Turkish Journal of Mathematics

http://journals.tubitak.gov.tr/math/

Research Article

Turk J Math
(2016) 40: 471 – 486
© TÜBİTAK
doi:10.3906/mat-1411-61

New inequalities for fractional integrals and their applications

Shiow-Ru HWANG[1], **Kuei-Lin TSENG**[2,*], **Kai-Chen HSU**[2]
[1]China University of Science and Technology, Nankang, Taipei, Taiwan
[2]Department of Applied Mathematics, Aletheia University, Tamsui, New Taipei City, Taiwan

Received: 25.11.2014 • **Accepted/Published Online:** 24.06.2015 • **Final Version:** 08.04.2016

Abstract: In this paper, we establish some Hermite–Hadamard-type, Bullen-type, and Simpson-type inequalities for fractional integrals. Some applications for the beta function are also given.

Key words: Hermite–Hadamard inequality, Bullen inequality, Simpson inequality, fractional integral, convex function

1. Introduction

Throughout this paper, let $a < b$ in $\mathbb{R}$.

The inequality

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)\,dx \leq \frac{f(a)+f(b)}{2} \tag{1.1}$$

which holds for all convex functions $f : [a,b] \to \mathbb{R}$, is known in the literature as the Hermite–Hadamard inequality [7].

For some results that generalize, improve, and extend the inequality (1.1), see $[1-6, 8-16]$.

Tseng et al. [12] established the following Hermite–Hadamard-type inequality that refines the inequality (1.1).

Theorem A *Suppose that* $f : [a,b] \to \mathbb{R}$ *is a convex function on* $[a,b]$. *Then we have the inequality*

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{2}\left[f\left(\frac{3a+b}{4}\right) + f\left(\frac{a+3b}{4}\right)\right] \tag{1.2}$$

$$\leq \frac{1}{b-a}\int_a^b f(x)\,dx$$

$$\leq \frac{1}{2}\left[f\left(\frac{a+b}{2}\right) + \frac{f(a)+f(b)}{2}\right] \leq \frac{f(a)+f(b)}{2}.$$

The third inequality in (1.2) is known in the literature as the Bullen inequality.

Dragomir and Agarwal [4] established the following results connected with the second inequality in the inequality (1.1).

*Correspondence: kltseng@mail.au.edu.tw
2010 *AMS Mathematics Subject Classification:* Primary 26D15. Secondary 26D99.

Theorem B *Let $f : [a,b] \to \mathbb{R}$ be a differentiable function on (a,b) with $a < b$. If $|f'|$ is convex on $[a,b]$, then we have*

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(t)\,dt\right| \le \frac{b-a}{8}\left(|f'(a)| + |f'(b)|\right) \tag{1.3}$$

which is the trapezoid inequality provided $|f'|$ is convex on $[a,b]$.

Kirmaci and Özdemir [10] established the following results connected with the first inequality in the inequality (1.1).

Theorem C *Under the assumptions of Theorem B, then we have*

$$\left|\frac{1}{b-a}\int_a^b f(t)\,dt - f\left(\frac{a+b}{2}\right)\right| \le \frac{b-a}{8}\left(|f'(a)| + |f'(b)|\right) \tag{1.4}$$

which is the midpoint inequality provided $|f'|$ is convex on $[a,b]$.

In what follows we recall the following definitions [11].

Definition 1.1 *Let $f \in L_1[a,b]$. The Riemann–Liouville integrals $J_{a+}^{\alpha} f$ and $J_{b-}^{\alpha} f$ of order $\alpha > 0$ with $a \ge 0$ are defined by*

$$J_{a+}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha-1} f(t)\,dt \quad (x > a)$$

and

$$J_{b-}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)}\int_x^b (t-x)^{\alpha-1} f(t)\,dt \quad (x < b),$$

respectively. Here, $\Gamma(\alpha)$ is the gamma function and $J_{a+}^{0} f(x) = J_{b-}^{0} f(x) = f(x)$.

Sarikaya et al. [11] established the following Hermite–Hadamard-type inequalities for fractional integrals:

Theorem D *Let $f : [a,b] \to \mathbb{R}$ be positive with $0 \le a < b$ and $f \in L_1[a,b]$. If f is a convex function on $[a,b]$, then*

$$f\left(\frac{a+b}{2}\right) \le \frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\left[J_{a+}^{\alpha} f(b) + J_{b-}^{\alpha} f(a)\right] \le \frac{f(a)+f(b)}{2} \tag{1.5}$$

for $\alpha > 0$.

Theorem E *Under the assumptions of Theorem B, then we have*

$$\begin{aligned} &\left|\frac{f(a)+f(b)}{2} - \frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\left[J_{a+}^{\alpha} f(b) + J_{b-}^{\alpha} f(a)\right]\right| \\ \le\ & \frac{2^{\alpha}-1}{2^{\alpha+1}(\alpha+1)}(b-a)\left(|f'(a)| + |f'(b)|\right) \end{aligned} \tag{1.6}$$

for $\alpha > 0$.

Zhu et al. [16] established the following fractional integral inequality with the first inequality of (1.5):

Theorem F *Under the assumptions of Theorem B, then we have*

$$\left|\frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\left[J_{a^+}^{\alpha}f(b)+J_{b^-}^{\alpha}f(a)\right]-f\left(\frac{a+b}{2}\right)\right| \tag{1.7}$$
$$\leq \frac{(b-a)}{4(\alpha+1)}\left(\alpha+3-\frac{1}{2^{\alpha-1}}\right)\left(|f'(a)|+|f'(b)|\right)$$

for $\alpha>0$.

Remark 1.2

1. *The assumption* $f:[a,b]\to\mathbb{R}$ *is positive with* $0\leq a<b$ *in Theorem D can be weakened as* $f:[a,b]\to\mathbb{R}$ *with* $a<b$.
2. *In Theorem F, let* $\alpha=1$. *Then we have*

$$\left|\frac{1}{b-a}\int_a^b f(t)\,dt-f\left(\frac{a+b}{2}\right)\right|\leq\frac{3(b-a)}{8}\left(|f'(a)|+|f'(b)|\right). \tag{1.8}$$

Thus, the upper bound in the inequality (1.4) is smaller than that in the inequality (1.8) and Theorem F cannot reduce to Theorem C.

In this paper, we establish some Hermite–Hadamard-type, Bullen-type, and Simpson-type inequalities for fractional integrals that improve the inequality (1.7) and generalize Theorem C.

2. Hermite–Hadamard-type inequality for fractional integrals

Theorem 2.1 *Under the assumptions of Theorem B, then we have the following Hermite–Hadamard-type inequality for fractional integrals:*

$$\left|\frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\left[J_{a^+}^{\alpha}f(b)+J_{b^-}^{\alpha}f(a)\right]-f\left(\frac{a+b}{2}\right)\right| \tag{2.1}$$
$$\leq \frac{(b-a)}{4(\alpha+1)}\left(\alpha-1+\frac{1}{2^{\alpha-1}}\right)\left(|f'(a)|+|f'(b)|\right)$$

for $\alpha>0$.

