose) is the eigenvector corresponding to the zero eigenvalue, $R\theta = 0$, $\lambda_1 < \lambda_2 < \cdots < \lambda_{2N-1}$ are the nonzero eigenvalues, and $\varphi^{(1)}, \varphi^{(2)}, \cdots, \varphi^{(2N-1)}$ are the corresponding eigenvectors of Eq. (4). It follows that we can write the general solution of differential equation (1) in the form

$$u(t) = \alpha\theta + \sum_{j=1}^{2N-1} \alpha_j e^{\lambda_j t} \varphi^{(j)}, \tag{13}$$

where $\alpha, \alpha_1, \cdots, \alpha_{2N-1}$ are arbitrary constants.

In the case of $b_0 = 0$, the vector

$$\psi = (0, -\frac{a_0}{r_1}, 0, \cdots, 0)^T$$

satisfies

$$J\theta + R\psi = 0$$

and hence $\psi$ is a generalized eigenvector associated with the eigenvector $\theta$. In [2], we showed that then the vectors

$$\Theta = [\theta, 0], \quad \Psi = [\psi, \theta], \quad \Phi_j = [\varphi^{(j)}, \lambda_j \varphi^{(j)}], \qquad j = 1, \cdots, 2N-2$$

form a basis of $\mathbb{C}^N \times \mathbb{C}^N$. In this case we can write the general solution of differential equation (1) in the form

$$u(t) = \alpha\theta + \beta(\psi + t\theta) + \sum_{j=1}^{2N-2} \alpha_j e^{\lambda_j t} \varphi^{(j)}, \tag{14}$$

where $\alpha, \beta, \alpha_1, \cdots, \alpha_{2N-2}$ are arbitrary constants.

Denote by $\{\varphi_n(\lambda)\}_{n=-1}^{N}$ the solution of Eq. (5) satisfying the initial conditions

$$\varphi_{-1}(\lambda) = 0, \quad \varphi_0(\lambda) = 1$$

so that the first condition in (6) is satisfied for this solution for all $\lambda$. Therefore, from the second condition in (6) we get that each nonzero eigenvalue $\lambda_j$ of Eq. (4) is a zero of the function (characteristic function) $\chi(\lambda) = \varphi_N(\lambda) + h\varphi_{N-1}(\lambda)$ and the vector $\varphi^{(j)} = \{\varphi_n(\lambda_j)\}_{n=0}^{N-1}$ is an eigenvector of Eq. (4) corresponding to $\lambda_j$.

For easy reference let us formulate here the following two theorems proved in [2].

**Theorem 2.1** *Suppose (9), (11), and $b_0 \neq 0$. Then the eigenvectors $\theta = (1, 0, \cdots, 0)^T$ and $\varphi^{(j)} = \{\varphi_n(\lambda_j)\}_{n=0}^{N-1}$, $j = 1, \cdots, 2N-1$ of Eq. (4) form a two-fold basis of $\mathbb{C}^N$; that is, for arbitrary vectors $f$ and $g$ belonging to $\mathbb{C}^N$ the expansions*

$$f = \alpha\theta + \sum_{j=1}^{2N-1} \alpha_j \varphi^{(j)}, \quad g = \sum_{j=1}^{2N-1} \alpha_j \lambda_j \varphi^{(j)} \tag{15}$$

*hold, where the coefficients $\alpha_j$ and $\alpha$ are determined by*

$$\alpha_j = \frac{1}{\rho_j} \sum_{k=0}^{N-1} (r_k f_k + \lambda_j c_k g_k) \varphi_k(\lambda_j), \quad j = 1, \cdots, 2N-1, \tag{16}$$

$$\rho_j = \sum_{k=0}^{N-1} (r_k + \lambda_j^2 c_k) \varphi_k^2(\lambda_j), \quad j = 1, \cdots, 2N-1, \tag{17}$$

$$\alpha = f_0 - \sum_{j=1}^{2N-1} \alpha_j. \tag{18}$$

**Theorem 2.2** *Suppose (9), (11), and $b_0 = 0$. Then the eigenvectors $\theta = (1, 0, \cdots, 0)^T$ and $\varphi^{(j)} = \{\varphi_n(\lambda_j)\}_{n=0}^{N-1}$, $j = 1, \cdots, 2N-2$ including the associated vector $\psi = (0, -a_0/r_1, 0, \cdots, 0)^T$ of Eq. (4) form a two-fold basis of $\mathbb{C}^N$ in the sense that for arbitrary vectors $f$ and $g$ belonging to $\mathbb{C}^N$ the expansions*

$$f = \alpha\theta + \beta\psi + \sum_{j=1}^{2N-2} \alpha_j \varphi^{(j)}, \quad g = \beta\theta + \sum_{j=1}^{2N-2} \alpha_j \lambda_j \varphi^{(j)} \tag{19}$$

*hold, where the coefficients $\alpha_j$, $\beta$ and $\alpha$ are determined by*

$$\alpha_j = \frac{1}{\rho_j} \sum_{k=0}^{N-1} (r_k f_k + \lambda_j c_k g_k) \varphi_k(\lambda_j), \qquad j = 1, \cdots, 2N-2, \tag{20}$$

$$\rho_j = \sum_{k=0}^{N-1} (r_k + \lambda_j^2 c_k) \varphi_k^2(\lambda_j), \qquad j = 1, \cdots, 2N-2, \tag{21}$$

$$\beta = \frac{1}{\rho}(c_0 g_0 - a_0 f_1), \tag{22}$$

$$\rho = \frac{a_0^2}{r_1} + c_0, \tag{23}$$

$$\alpha = f_0 - \sum_{j=1}^{2N-2} \alpha_j. \tag{24}$$

## 3. Determination of the negative and positive eigenvalues

We investigate Eq. (4) in the space of $\mathbb{C}^N$ with the inner product

$$(y,z) \quad = \quad \sum_{j=0}^{N-1} y_j \bar{z}_j, \tag{25}$$

where the bar denotes complex conjugation.

We know that there exist one zero eigenvalue and $2N-1$ nonzero distinct real eigenvalues provided that $r_0 = 0$, $r_j > 0$ for $j = 1, \cdots, N-1$ and $b_0 \neq 0$; see [2]. We denote the nonzero eigenvalues by

$$\lambda_1 < \cdots < \lambda_{2N-1}. \tag{26}$$

**Lemma 3.1** *If $b_0 < 0$, the first $N$ eigenvalues in (26) are negative and the remaining $N-1$ eigenvalues are positive. If $b_0 > 0$, the first $N-1$ eigenvalues are negative and the remaining $N$ eigenvalues are positive.*

**Proof** If

$$R(\epsilon) = \begin{bmatrix} \epsilon & 0 & 0 & \cdots & 0 \\ 0 & r_1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & r_{N-1} \end{bmatrix},$$

for all $\epsilon > 0$, it is proved in [7] that there exist $2N$ nonzero distinct eigenvalues of the problem $(\lambda^2 C - \lambda J - R(\epsilon))y = 0$. When $\epsilon = 0$ ($b_0 \neq 0$), we know from [2] that zero is an eigenvalue, and besides there are $2N-1$ nonzero eigenvalues $\lambda_1 < \cdots < \lambda_{2N-1}$. When we include the parameter $\epsilon > 0$ into the problem these eigenvalues $\lambda_j$ become functions of $\epsilon$: $\lambda_j = \lambda_j(\epsilon)$ and the zero eigenvalue is deformed to a nonzero eigenvalue. It is denoted by $\mu(\epsilon)$. The eigenvector $\theta = (1, 0, \cdots, 0)^T$ corresponding to the zero eigenvalue turns out to be $\theta(\epsilon)$ corresponding to $\mu(\epsilon)$. Since $\mu(\epsilon)$ is an analytic function depending on $\epsilon$ (see [8], Chapter 2), its Taylor expansion gives

$$\mu(\epsilon) = \mu(0) + \mu'(0)\epsilon + O(\epsilon^2) \ \ (\epsilon \to 0).$$

Let us denote $\mu'(0) = a$. Since $\mu(0) = 0$,

$$\mu(\epsilon) = a\epsilon + O(\epsilon^2). \tag{27}$$

The eigenvector $\theta(\epsilon)$ satisfies the equation

$$(\mu^2(\epsilon)C - \mu(\epsilon)J - R(\epsilon))\theta(\epsilon) \quad = \quad 0. \tag{28}$$