Proof Define

$$h_1(x)=\begin{cases}(b-x)^{\alpha}-(x-a)^{\alpha}-(b-a)^{\alpha}, & x\in\left[a,\frac{a+b}{2}\right)\\ (b-x)^{\alpha}-(x-a)^{\alpha}+(b-a)^{\alpha}, & x\in\left[\frac{a+b}{2},b\right]\end{cases}.$$

Using integration by parts, we have the following identities:

$$\begin{aligned} & \frac{1}{2(b-a)^{\alpha}} \int_a^b h_1(x) f'(x)\,dx \\ = & \frac{\alpha}{2(b-a)^{\alpha}} \int_a^b \left[(x-a)^{\alpha-1} + (b-x)^{\alpha-1}\right] f(x)\,dx - f\left(\frac{a+b}{2}\right) \\ = & \frac{\alpha\Gamma(\alpha)}{2(b-a)^{\alpha}} \left[J_{a+}^{\alpha} f(b) + J_{b-}^{\alpha} f(a)\right] - f\left(\frac{a+b}{2}\right) \\ = & \frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}} \left[J_{a+}^{\alpha} f(b) + J_{b-}^{\alpha} f(a)\right] - f\left(\frac{a+b}{2}\right). \end{aligned} \tag{2.2}$$

$$x = \frac{b-x}{b-a}a + \frac{x-a}{b-a}b \quad (x \in [a,b]). \tag{2.3}$$

$$\begin{aligned} & \int_a^{\frac{a+b}{2}} \left[(b-a)^{\alpha} - (b-x)^{\alpha} + (x-a)^{\alpha}\right] \frac{b-x}{b-a} \left|f'(a)\right| dx \\ & + \int_{\frac{a+b}{2}}^b \left[(b-x)^{\alpha} - (x-a)^{\alpha} + (b-a)^{\alpha}\right] \frac{b-x}{b-a} \left|f'(a)\right| dx \\ = & \int_a^{\frac{a+b}{2}} \left[(b-a)^{\alpha} - (b-x)^{\alpha} + (x-a)^{\alpha}\right] \frac{b-x}{b-a} \left|f'(a)\right| dx \\ & + \int_a^{\frac{a+b}{2}} \left[(b-a)^{\alpha} - (b-x)^{\alpha} + (x-a)^{\alpha}\right] \frac{x-a}{b-a} \left|f'(a)\right| dx \\ = & \left|f'(a)\right| \int_a^{\frac{a+b}{2}} \left[(b-a)^{\alpha} - (b-x)^{\alpha} + (x-a)^{\alpha}\right] dx := M_1. \end{aligned} \tag{2.4}$$

$$\begin{aligned} & \int_a^{\frac{a+b}{2}} \left[(b-a)^{\alpha} - (b-x)^{\alpha} + (x-a)^{\alpha}\right] \frac{x-a}{b-a} \left|f'(b)\right| dx \\ & + \int_{\frac{a+b}{2}}^b \left[(b-x)^{\alpha} - (x-a)^{\alpha} + (b-a)^{\alpha}\right] \frac{x-a}{b-a} \left|f'(b)\right| dx \\ = & \int_a^{\frac{a+b}{2}} \left[(b-a)^{\alpha} - (b-x)^{\alpha} + (x-a)^{\alpha}\right] \frac{x-a}{b-a} \left|f'(b)\right| dx \\ & + \int_a^{\frac{a+b}{2}} \left[(b-a)^{\alpha} - (b-x)^{\alpha} + (x-a)^{\alpha}\right] \frac{b-x}{b-a} \left|f'(b)\right| dx \\ = & \left|f'(b)\right| \int_a^{\frac{a+b}{2}} \left[(b-a)^{\alpha} - (b-x)^{\alpha} + (x-a)^{\alpha}\right] dx := M_2. \end{aligned} \tag{2.5}$$

Now, using simple computation, the identities (2.3) − (2.5) and the convexity of $|f'|$, we have the inequality

$$\begin{aligned}
&\left|\frac{1}{2(b-a)^{\alpha}}\int_a^b h_1(x) f'(x)\,dx\right| \qquad (2.6)\\
\le\;& \frac{1}{2(b-a)^{\alpha}}\int_a^b |h_1(x)|\,|f'(x)|\,dx\\
\le\;& \frac{1}{2(b-a)^{\alpha}}\int_a^{\frac{a+b}{2}} \left[(b-a)^{\alpha}-(b-x)^{\alpha}+(x-a)^{\alpha}\right]|f'(x)|\,dx\\
&+\frac{1}{2(b-a)^{\alpha}}\int_{\frac{a+b}{2}}^b \left[(b-x)^{\alpha}-(x-a)^{\alpha}+(b-a)^{\alpha}\right]|f'(x)|\,dx\\
\le\;& \frac{M_1+M_2}{2(b-a)^{\alpha}}\\
=\;& \frac{|f'(a)|+|f'(b)|}{2(b-a)^{\alpha}}\int_a^{\frac{a+b}{2}}\left[(b-a)^{\alpha}-(b-x)^{\alpha}+(x-a)^{\alpha}\right]dx\\
=\;& \frac{(b-a)}{4(\alpha+1)}\left(\alpha-1+\frac{1}{2^{\alpha-1}}\right)\left(|f'(a)|+|f'(b)|\right).
\end{aligned}$$

The inequality (2.1) follows from the identity (2.2) and the inequality (2.6).

This completes the proof. □

Remark 2.2

1. *The inequality* (2.1) *improves the inequality* (1.5) since

$$\begin{aligned}
&\alpha-1+\frac{1}{2^{\alpha-1}}\\
=\;& \alpha+3-\frac{2^{\alpha+1}-1}{2^{\alpha-1}}\\
<\;& \alpha+3-\frac{1}{2^{\alpha-1}} \qquad (\alpha>0).
\end{aligned}$$

2. *In Theorem 2.1, let* $\alpha=1$*. Then Theorem 2.1 reduces to Theorem C.*

3. The inequality for fractional integrals with $\frac{1}{2}\left[f\left(\frac{3a+b}{4}\right)+f\left(\frac{a+3b}{4}\right)\right]$

Theorem 3.1 *Under the assumptions of Theorem B, then we have the following inequality for fractional integrals with* $\frac{f\left(\frac{3a+b}{4}\right)+f\left(\frac{a+3b}{4}\right)}{2}$*:*

$$\begin{aligned}
&\left|\frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\left[J_{a^+}^{\alpha}f(b)+J_{b^-}^{\alpha}f(a)\right]-\frac{1}{2}\left[f\left(\frac{3a+b}{4}\right)+f\left(\frac{a+3b}{4}\right)\right]\right| \qquad (3.1)\\
\le\;& \left(\frac{1}{8}+\frac{3^{\alpha+1}-2^{\alpha+1}+1}{4^{\alpha+1}(\alpha+1)}-\frac{1}{2(\alpha+1)}\right)(b-a)\left(|f'(a)|+|f'(b)|\right)
\end{aligned}$$

for $\alpha>0$.