Since the eigenvector $\theta(\epsilon)$ is an analytic function depending on $\epsilon$ (see [8], Chapter 2), its Taylor expansion gives

$$\theta(\epsilon) = \theta(0) + \theta'(0)\epsilon + O(\epsilon^2) \ \ (\epsilon \to 0).$$

Let us denote $\theta'(0) = \theta^{(1)}$. Then we have

$$\theta(\epsilon) \quad = \quad \theta + \theta^{(1)}\epsilon + O(\epsilon^2). \tag{29}$$

In equation (28), $R(\epsilon) = R + \epsilon T$, where

$$T = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}.$$

Note that since

$$(\mu(\epsilon))^2 = (a\epsilon + O(\epsilon^2))^2 = O(\epsilon^2),$$

Eq. (28) gives

$$\mu(\epsilon)J\theta(\epsilon) + R(\epsilon)\theta(\epsilon) \quad = \quad O(\epsilon^2), \ \ \forall\epsilon > 0. \tag{30}$$

Substituting (27) and (29) in (30) and using $R\theta = 0$, we get

$$\epsilon(aJ\theta + R\theta^{(1)} + T\theta) \quad = \quad O(\epsilon^2). \tag{31}$$

Dividing both sides of (31) by $\epsilon$ and then passing to the limit as $\epsilon \to 0$, we obtain

$$aJ\theta + R\theta^{(1)} + T\theta \quad = \quad 0. \tag{32}$$

To find the number $a$, multiply (32) in the sense of the inner product by $\theta$:

$$a(J\theta, \theta) + (R\theta^{(1)}, \theta) + (T\theta, \theta) \quad = \quad 0. \tag{33}$$

It is easily seen that

$$(J\theta, \theta) = b_0, \quad (R\theta^{(1)}, \theta) = 0, \quad (T\theta, \theta) = 1,$$

where $b_0 \neq 0$ is the first element of the matrix $J$. It follows from (33) that

$$a = -\frac{1}{b_0}.$$

Therefore, (27) takes the form

$$\mu(\epsilon) = -\frac{1}{b_0}\epsilon + O(\epsilon^2). \tag{34}$$

There are two possible cases: $b_0$ is either negative or positive.

If $b_0 < 0$, then from (34), $\mu(\epsilon) > 0$ for very small values of $\epsilon$. On the other hand, we know from [7] that there exist $2N$ nonzero eigenvalues of the problem $(\lambda^2 C - \lambda J - R(\epsilon))y = 0$, and furthermore, the first half of them are negative and the other half are positive. Hence, we have

$$\lambda_1(\epsilon) < \cdots < \lambda_N(\epsilon) < 0 < \mu(\epsilon) < \lambda_{N+1}(\epsilon) < \cdots < \lambda_{2N-1}(\epsilon).$$

Passing here to the limit as $\epsilon \to 0$ and taking into account that the eigenvalues at $\epsilon = 0$ are simple, we get

$$\lambda_1 < \cdots < \lambda_N < 0 < \lambda_{N+1} < \cdots < \lambda_{2N-1}.$$

If $b_0 > 0$, then from (34), $\mu(\epsilon) < 0$ for small enough values of $\epsilon$. We know from [7] that the first half of nonzero $2N$ eigenvalues are negative and the other half are positive:

$$\lambda_1(\epsilon) < \cdots < \lambda_{N-1}(\epsilon) < \mu(\epsilon) < 0 < \lambda_N(\epsilon) < \cdots < \lambda_{2N-1}(\epsilon).$$

Passing here to the limit as $\epsilon \to 0$, and taking into account that the eigenvalues at $\epsilon = 0$ are simple, we get

$$\lambda_1 < \cdots < \lambda_{N-1} < 0 < \lambda_N < \cdots < \lambda_{2N-1}.$$

The lemma is proved. □

Suppose that $r_0 = 0$, $r_j > 0$, for $j = 1, \cdots, N-1$ and $b_0 = 0$. In this case, we know from [2] that there exist one zero eigenvalue and $2N-2$ nonzero distinct real eigenvalues. We denote the nonzero eigenvalues by

$$\lambda_1 < \cdots < \lambda_{2N-2}. \tag{35}$$

**Lemma 3.2** *Suppose that $b_0 = 0$. Then, in (35), the first $N-1$ eigenvalues are negative and the other $N-1$ eigenvalues are positive.*