Proof Define

$$h_2(x) = \begin{cases} (b-x)^\alpha - (x-a)^\alpha - (b-a)^\alpha, & x \in \left[a, \frac{3a+b}{4}\right) \\ (b-x)^\alpha - (x-a)^\alpha, & x \in \left[\frac{3a+b}{4}, \frac{a+3b}{4}\right) \\ (b-x)^\alpha - (x-a)^\alpha + (b-a)^\alpha, & x \in \left[\frac{a+3b}{4}, b\right] \end{cases}.$$

Using integration by parts, we have the following identities:

$$\begin{aligned} & \frac{1}{2(b-a)^\alpha} \int_a^b h_2(x) f'(x)\, dx \\ = \ & \frac{\alpha}{2(b-a)^\alpha} \int_a^b \left[(x-a)^{\alpha-1} + (b-x)^{\alpha-1}\right] f(x)\, dx - \frac{f\left(\frac{3a+b}{4}\right) + f\left(\frac{a+3b}{4}\right)}{2} \\ = \ & \frac{\alpha\Gamma(\alpha)}{2(b-a)^\alpha} \left[J_{a^+}^\alpha f(b) + J_{b^-}^\alpha f(a)\right] - \frac{f\left(\frac{3a+b}{4}\right) + f\left(\frac{a+3b}{4}\right)}{2} \\ = \ & \frac{\Gamma(\alpha+1)}{2(b-a)^\alpha} \left[J_{a^+}^\alpha f(b) + J_{b^-}^\alpha f(a)\right] - \frac{1}{2}\left[f\left(\frac{3a+b}{4}\right) + f\left(\frac{a+3b}{4}\right)\right]. \end{aligned} \tag{3.2}$$

$$\begin{aligned} & \int_a^{\frac{3a+b}{4}} \left[(b-a)^\alpha - (b-x)^\alpha + (x-a)^\alpha\right] \frac{b-x}{b-a} |f'(a)|\, dx \\ & + \int_{\frac{a+3b}{4}}^b \left[(b-x)^\alpha - (x-a)^\alpha + (b-a)^\alpha\right] \frac{b-x}{b-a} |f'(a)|\, dx \\ = \ & \int_a^{\frac{3a+b}{4}} \left[(b-a)^\alpha - (b-x)^\alpha + (x-a)^\alpha\right] \frac{b-x}{b-a} |f'(a)|\, dx \\ & + \int_a^{\frac{3a+b}{4}} \left[(b-a)^\alpha - (b-x)^\alpha + (x-a)^\alpha\right] \frac{x-a}{b-a} |f'(a)|\, dx \\ = \ & |f'(a)| \int_a^{\frac{3a+b}{4}} \left[(b-a)^\alpha - (b-x)^\alpha + (x-a)^\alpha\right] dx := N_1. \end{aligned} \tag{3.3}$$

$$\begin{aligned} & \int_a^{\frac{3a+b}{4}} \left[(b-a)^\alpha - (b-x)^\alpha + (x-a)^\alpha\right] \frac{x-a}{b-a} |f'(b)|\, dx \\ & + \int_{\frac{a+3b}{4}}^b \left[(b-x)^\alpha - (x-a)^\alpha + (b-a)^\alpha\right] \frac{x-a}{b-a} |f'(b)|\, dx \\ = \ & \int_a^{\frac{3a+b}{4}} \left[(b-a)^\alpha - (b-x)^\alpha + (x-a)^\alpha\right] \frac{x-a}{b-a} |f'(b)|\, dx \\ & + \int_a^{\frac{3a+b}{4}} \left[(b-a)^\alpha - (b-x)^\alpha + (x-a)^\alpha\right] \frac{b-x}{b-a} |f'(b)|\, dx \\ = \ & |f'(b)| \int_a^{\frac{3a+b}{4}} \left[(b-a)^\alpha - (b-x)^\alpha + (x-a)^\alpha\right] dx := N_2. \end{aligned} \tag{3.4}$$

$$\begin{aligned}
&\int_{\frac{3a+b}{4}}^{\frac{a+b}{2}}\left[(b-x)^{\alpha}-(x-a)^{\alpha}\right]\frac{b-x}{b-a}\left|f'(a)\right|dx \\
&+\int_{\frac{a+b}{2}}^{\frac{a+3b}{4}}\left[(x-a)^{\alpha}-(b-x)^{\alpha}\right]\frac{b-x}{b-a}\left|f'(a)\right|dx \\
=&\int_{\frac{3a+b}{4}}^{\frac{a+b}{2}}\left[(b-x)^{\alpha}-(x-a)^{\alpha}\right]\frac{b-x}{b-a}\left|f'(a)\right|dx \\
&+\int_{\frac{3a+b}{4}}^{\frac{a+b}{2}}\left[(b-x)^{\alpha}-(x-a)^{\alpha}\right]\frac{x-a}{b-a}\left|f'(a)\right|dx \\
=&\left|f'(a)\right|\int_{\frac{3a+b}{4}}^{\frac{a+b}{2}}\left[(b-x)^{\alpha}-(x-a)^{\alpha}\right]dx:=N_3.
\end{aligned} \tag{3.5}$$

$$\begin{aligned}
&\int_{\frac{3a+b}{4}}^{\frac{a+b}{2}}\left[(b-x)^{\alpha}-(x-a)^{\alpha}\right]\frac{x-a}{b-a}\left|f'(b)\right|dx \\
&+\int_{\frac{a+b}{2}}^{\frac{a+3b}{4}}\left[(x-a)^{\alpha}-(b-x)^{\alpha}\right]\frac{x-a}{b-a}\left|f'(b)\right|dx \\
=&\int_{\frac{3a+b}{4}}^{\frac{a+b}{2}}\left[(b-x)^{\alpha}-(x-a)^{\alpha}\right]\frac{x-a}{b-a}\left|f'(b)\right|dx \\
&+\int_{\frac{3a+b}{4}}^{\frac{a+b}{2}}\left[(b-x)^{\alpha}-(x-a)^{\alpha}\right]\frac{b-x}{b-a}\left|f'(b)\right|dx \\
=&\left|f'(b)\right|\int_{\frac{3a+b}{4}}^{\frac{a+b}{2}}\left[(b-x)^{\alpha}-(x-a)^{\alpha}\right]dx:=N_4.
\end{aligned} \tag{3.6}$$

Now, using simple computation and the identities (2.3) and (3.3) – (3.6), we have the inequality

$$\begin{aligned}
&\left|\frac{1}{2(b-a)^{\alpha}}\int_a^b h_2(x)\,f'(x)\,dx\right| \\
\leq&\frac{1}{2(b-a)^{\alpha}}\int_a^b\left|h_2(x)\right|\left|f'(x)\right|dx \\
=&\frac{1}{2(b-a)^{\alpha}}\int_a^{\frac{3a+b}{4}}\left[(b-a)^{\alpha}-(b-x)^{\alpha}+(x-a)^{\alpha}\right]\left|f'(x)\right|dx \\
&+\frac{1}{2(b-a)^{\alpha}}\int_{\frac{3a+b}{4}}^{\frac{a+b}{2}}\left[(b-x)^{\alpha}-(x-a)^{\alpha}\right]\left|f'(x)\right|dx \\
&+\frac{1}{2(b-a)^{\alpha}}\int_{\frac{a+b}{2}}^{\frac{a+3b}{4}}\left[(x-a)^{\alpha}-(b-x)^{\alpha}\right]\left|f'(x)\right|dx \\
&+\frac{1}{2(b-a)^{\alpha}}\int_{\frac{a+3b}{4}}^{b}\left[(b-x)^{\alpha}-(x-a)^{\alpha}+(b-a)^{\alpha}\right]\left|f'(x)\right|dx
\end{aligned} \tag{3.7}$$

$$\begin{aligned}
&\leq \frac{N_1+N_2+N_3+N_4}{2(b-a)^{\alpha}} \\
&= \frac{|f'(a)|+|f'(b)|}{2(b-a)^{\alpha}}\int_a^{\frac{3a+b}{4}}\left[(b-a)^{\alpha}-(b-x)^{\alpha}+(x-a)^{\alpha}\right]dx \\
&\quad+\frac{|f'(a)|+|f'(b)|}{2(b-a)^{\alpha}}\int_{\frac{3a+b}{4}}^{\frac{a+b}{2}}\left[(b-x)^{\alpha}-(x-a)^{\alpha}\right]dx \\
&= \left(\frac{1}{8}+\frac{3^{\alpha+1}-2^{\alpha+1}+1}{4^{\alpha+1}(\alpha+1)}-\frac{1}{2(\alpha+1)}\right)(b-a)\left(|f'(a)|+|f'(b)|\right).
\end{aligned}$$

The inequality (3.1) follows from the identity (3.2) and the inequality (3.7).

This completes the proof. □

Remark 3.2 *In Theorem 3.1, let* $\alpha=1$. *Then the inequality* (3.1) *reduces to the inequality*

$$\begin{aligned}
&\left|\frac{1}{b-a}\int_a^b f(t)\,dt-\frac{f\left(\frac{3a+b}{4}\right)+f\left(\frac{a+3b}{4}\right)}{2}\right| \\
&\leq \frac{(b-a)\left(|f'(a)|+|f'(b)|\right)}{16}
\end{aligned}$$

which is the second inequality in (1.2) *provided* $|f'|$ *is convex on* $[a,b]$.

4. Bullen-type inequality for fractional integrals

Theorem 4.1 *Under the assumptions of Theorem B, then we have the following Bullen-type inequality for fractional integrals:*

$$\begin{aligned}
&\left|\frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\left[J_{a^+}^{\alpha}f(b)+J_{b^-}^{\alpha}f(a)\right]\right. \\
&\left.-\left[\frac{3^{\alpha}-1}{4^{\alpha}}f\left(\frac{a+b}{2}\right)+\frac{4^{\alpha}-3^{\alpha}+1}{4^{\alpha}}\frac{f(a)+f(b)}{2}\right]\right| \\
&\leq \frac{1}{\alpha+1}\left(\frac{2^{\alpha}+1}{2^{\alpha+1}}-\frac{3^{\alpha+1}+1}{4^{\alpha+1}}\right)(b-a)\left(|f'(a)|+|f'(b)|\right)
\end{aligned} \tag{4.1}$$

for $\alpha>0$.

Proof Define

$$h_3(x)=\begin{cases}(b-x)^{\alpha}-(x-a)^{\alpha}-\frac{3^{\alpha}-1}{4^{\alpha}}(b-a)^{\alpha}, & x\in\left[a,\frac{a+b}{2}\right)\\ (b-x)^{\alpha}-(x-a)^{\alpha}+\frac{3^{\alpha}-1}{4^{\alpha}}(b-a)^{\alpha}, & x\in\left[\frac{a+b}{2},b\right]\end{cases}.$$

Using integration by parts, we have the following identities:

$$\begin{aligned}
& \frac{1}{2(b-a)^{\alpha}} \int_a^b h_3(x) f'(x)\,dx \qquad (4.2)\\
= & \frac{\alpha}{2(b-a)^{\alpha}} \int_a^b \left[(x-a)^{\alpha-1} + (b-x)^{\alpha-1}\right] f(x)\,dx \\
& - \left[\frac{3^{\alpha}-1}{4^{\alpha}} f\left(\frac{a+b}{2}\right) + \frac{4^{\alpha}-3^{\alpha}+1}{4^{\alpha}} \frac{f(a)+f(b)}{2}\right] \\
= & \frac{\alpha\Gamma(\alpha)}{2(b-a)^{\alpha}} \left[J_{a+}^{\alpha} f(b) + J_{b-}^{\alpha} f(a)\right] \\
& - \left[\frac{3^{\alpha}-1}{4^{\alpha}} f\left(\frac{a+b}{2}\right) + \frac{4^{\alpha}-3^{\alpha}+1}{4^{\alpha}} \frac{f(a)+f(b)}{2}\right] \\
= & \frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}} \left[J_{a+}^{\alpha} f(b) + J_{b-}^{\alpha} f(a)\right] \\
& - \left[\frac{3^{\alpha}-1}{4^{\alpha}} f\left(\frac{a+b}{2}\right) + \frac{4^{\alpha}-3^{\alpha}+1}{4^{\alpha}} \frac{f(a)+f(b)}{2}\right].
\end{aligned}$$

$$\begin{aligned}
& \int_a^{\frac{3a+b}{4}} \left[(b-x)^{\alpha} - (x-a)^{\alpha} - \frac{3^{\alpha}-1}{4^{\alpha}}(b-a)^{\alpha}\right] \frac{b-x}{b-a} |f'(a)|\,dx \qquad (4.3)\\
& + \int_{\frac{a+3b}{4}}^b \left[(x-a)^{\alpha} - (b-x)^{\alpha} - \frac{3^{\alpha}-1}{4^{\alpha}}(b-a)^{\alpha}\right] \frac{b-x}{b-a} |f'(a)|\,dx \\
= & \int_a^{\frac{3a+b}{4}} \left[(b-x)^{\alpha} - (x-a)^{\alpha} - \frac{3^{\alpha}-1}{4^{\alpha}}(b-a)^{\alpha}\right] \frac{b-x}{b-a} |f'(a)|\,dx \\
& + \int_a^{\frac{3a+b}{4}} \left[(b-x)^{\alpha} - (x-a)^{\alpha} - \frac{3^{\alpha}-1}{4^{\alpha}}(b-a)^{\alpha}\right] \frac{x-a}{b-a} |f'(a)|\,dx \\
= & |f'(a)| \int_a^{\frac{3a+b}{4}} \left[(b-x)^{\alpha} - (x-a)^{\alpha} - \frac{3^{\alpha}-1}{4^{\alpha}}(b-a)^{\alpha}\right] dx := P_1.
\end{aligned}$$