**Proof** Consider the eigenvalue problem (4) with the variable $b_0 = \gamma < 0$ under conditions (9) and (11). In this case, our problem turns out to be the one in Lemma 3.1. Therefore, we have

$$\lambda_1(\gamma) < \cdots < \lambda_{N-1}(\gamma) < \mu(\gamma) < 0 < \lambda_N(\gamma) < \cdots < \lambda_{2N-2}(\gamma).$$

Passing to the limit as $\gamma \to 0$ and taking into account that the eigenvalues at $\gamma = 0$ are simple, we obtain that

$$\lambda_1 < \cdots < \lambda_{N-1} < 0 < \lambda_N < \cdots < \lambda_{2N-2}.$$

□

We summarize the results obtained so far in the following theorem:

**Theorem 3.3** **(i)** *Suppose (9), (11), and $b_0 \neq 0$. Then Eq. (4) has precisely one zero eigenvalue and $2N-1$ nonzero distinct real eigenvalues. If $b_0 < 0$, then the first $N$ nonzero eigenvalues $\lambda_j$ $(j = 1, \cdots, 2N-1)$ are negative and the other $N-1$ are positive. If $b_0 > 0$, the first $N-1$ nonzero eigenvalues are negative and the other $N$ are positive. To the zero eigenvalue there corresponds the eigenvector $\theta = (1, 0, \cdots, 0)^T$, and to each nonzero eigenvalue $\lambda_j$ there corresponds a single, up to a scalar factor, eigenvector that can be taken as $\varphi^{(j)} = \{\varphi_n(\lambda_j)\}_{n=0}^{N-1}$.*

**(ii)** *Suppose (9), (11), and $b_0 = 0$. Then Eq. (4) has precisely one zero eigenvalue and $2N-2$ nonzero distinct real eigenvalues. The first $N-1$ of nonzero eigenvalues $\lambda_j$ $(j = 1, \cdots, 2N-2)$ are negative and the other $N-1$ are positive. To the zero eigenvalue there corresponds the eigenvector $\theta = (1, 0, \cdots, 0)^T$, and to each nonzero eigenvalue $\lambda_j$ there corresponds a single, up to a scalar factor, eigenvector that can be taken as $\varphi^{(j)} = \{\varphi_n(\lambda_j)\}_{n=0}^{N-1}$. Moreover, $J\theta + R\psi = 0$, where $\psi = (0, -a_0/r_1, 0, \cdots, 0)^T$ forms a generalized eigenvector associated with the eigenvector $\theta$.*

## 4. Bases consisting of half of the eigenvectors

We will make use of Theorem 2.1 and Theorem 2.2 given above in Section 2 and proved in [2].

**Theorem 4.1** *Suppose (9), (11), and* $b_0 \neq 0$*. Then:*

*(i) If* $b_0 < 0$*, then the eigenvectors* $\theta = (1, 0, \cdots, 0)^T$ *and* $\varphi^{(j)}$ *for* $j = N+1, \cdots, 2N-1$ *(corresponding to the positive eigenvalues) form a basis of* $\mathbb{C}^N$*. Also, the eigenvectors* $\varphi^{(j)}$ *for* $j = 1, \cdots, N$ *(corresponding to the negative eigenvalues) form a basis of* $\mathbb{C}^N$*.*

*(ii) If* $b_0 > 0$*, then the eigenvectors* $\theta = (1, 0, \cdots, 0)^T$ *and* $\varphi^{(j)}$*, for* $j = 1, \cdots, N-1$ *(corresponding to the negative eigenvalues), form a basis of* $\mathbb{C}^N$*. Also, the eigenvectors* $\varphi^{(j)}$ *for* $j = N, \cdots, 2N-1$ *(corresponding to the positive eigenvalues) form a basis of* $\mathbb{C}^N$*.*