$$\begin{aligned}
& \int_a^{\frac{3a+b}{4}} \left[(b-x)^{\alpha} - (x-a)^{\alpha} - \frac{3^{\alpha}-1}{4^{\alpha}}(b-a)^{\alpha}\right] \frac{x-a}{b-a} |f'(b)|\,dx \qquad (4.4)\\
& + \int_{\frac{a+3b}{4}}^b \left[(x-a)^{\alpha} - (b-x)^{\alpha} - \frac{3^{\alpha}-1}{4^{\alpha}}(b-a)^{\alpha}\right] \frac{x-a}{b-a} |f'(b)|\,dx \\
= & \int_a^{\frac{3a+b}{4}} \left[(b-x)^{\alpha} - (x-a)^{\alpha} - \frac{3^{\alpha}-1}{4^{\alpha}}(b-a)^{\alpha}\right] \frac{x-a}{b-a} |f'(b)|\,dx \\
& + \int_a^{\frac{3a+b}{4}} \left[(b-x)^{\alpha} - (x-a)^{\alpha} - \frac{3^{\alpha}-1}{4^{\alpha}}(b-a)^{\alpha}\right] \frac{b-x}{b-a} |f'(b)|\,dx \\
= & |f'(b)| \int_a^{\frac{3a+b}{4}} \left[(b-x)^{\alpha} - (x-a)^{\alpha} - \frac{3^{\alpha}-1}{4^{\alpha}}(b-a)^{\alpha}\right] dx := P_2.
\end{aligned}$$

$$\begin{aligned}
&\int_{\frac{3a+b}{4}}^{\frac{a+b}{2}}\left[(x-a)^{\alpha}-(b-x)^{\alpha}+\frac{3^{\alpha}-1}{4^{\alpha}}(b-a)^{\alpha}\right]\frac{b-x}{b-a}\left|f'(a)\right|dx \\
&+\int_{\frac{a+b}{2}}^{\frac{a+3b}{4}}\left[(b-x)^{\alpha}-(x-a)^{\alpha}+\frac{3^{\alpha}-1}{4^{\alpha}}(b-a)^{\alpha}\right]\frac{b-x}{b-a}\left|f'(a)\right|dx \\
=&\int_{\frac{3a+b}{4}}^{\frac{a+b}{2}}\left[(x-a)^{\alpha}-(b-x)^{\alpha}+\frac{3^{\alpha}-1}{4^{\alpha}}(b-a)^{\alpha}\right]\frac{b-x}{b-a}\left|f'(a)\right|dx \\
&+\int_{\frac{3a+b}{4}}^{\frac{a+b}{2}}\left[(x-a)^{\alpha}-(b-x)^{\alpha}+\frac{3^{\alpha}-1}{4^{\alpha}}(b-a)^{\alpha}\right]\frac{x-a}{b-a}\left|f'(a)\right|dx \\
=&\left|f'(a)\right|\int_{\frac{3a+b}{4}}^{\frac{a+b}{2}}\left[(x-a)^{\alpha}-(b-x)^{\alpha}+\frac{3^{\alpha}-1}{4^{\alpha}}(b-a)^{\alpha}\right]dx:=P_3.
\end{aligned}\tag{4.5}$$

$$\begin{aligned}
&\int_{\frac{3a+b}{4}}^{\frac{a+b}{2}}\left[(x-a)^{\alpha}-(b-x)^{\alpha}+\frac{3^{\alpha}-1}{4^{\alpha}}(b-a)^{\alpha}\right]\frac{x-a}{b-a}\left|f'(b)\right|dx \\
&+\int_{\frac{a+b}{2}}^{\frac{a+3b}{4}}\left[(b-x)^{\alpha}-(x-a)^{\alpha}+\frac{3^{\alpha}-1}{4^{\alpha}}(b-a)^{\alpha}\right]\frac{x-a}{b-a}\left|f'(b)\right|dx \\
=&\int_{\frac{3a+b}{4}}^{\frac{a+b}{2}}\left[(x-a)^{\alpha}-(b-x)^{\alpha}+\frac{3^{\alpha}-1}{4^{\alpha}}(b-a)^{\alpha}\right]\frac{x-a}{b-a}\left|f'(b)\right|dx \\
&+\int_{\frac{3a+b}{4}}^{\frac{a+b}{2}}\left[(x-a)^{\alpha}-(b-x)^{\alpha}+\frac{3^{\alpha}-1}{4^{\alpha}}(b-a)^{\alpha}\right]\frac{b-x}{b-a}\left|f'(b)\right|dx \\
=&\left|f'(b)\right|\int_{\frac{3a+b}{4}}^{\frac{a+b}{2}}\left[(x-a)^{\alpha}-(b-x)^{\alpha}+\frac{3^{\alpha}-1}{4^{\alpha}}(b-a)^{\alpha}\right]dx:=P_4.
\end{aligned}\tag{4.6}$$

Now, using simple computation and the identities (2.3) and (4.3) − (4.6), we have the inequality

$$\begin{aligned}
&\left|\frac{1}{2(b-a)^{\alpha}}\int_a^b h_3(x)f'(x)\,dx\right| \\
\le&\frac{1}{2(b-a)^{\alpha}}\int_a^b\left|h_3(x)\right|\left|f'(x)\right|dx \\
=&\frac{1}{2(b-a)^{\alpha}}\int_a^{\frac{3a+b}{4}}\left[(b-x)^{\alpha}-(x-a)^{\alpha}-\frac{3^{\alpha}-1}{4^{\alpha}}\right](b-a)^{\alpha}\left|f'(x)\right|dx \\
&+\frac{1}{2(b-a)^{\alpha}}\int_{\frac{3a+b}{4}}^{\frac{a+b}{2}}\left[(x-a)^{\alpha}-(b-x)^{\alpha}+\frac{3^{\alpha}-1}{4^{\alpha}}(b-a)^{\alpha}\right]\left|f'(x)\right|dx \\
&+\frac{1}{2(b-a)^{\alpha}}\int_{\frac{a+b}{2}}^{\frac{a+3b}{4}}\left[(b-x)^{\alpha}-(x-a)^{\alpha}+\frac{3^{\alpha}-1}{4^{\alpha}}(b-a)^{\alpha}\right]\left|f'(x)\right|dx
\end{aligned}\tag{4.7}$$

$$
\begin{aligned}
&+\frac{1}{2(b-a)^{\alpha}}\int_{\frac{a+3b}{4}}^{b}\left[(x-a)^{\alpha}-(b-x)^{\alpha}-\frac{3^{\alpha}-1}{4^{\alpha}}(b-a)^{\alpha}\right]|f'(x)|\,dx \\
\leq\;& \frac{P_1+P_2+P_3+P_4}{2(b-a)^{\alpha}} \\
=\;& \frac{|f'(a)|+|f'(b)|}{2(b-a)^{\alpha}}\int_{a}^{\frac{3a+b}{4}}\left[(b-x)^{\alpha}-(x-a)^{\alpha}-\frac{3^{\alpha}-1}{4^{\alpha}}(b-a)^{\alpha}\right]dx \\
&+\frac{|f'(a)|+|f'(b)|}{2(b-a)^{\alpha}}\int_{\frac{3a+b}{4}}^{\frac{a+b}{2}}\left[(x-a)^{\alpha}-(b-x)^{\alpha}+\frac{3^{\alpha}-1}{4^{\alpha}}(b-a)^{\alpha}\right]dx \\
=\;& \frac{1}{\alpha+1}\left(\frac{2^{\alpha}+1}{2^{\alpha+1}}-\frac{3^{\alpha+1}+1}{4^{\alpha+1}}\right)(b-a)\left(|f'(a)|+|f'(b)|\right).
\end{aligned}
$$

The inequality (4.1) follows from the identity (4.2) and the inequality (4.7).

This completes the proof. □

Remark 4.2 *In Theorem 4.1, let $\alpha=1$. Then the inequality (4.1) reduces to the inequality*

$$
\begin{aligned}
&\left|\frac{1}{b-a}\int_a^b f(t)\,dt-\frac{1}{2}\left[f\left(\frac{a+b}{2}\right)+\frac{f(a)+f(b)}{2}\right]\right| \\
\leq\;& \frac{(b-a)\left(|f'(a)|+|f'(b)|\right)}{16}
\end{aligned}
$$

which is the Bullen-type inequality provided $|f'|$ is convex on $[a,b]$.

5. Simpson-type inequality for fractional integrals

Theorem 5.1 *Under the assumptions of Theorem B, then we have the following Bullen-type inequality for fractional integrals:*

$$
\begin{aligned}
&\left|\frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\left[J_{a+}^{\alpha}f(b)+J_{b-}^{\alpha}f(a)\right]\right. \\
&\qquad\left.-\left[\frac{5^{\alpha}-1}{6^{\alpha}}f\left(\frac{a+b}{2}\right)+\frac{6^{\alpha}-5^{\alpha}+1}{6^{\alpha}}\frac{f(a)+f(b)}{2}\right]\right| \\
\leq\;& \left[\frac{1}{\alpha+1}\left(\frac{2^{\alpha}+1}{2^{\alpha+1}}-\frac{5^{\alpha+1}+1}{6^{\alpha+1}}\right)+\left(\frac{5^{\alpha}-1}{12\cdot 6^{\alpha}}\right)\right](b-a)\left(|f'(a)|+|f'(b)|\right)
\end{aligned}
\tag{5.1}
$$

for $\alpha>0$.

Proof Define

$$
h_4(x)=\begin{cases}(b-x)^{\alpha}-(x-a)^{\alpha}-\frac{5^{\alpha}-1}{6^{\alpha}}(b-a)^{\alpha}, & x\in\left[a,\frac{a+b}{2}\right) \\ (b-x)^{\alpha}-(x-a)^{\alpha}+\frac{5^{\alpha}-1}{6^{\alpha}}(b-a)^{\alpha}, & x\in\left[\frac{a+b}{2},b\right]\end{cases}.
$$

Using integration by parts, we have the following identities:

$$\begin{aligned}
& \frac{1}{2(b-a)^{\alpha}} \int_a^b h_4(x) f'(x)\, dx \\
= \; & \frac{\alpha}{2(b-a)^{\alpha}} \int_a^b \left[(x-a)^{\alpha-1} + (b-x)^{\alpha-1} \right] f(x)\, dx \\
& - \left[\frac{5^{\alpha}-1}{6^{\alpha}} f\left(\frac{a+b}{2}\right) + \frac{6^{\alpha}-5^{\alpha}+1}{6^{\alpha}} \frac{f(a)+f(b)}{2} \right] \\
= \; & \frac{\alpha\Gamma(\alpha)}{2(b-a)^{\alpha}} \left[J_{a+}^{\alpha} f(b) + J_{b-}^{\alpha} f(a) \right] \\
& - \left[\frac{5^{\alpha}-1}{6^{\alpha}} f\left(\frac{a+b}{2}\right) + \frac{6^{\alpha}-5^{\alpha}+1}{6^{\alpha}} \frac{f(a)+f(b)}{2} \right] \\
= \; & \frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}} \left[J_{a+}^{\alpha} f(b) + J_{b-}^{\alpha} f(a) \right] \\
& - \left[\frac{5^{\alpha}-1}{6^{\alpha}} f\left(\frac{a+b}{2}\right) + \frac{6^{\alpha}-5^{\alpha}+1}{6^{\alpha}} \frac{f(a)+f(b)}{2} \right].
\end{aligned} \tag{5.2}$$

$$\begin{aligned}
& \int_a^{\frac{5a+b}{6}} \left[(b-x)^{\alpha} - (x-a)^{\alpha} - \frac{5^{\alpha}-1}{6^{\alpha}} (b-a)^{\alpha} \right] \frac{b-x}{b-a} |f'(a)|\, dx \\
& + \int_{\frac{a+5b}{6}}^{b} \left[(x-a)^{\alpha} - (b-x)^{\alpha} - \frac{5^{\alpha}-1}{6^{\alpha}} (b-a)^{\alpha} \right] \frac{b-x}{b-a} |f'(a)|\, dx \\
= \; & \int_a^{\frac{5a+b}{6}} \left[(b-x)^{\alpha} - (x-a)^{\alpha} - \frac{5^{\alpha}-1}{6^{\alpha}} (b-a)^{\alpha} \right] \frac{b-x}{b-a} |f'(a)|\, dx \\
& + \int_a^{\frac{5a+b}{6}} \left[(b-x)^{\alpha} - (x-a)^{\alpha} - \frac{5^{\alpha}-1}{6^{\alpha}} (b-a)^{\alpha} \right] \frac{x-a}{b-a} |f'(a)|\, dx \\
= \; & |f'(a)| \int_a^{\frac{5a+b}{6}} \left[(b-x)^{\alpha} - (x-a)^{\alpha} - \frac{5^{\alpha}-1}{6^{\alpha}} (b-a)^{\alpha} \right] dx := Q_1.
\end{aligned} \tag{5.3}$$