**Proof**

(i) Let $x = \{x_n\}_{n=0}^{N-1} \in \mathbb{C}^N$. We assume that $(x, \theta) = 0$ and $(x, \varphi^{(j)}) = 0$, for $j = N+1, \cdots, 2N-1$. We need to show that then $x = 0$. Applying (15), (16), (17), and (18) to the vectors $f = 0$ and $g = C^{-1}x$, we get

$$0 = \alpha\theta + \sum_{j=1}^{2N-1} \alpha_j \varphi^{(j)}, \quad C^{-1}x = \sum_{j=1}^{2N-1} \alpha_j \lambda_j \varphi^{(j)}, \tag{36}$$

where

$$\alpha_j = \frac{1}{\rho_j} \lambda_j (x, \varphi^{(j)}), \quad j = 1, \cdots, 2N-1. \tag{37}$$

Since $(x, \varphi^{(j)}) = 0$ for $j = N+1, \cdots, 2N-1$, it follows from (37) that $\alpha_j = 0$ for $j = N+1, \cdots, 2N-1$. Therefore, equations in (36) can be rewritten as

$$0 = \alpha\theta + \sum_{j=1}^{N} \alpha_j \varphi^{(j)}, \tag{38}$$

$$C^{-1}x = \sum_{j=1}^{N} \alpha_j \lambda_j \varphi^{(j)}. \tag{39}$$

Taking the inner product of (38) with $x$, we obtain that

$$0 = \alpha(\theta, x) + \sum_{j=1}^{N} \alpha_j (\varphi^{(j)}, x). \tag{40}$$

Since

$$(\varphi^{(j)}, x) = \rho_j \frac{1}{\lambda_j} \bar{\alpha}_j \;\; for \;\; j = 1, \cdots, N$$

from (37), and $(x,\theta)=0$, (40) takes the form:

$$0=\sum_{j=1}^{N}\rho_j\frac{1}{\lambda_j}|\alpha_j|^2.$$

Since $\rho_j>0$ by (17) and $\lambda_j<0$, $j=1,\cdots,N$, we have from the latter equality that $\alpha_j=0$ for $j=1,\cdots,N$. Hence, $C^{-1}x=0$ by (39), and so $x=0$.

Now we show that the eigenvectors $\varphi^{(j)}$ for $j=1,\cdots,N$ corresponding to the negative eigenvalues also form a basis of $\mathbb{C}^N$. Reasoning similarly, let $x=\{x_n\}_{n=0}^{N-1}\in\mathbb{C}^N$ and assume that $(x,\varphi^{(j)})=0,\ j=1,\cdots,N$. We must show that $x=0$. Again applying (15), (16), (17), and (18) to the vectors $f=0$ and $g=C^{-1}x$, we have (36) and (37). Since $(x,\varphi^{(j)})=0$ for $j=1,\cdots,N$, from (37), we get $\alpha_j=0,\ j=1\cdots,N$. Equations in (36) take the form:

$$0=\alpha\theta+\sum_{j=N+1}^{2N-1}\alpha_j\varphi^{(j)}, \tag{41}$$

$$C^{-1}x=\sum_{j=N+1}^{2N-1}\alpha_j\lambda_j\varphi^{(j)}. \tag{42}$$

We have proved above that the eigenvectors $\{\theta,\varphi^{(N+1)},\cdots,\varphi^{(2N-1)}\}$ form a basis of $\mathbb{C}^N$. It follows from (41) that $\alpha=\alpha_{N+1}=\cdots=\alpha_{2N-1}=0$. Then from (42), we get $C^{-1}x=0$. Hence, $x=0$.

(ii) Let $x=\{x_n\}_{n=0}^{N-1}\in\mathbb{C}^N$ and assume that $(x,\theta)=0$ and $(x,\varphi^{(j)})=0$, for $j=1,\cdots,N-1$. We must show that $x=0$. Applying (15), (16), (17), and (18) to the vectors $f=0$ and $g=C^{-1}x$, we get equations (36) and (37). Since $(x,\varphi^{(j)})=0$ for $j=1,\cdots,N-1$, it follows from (37) that $\alpha_j=0$ for $j=1,\cdots,N-1$. Therefore, equations in (36) can be rewritten as

$$0=\alpha\theta+\sum_{j=N}^{2N-1}\alpha_j\varphi^{(j)}, \tag{43}$$

$$C^{-1}x=\sum_{j=N}^{2N-1}\alpha_j\lambda_j\varphi^{(j)}. \tag{44}$$

Taking the inner product of (43) with $x$, we obtain that

$$0=\alpha(\theta,x)+\sum_{j=N}^{2N-1}\alpha_j(\varphi^{(j)},x). \tag{45}$$

Since $(x,\theta)=0$ and

$$(\varphi^{(j)},x)=\rho_j\frac{1}{\lambda_j}\bar{\alpha}_j\ \ for\ \ j=N,\cdots,2N-1$$

by (37), Eq. (45) gives

$$0 = \sum_{j=N}^{2N-1} \alpha_j \rho_j \frac{1}{\lambda_j} \bar{\alpha_j} = \sum_{j=N}^{2N-1} \rho_j \frac{1}{\lambda_j} |\alpha_j|^2. \tag{46}$$

Since $\rho_j > 0$ by (17) and $\lambda_j > 0$, $j = N, \cdots, 2N-1$, we have from (46) that $\alpha_j = 0$ for $j = N, \cdots, 2N-1$. Hence, $C^{-1}x = 0$ by (44), and so $x = 0$. Now, we will show that eigenvectors $\varphi^{(j)}$ for $j = N, \cdots, 2N-1$ (corresponding to the positive eigenvalues) also form a basis of $\mathbb{C}^N$. Let $x = \{x_n\}_{n=0}^{N-1} \in \mathbb{C}^N$ and assume that $(x, \varphi^{(j)}) = 0$, $j = N, \cdots, 2N-1$. We must show that $x = 0$. Applying (15), (16), (17), and (18) to the vectors $f = 0$ and $g = C^{-1}x$, we get (36) and (37). Since $(x, \varphi^{(j)}) = 0$ for $j = N, \cdots, 2N-1$, we have from (37) that $\alpha_j = 0$, $j = N \cdots, 2N-1$. Therefore, equations in (36) take the form:

$$0 = \alpha\theta + \sum_{j=1}^{N-1} \alpha_j \varphi^{(j)}, \tag{47}$$

$$C^{-1}x = \sum_{j=1}^{N-1} \alpha_j \lambda_j \varphi^{(j)}. \tag{48}$$

Above, we have showed that the eigenvectors $\{\theta, \varphi^{(1)}, \cdots, \varphi^{(N-1)}\}$ form a basis of $\mathbb{C}^N$. Then it follows from (47) that $\alpha = \alpha_1 = \cdots = \alpha_{N-1} = 0$. Therefore, (48) implies $C^{-1}x = 0$. Hence, we obtain that $x = 0$. This completes the proof of the theorem. □

**Theorem 4.2** *Suppose (9), (11), and $b_0 = 0$. Then each of the systems $\{\varphi^{(1)}, \cdots, \varphi^{(N-1)}, \theta\}$ and $\{\varphi^{(N)}, \cdots, \varphi^{(2N-2)}, \theta\}$ forms a basis of $\mathbb{C}^N$.*

**Proof** Let us show that the system $\{\varphi^{(1)}, \cdots, \varphi^{(N-1)}, \theta\}$ forms a basis of $\mathbb{C}^N$. Let $x = \{x_n\}_{n=0}^{N-1} \in \mathbb{C}^N$ and assume that $(x, \varphi^{(j)}) = 0$ for $j = 1, \cdots, N-1$ and $(x, \theta) = 0$. We need to show that $x = 0$.

Applying (19), (20), (21), (22), (23), and (24) to the vectors $f = 0$ and $g = C^{-1}x$, we get

$$0 = \alpha\theta + \beta\psi + \sum_{j=1}^{2N-2} \alpha_j \varphi^{(j)}, \quad C^{-1}x = \beta\theta + \sum_{j=1}^{2N-2} \alpha_j \lambda_j \varphi^{(j)}, \tag{49}$$

where

$$\alpha_j = \frac{1}{\rho_j} \lambda_j (x, \varphi^{(j)}), \quad j = 1, \cdots, 2N-2. \tag{50}$$