$$\begin{aligned}
& \int_a^{\frac{5a+b}{6}} \left[(b-x)^{\alpha} - (x-a)^{\alpha} - \frac{5^{\alpha}-1}{6^{\alpha}} (b-a)^{\alpha} \right] \frac{x-a}{b-a} |f'(b)|\, dx \\
& + \int_{\frac{a+5b}{6}}^{b} \left[(x-a)^{\alpha} - (b-x)^{\alpha} - \frac{5^{\alpha}-1}{6^{\alpha}} (b-a)^{\alpha} \right] \frac{x-a}{b-a} |f'(b)|\, dx \\
= \; & \int_a^{\frac{5a+b}{6}} \left[(b-x)^{\alpha} - (x-a)^{\alpha} - \frac{5^{\alpha}-1}{6^{\alpha}} (b-a)^{\alpha} \right] \frac{x-a}{b-a} |f'(b)|\, dx \\
& + \int_a^{\frac{5a+b}{6}} \left[(b-x)^{\alpha} - (x-a)^{\alpha} - \frac{5^{\alpha}-1}{6^{\alpha}} (b-a)^{\alpha} \right] \frac{b-x}{b-a} |f'(b)|\, dx \\
= \; & |f'(b)| \int_a^{\frac{5a+b}{6}} \left[(b-x)^{\alpha} - (x-a)^{\alpha} - \frac{5^{\alpha}-1}{6^{\alpha}} (b-a)^{\alpha} \right] dx := Q_2.
\end{aligned} \tag{5.4}$$

$$\int_{\frac{5a+b}{6}}^{\frac{a+b}{2}}\left[(x-a)^{\alpha}-(b-x)^{\alpha}+\frac{5^{\alpha}-1}{6^{\alpha}}(b-a)^{\alpha}\right]\frac{b-x}{b-a}\left|f'(a)\right|dx \tag{5.5}$$
$$+\int_{\frac{a+b}{2}}^{\frac{a+5b}{6}}\left[(b-x)^{\alpha}-(x-a)^{\alpha}+\frac{5^{\alpha}-1}{6^{\alpha}}(b-a)^{\alpha}\right]\frac{b-x}{b-a}\left|f'(a)\right|dx$$
$$+\int_{\frac{5a+b}{6}}^{\frac{a+b}{2}}\left[(x-a)^{\alpha}-(b-x)^{\alpha}+\frac{5^{\alpha}-1}{6^{\alpha}}(b-a)^{\alpha}\right]\frac{b-x}{b-a}\left|f'(a)\right|dx$$
$$+\int_{\frac{5a+b}{6}}^{\frac{a+b}{2}}\left[(x-a)^{\alpha}-(b-x)^{\alpha}+\frac{5^{\alpha}-1}{6^{\alpha}}(b-a)^{\alpha}\right]\frac{x-a}{b-a}\left|f'(a)\right|dx$$
$$=\left|f'(a)\right|\int_{\frac{5a+b}{6}}^{\frac{a+b}{2}}\left[(x-a)^{\alpha}-(b-x)^{\alpha}+\frac{5^{\alpha}-1}{6^{\alpha}}(b-a)^{\alpha}\right]dx:=Q_3.$$

$$\int_{\frac{5a+b}{6}}^{\frac{a+b}{2}}\left[(x-a)^{\alpha}-(b-x)^{\alpha}+\frac{5^{\alpha}-1}{6^{\alpha}}(b-a)^{\alpha}\right]\frac{x-a}{b-a}\left|f'(b)\right|dx \tag{5.6}$$
$$+\int_{\frac{a+b}{2}}^{\frac{a+5b}{6}}\left[(b-x)^{\alpha}-(x-a)^{\alpha}+\frac{5^{\alpha}-1}{6^{\alpha}}(b-a)^{\alpha}\right]\frac{x-a}{b-a}\left|f'(b)\right|dx$$
$$+\int_{\frac{5a+b}{6}}^{\frac{a+b}{2}}\left[(x-a)^{\alpha}-(b-x)^{\alpha}+\frac{5^{\alpha}-1}{6^{\alpha}}(b-a)^{\alpha}\right]\frac{x-a}{b-a}\left|f'(b)\right|dx$$
$$+\int_{\frac{5a+b}{6}}^{\frac{a+b}{2}}\left[(x-a)^{\alpha}-(b-x)^{\alpha}+\frac{5^{\alpha}-1}{6^{\alpha}}(b-a)^{\alpha}\right]\frac{b-x}{b-a}\left|f'(b)\right|dx$$
$$=\left|f'(b)\right|\int_{\frac{5a+b}{6}}^{\frac{a+b}{2}}\left[(x-a)^{\alpha}-(b-x)^{\alpha}+\frac{5^{\alpha}-1}{6^{\alpha}}(b-a)^{\alpha}\right]dx:=Q_4.$$

Now, using simple computation and the identities (2.3) and (5.3) – (5.6), we have the inequality

$$\left|\frac{1}{2(b-a)^{\alpha}}\int_a^b h_4(x)\,f'(x)\,dx\right| \tag{5.7}$$
$$\leq\frac{1}{2(b-a)^{\alpha}}\int_a^b\left|h_4(x)\right|\left|f'(x)\right|dx$$
$$=\frac{1}{2(b-a)^{\alpha}}\int_a^{\frac{5a+b}{6}}\left[(b-x)^{\alpha}-(x-a)^{\alpha}-\frac{5^{\alpha}-1}{6^{\alpha}}(b-a)^{\alpha}\right]\left|f'(x)\right|dx$$
$$+\frac{1}{2(b-a)^{\alpha}}\int_{\frac{5a+b}{6}}^{\frac{a+b}{2}}\left[(x-a)^{\alpha}-(b-x)^{\alpha}+\frac{5^{\alpha}-1}{6^{\alpha}}(b-a)^{\alpha}\right]\left|f'(x)\right|dx$$
$$+\frac{1}{2(b-a)^{\alpha}}\int_{\frac{a+b}{2}}^{\frac{a+5b}{6}}\left[(b-x)^{\alpha}-(x-a)^{\alpha}+\frac{5^{\alpha}-1}{6^{\alpha}}(b-a)^{\alpha}\right]\left|f'(x)\right|dx$$
$$+\frac{1}{2(b-a)^{\alpha}}\int_{\frac{a+5b}{6}}^{b}\left[(x-a)^{\alpha}-(b-x)^{\alpha}-\frac{5^{\alpha}-1}{6^{\alpha}}(b-a)^{\alpha}\right]\left|f'(x)\right|dx$$

$$\begin{aligned}
&\leq \frac{Q_1+Q_2+Q_3+Q_4}{2(b-a)^{\alpha}} \\
&= \frac{|f'(a)|+|f'(b)|}{2(b-a)^{\alpha}} \int_a^{\frac{5a+b}{6}} \left[(b-x)^{\alpha}-(x-a)^{\alpha}-\frac{5^{\alpha}-1}{6^{\alpha}}(b-a)^{\alpha}\right] dx \\
&\quad + \frac{|f'(a)|+|f'(b)|}{2(b-a)^{\alpha}} \int_{\frac{5a+b}{6}}^{\frac{a+b}{2}} \left[(x-a)^{\alpha}-(b-x)^{\alpha}+\frac{5^{\alpha}-1}{6^{\alpha}}(b-a)^{\alpha}\right] dx \\
&= \left[\frac{1}{\alpha+1}\left(\frac{2^{\alpha}+1}{2^{\alpha+1}}-\frac{5^{\alpha+1}+1}{6^{\alpha+1}}\right)+\left(\frac{5^{\alpha}-1}{12\cdot 6^{\alpha}}\right)\right](b-a)\left(|f'(a)|+|f'(b)|\right).
\end{aligned}$$

The inequality (5.1) follows from the identity (5.2) and the inequality (5.7).