Since $(x, \varphi^{(j)}) = 0$ for $j = 1, \cdots, N-1$, it follows from (50) that $\alpha_j = 0$ for $j = 1, \cdots, N-1$. Therefore, equations in (49) can be rewritten as

$$0 = \alpha\theta + \beta\psi + \sum_{j=N}^{2N-2} \alpha_j \varphi^{(j)}, \tag{51}$$

$$C^{-1}x = \beta\theta + \sum_{j=N}^{2N-2} \alpha_j \lambda_j \varphi^{(j)}. \tag{52}$$

Taking the inner product of (51) with $x$, we obtain that

$$0 = \alpha(\theta, x) + \beta(\psi, x) + \sum_{j=N}^{2N-2} \alpha_j (\varphi^{(j)}, x). \tag{53}$$

Since $(x, \theta) = 0$, it follows that

$$0 = \beta(\psi, x) + \sum_{j=N}^{2N-2} \alpha_j (\varphi^{(j)}, x). \tag{54}$$

Putting the vectors $f = 0$ and $g = C^{-1}x$ into (22) and taking into account $(\theta, x) = 0$, we get $\beta = 0$. Now by (50), Eq. (54) gives

$$0 = \sum_{j=N}^{2N-2} \alpha_j \rho_j \frac{1}{\lambda_j} \bar{\alpha_j} = \sum_{j=N}^{2N-2} \rho_j \frac{1}{\lambda_j} |\alpha_j|^2. \tag{55}$$

Since $\rho_j > 0$ and $\lambda_j > 0$ for $j = N, \cdots, 2N-2$, we obtain from (55) that $\alpha_j = 0$ for $j = N, \cdots, 2N-2$. Therefore, from Eq. (52), we get $C^{-1}x = 0$. It follows that $x = 0$.

Now we show that the system $\{\varphi^{(N)}, \cdots, \varphi^{(2N-2)}, \theta\}$ also forms a basis of $\mathbb{C}^N$. Let $x = \{x_n\}_{n=0}^{N-1} \in \mathbb{C}^N$ and assume that $(x, \varphi^{(j)}) = 0$ for $j = N, \cdots, 2N-2$ and $(x, \theta) = 0$. We need to show that $x = 0$.

We apply (19), (20), (21), (22), (23), and (24) to the vectors $f = 0$ and $g = C^{-1}x$, to get equations (49) and (50). Since $(x, \varphi^{(j)}) = 0$ for $j = N, \cdots, 2N-2$, it follows from (50) that $\alpha_j = 0$ for $j = N, \cdots, 2N-2$. Therefore, equations in (49) can be rewritten as

$$0 = \alpha\theta + \beta\psi + \sum_{j=1}^{N-1} \alpha_j \varphi^{(j)}, \tag{56}$$

$$C^{-1}x = \beta\theta + \sum_{j=1}^{N-1} \alpha_j \lambda_j \varphi^{(j)}. \tag{57}$$

Taking the inner product of (56) with $x$, we obtain that

$$0 = \alpha(\theta, x) + \beta(\psi, x) + \sum_{j=1}^{N-1} \alpha_j (\varphi^{(j)}, x). \tag{58}$$

Since $(x, \theta) = 0$, it follows that

$$0 = \beta(\psi, x) + \sum_{j=1}^{N-1} \alpha_j (\varphi^{(j)}, x). \tag{59}$$

Putting the vectors $f = 0$ and $g = C^{-1}x$ into (22) and using $(\theta, x) = 0$, we find $\beta = 0$. Now by (50), Eq. (59) yields

$$0 = \sum_{j=1}^{N-1} \alpha_j \rho_j \frac{1}{\lambda_j} \bar{\alpha_j} = \sum_{j=1}^{N-1} \rho_j \frac{1}{\lambda_j} |\alpha_j|^2. \tag{60}$$

Since $\rho_j > 0$ and $\lambda_j < 0$ for $j = 1, \cdots, N-1$, we obtain from (60) that $\alpha_j = 0$ for $j = 1, \cdots, N-1$. Therefore, from Eq. (57), we get $C^{-1}x = 0$. It follows that $x = 0$. This finishes the proof of the theorem. □

## 5. Application

In this section, we give some applications of the results obtained above.

**Theorem 5.1** *Assume (9), (11), and* $b_0 \neq 0$*.*

*(i) If* $b_0 < 0$*, then for an arbitrary vector* $f = \{f_n\}_{n=0}^{N-1} \in \mathbb{C}^N$ *Eq. (1) has a unique solution* $u(t)$ *that satisfies the conditions*

$$u(0) = f, \quad \lim_{t\to\infty} u(t) = 0. \tag{61}$$

*(ii) If* $b_0 > 0$*, then for an arbitrary vector* $f = \{f_n\}_{n=0}^{N-1} \in \mathbb{C}^N$ *Eq. (1) has a unique solution* $u(t)$ *that satisfies the conditions*

$$u(0) = f, \quad \lim_{t\to\infty} u(t) = \sigma\theta, \tag{62}$$

*where* $\theta = (1, 0, \cdots, 0)^T$ *and* $\sigma$ *is defined by means of expansion (63) below.*

**Proof**

(i) The function

$$u(t) \quad = \quad \sum_{j=1}^{N} \sigma_j e^{\lambda_j t} \varphi^{(j)}$$

is a solution of problem (1), (61), where $\lambda_1, \cdots, \lambda_N$ are negative eigenvalues of Eq. (4) and $\sigma_1, \cdots, \sigma_N$ are defined by means of the expansion

$$f \quad = \quad \sum_{j=1}^{N} \sigma_j \varphi^{(j)}.$$

Such numbers $\sigma_j$ exist and are determined uniquely by Theorem 4.1 (i).

To prove the uniqueness of the solution, note that the general solution of Eq. (1) has the form (13). It follows from (13) and the second condition in (61) that $\alpha = \alpha_{N+1} = \cdots = \alpha_{2N-1} = 0$. Setting now $t = 0$ in (13), we obtain that $\alpha_j = \sigma_j$, for $j = 1, \cdots, N$. This finishes the proof of the part (i) of the theorem.

(ii) By Theorem 4.1 (ii), the function

$$u(t) = \sigma\theta + \sum_{j=1}^{N-1} \sigma_j e^{\lambda_j t} \varphi^{(j)}$$

is a solution of problem (1), (61), where $\lambda_1, \cdots, \lambda_{N-1}$ are negative eigenvalues of Eq. (4) and $\sigma_1, \cdots, \sigma_{N-1}$ and $\sigma$ are defined by means of the expansion

$$f = \sigma\theta + \sum_{j=1}^{N-1} \sigma_j \varphi^{(j)}. \tag{63}$$

The proof of the uniqueness is similar to that given in the part (i). □

**Theorem 5.2** *Assume (9), (11), and $b_0 = 0$. Then for an arbitrary vector $f = \{f_n\}_{n=0}^{N-1} \in \mathbb{C}^N$ and an arbitrary number $\sigma^*$ Eq. (1) has a unique solution $u(t)$ that satisfies the conditions*

$$u(0) = f, \quad \lim_{t\to\infty} |u(t) - \sigma\theta - \sigma^*(\psi + t\theta)| = 0, \tag{64}$$

*where $\sigma$ is defined by means of expansion (65) below.*

**Proof** The function

$$u(t) = \sigma\theta + \sigma^*(\psi + t\theta) + \sum_{j=1}^{N-1} \sigma_j e^{\lambda_j t} \varphi^{(j)},$$

is a solution of problem (1) and (64), where $\lambda_1, \cdots, \lambda_{N-1}$ are negative eigenvalues of Eq. (4), and $\sigma_1, \cdots, \sigma_{N-1}$ and $\sigma$ are defined by means of the expansion

$$f - \sigma^*\psi = \sigma\theta + \sum_{j=1}^{N-1} \sigma_j \varphi^{(j)}. \tag{65}$$

Such numbers $\sigma$ and $\sigma_j$ exist and are determined uniquely by Theorem 4.2.

To prove the uniqueness of the solution, note that the general solution of Eq. (1) has the form (14). It follows from (14) and the second condition in (64) that $\alpha_N = \cdots = \alpha_{2N-2} = 0$ and $\beta = \sigma^*$. Setting now $t = 0$ in (14), we obtain that $\alpha = \sigma$ and $\alpha_j = \sigma_j$ for $j = 1, \cdots, N-1$. This finishes the proof. □

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