This completes the proof. □

Remark 5.2 *In Theorem 5.1, let* $\alpha = 1$. *Then the inequality* (5.1) *reduces to the inequality*

$$\begin{aligned}
&\left|\frac{1}{b-a}\int_a^b f(t)\,dt - \frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]\right| \\
&\leq \frac{5(b-a)\left(|f'(a)|+|f'(b)|\right)}{72}
\end{aligned}$$

which is the Simpson-type inequality provided $|f'|$ *is convex on* $[a,b]$.

6. Applications for the beta function

Throughout this section, let $\alpha > 0$, $\rho \geq 3$, $a = 0$, $b = 1$, $\Gamma(\alpha)$ be the gamma function, and $f(x) = x^{\rho-1}$ $(x \in [0,1])$. Then $|f'|$ is convex on $[0,1]$.

Let us recall the *beta function*

$$B(p,q) = \int_0^1 x^{p-1}(1-x)^{q-1}\,dx \quad (p,q>0).$$

Remark 6.1 *In Sections 2–5, we get*

$$\begin{aligned}
&\frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}} J_{a+}^{\alpha} f(b) \\
&= \frac{\alpha}{2}\int_0^1 (1-x)^{\alpha-1} x^{\rho-1}dx = \frac{\alpha}{2}B(\rho,\alpha)
\end{aligned}$$

and

$$\begin{aligned}
&\frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}} J_{b-}^{\alpha} f(a) \\
&= \frac{\alpha}{2}\int_0^1 x^{\alpha+\rho-2}dx = \frac{\alpha}{2(\alpha+\rho-1)}.
\end{aligned}$$

Using Theorems 2.1–5.1 and Remark 6.1, we have the following propositions:

Proposition 1 *In Theorem 2.1, the following inequality holds:*

$$\left|\frac{\alpha}{2}B(\rho,\alpha)+\frac{\alpha}{2(\alpha+\rho-1)}-\frac{1}{2^{\rho-1}}\right| \leq \left(\frac{1}{4}-\frac{2^{\alpha}-1}{2^{\alpha+1}(\alpha+1)}\right)(\rho-1).$$

Proposition 2 *In Theorem 3.1, the following inequality holds:*

$$\left|\frac{\alpha}{2}B(\rho,\alpha)+\frac{\alpha}{2(\alpha+\rho-1)}-\frac{3^{\rho-1}+1}{2\cdot 4^{\rho-1}}\right| \leq \left[\frac{3^{\alpha+1}+1}{4^{\alpha+1}(\alpha+1)}+\frac{1}{8}-\frac{2^{\alpha}+1}{2^{\alpha+1}(\alpha+1)}\right](\rho-1).$$

Proposition 3 *In Theorem 4.1, the following inequality holds:*

$$\left|\frac{\alpha}{2}B(\rho,\alpha)+\frac{\alpha}{2(\alpha+\rho-1)}-\left(\frac{3^{\alpha}-1}{2^{\rho-1}4^{\alpha}}+\frac{4^{\alpha}-3^{\alpha}+1}{2\cdot 4^{\alpha}}\right)\right| \leq \frac{\rho-1}{\alpha+1}\left(\frac{2^{\alpha}+1}{2^{\alpha+1}}-\frac{3^{\alpha+1}+1}{4^{\alpha+1}}\right).$$

Proposition 4 *In Theorem 5.1, the following inequality holds:*

$$\left|\frac{\alpha}{2}B(\rho,\alpha)+\frac{\alpha}{2(\alpha+\rho-1)}-\left(\frac{5^{\alpha}-1}{2^{\rho-1}6^{\alpha}}+\frac{6^{\alpha}-5^{\alpha}+1}{2\cdot 6^{\alpha}}\right)\right| \leq \frac{\rho-1}{\alpha+1}\left(\frac{2^{\alpha}+1}{2^{\alpha+1}}-\frac{5^{\alpha+1}+1}{6^{\alpha+1}}\right)+\left(\frac{5^{\alpha}-1}{12\cdot 6^{\alpha}}\right).$$

References

[1] Alomari M, Darus M. On the Hadamard's inequality for log-convex functions on the coordinates. J Ineq Appl 2009; Art ID 283147: 13 pages.

[2] Dragomir SS. Two mappings in connection to Hadamard's inequalities, J Math Anal Appl 1992; 167: 49-56.

[3] Dragomir SS. On the Hadamard's inequality for convex on the co-ordinates in a rectangle from the plane. Taiwanese J Math 2001; 5: 775-788.

[4] Dragomir SS, Agarwal RP. Two inequalities for differentiable mappings and applications to special means of real numbers and to trapezoidal formula. Appl Math Lett 1998; 11: 91-95.

[5] Dragomir SS. Cho YJ, Kim SS. Inequalities of Hadamard's type for lipschitzian mappings and their applications. J Math Anal Appl 2000; 245: 489-501.

[6] Fejér L. Über die fourierreihen II. Math Naturwiss Anz Ungar Akad Wiss 1906; 24: 369-390 (in Hungarian).

[7] Hadamard J. Étude sur les propriétés des fonctions entières en particulier d'une fonction considérée par Riemann. J Math Pures Appl 1893; 58: 171-215 (in French).

[8] Hwang SR, Hsu KC, Tseng KL. Hadamard-type inequalities for lipschitzian functions in one and two variables with their applications. J Math Anal Appl 2013; 405: 546-554.

[9] Kirmaci US. Inequalities for differentiable mappings and applications to special means of real numbers and to midpoint formula. Appl Math Comp 2004; 147: 137-146.

[10] Kirmaci US, Özdemir ME. On some inequalities for differentiable mappings and applications to special means of real numbers and to midpoint formula. Appl Math Comp 2004; 153: 361-368.

[11] Sarikaya MZ, Set E, Yaldiz H, Başak N. Hermite-Hadamard's inequalities for fractional integrals and related fractional inequalities. Math Comp Model 2013; 57: 2403-2407.

[12] Tseng KL, Hwang SR, Dragomir SS. Fejér-type inequalities (I). J Ineq Appl 2010; Art ID 531976: 7 pages.

[13] Tseng KL, Yang GS, Hsu KC. On some inequalities of Hadamard's type and applications. Taiwanese J Math 2009; 13: 1929-1948.

[14] Yang GS, Tseng KL. On certain integral inequalities related to Hermite-Hadamard inequalities. J Math Anal Appl 1999; 239: 180-187.

[15] Yang GS, Tseng KL. Inequalities of Hadamard's type for lipschitzian mappings. J Math Anal Appl 2001; 260: 230-238.

[16] Zhu C, Fečkan M, Wang JR. Fractional integral inequalities for differentiable convex mappings and applications to special means and a midpoint formula. JAMSI 2012; 8: 21-28.