Turkish Journal of Mathematics

http://journals.tubitak.gov.tr/math/

Research Article

Turk J Math
(2016) 40: 605 – 619

doi:10.3906/mat-1502-23

The inclusion theorems for variable exponent Lorentz spaces

Öznur KULAK*
Department of Banking and Finance, Görele Applied Sciences Academy, Giresun University, Görele, Giresun, Turkey

Received: 06.02.2015 • **Accepted/Published Online:** 20.09.2015 • **Final Version:** 08.04.2016

Abstract: Let (X,Σ,μ) and (X,Σ,ν) be measure spaces. Assume that $L^{p_1(.),q_1(.)}(X,\mu)$ and $L^{p_2(.),q_2(.)}(X,\nu)$ are two variable exponent Lorentz spaces where $p,q \in P_0([0,l])$. In this paper we investigated the existence of the inclusion $L^{p_1(.),q_1(.)}(X,\mu) \subset L^{p_2(.),q_2(.)}(X,\nu)$ under what conditions for two measures μ and ν on (X,Σ).

Key words: Inclusion, variable exponent Lorentz space

1. Introduction

Let (X,Σ,μ) be a measure space. The distribution function of f is defined by

$$\lambda_f(y) = \mu(\{x \in X : |f(x)| > y\}) = \int\limits_{\{x\in X:|f(x)|>y\}} d\mu(x) \quad [4,6].$$

The rearrangement function of f is defined by

$$f^*(t) = \inf\{y > 0 : \lambda_f(y) \le t\} = \sup\{y > 0 : \lambda_f(y) > t\}, t \ge 0 \;\; [4,6].$$

Moreover, the average function of f^* is given by

$$f^{**}(t) = \frac{1}{t}\int\limits_0^t f^*(s)\,ds.$$

Let $0 < l \le \infty$. We put

$$p_- = \inf_{x\in[0,l]} p(x), \qquad p^+ = \sup_{x\in[0,l]} p(x).$$

In this paper, we shall also use the notation

$$P_a = \left\{p : a < p_- \le p^+ < \infty\right\}, \; a \in \mathbb{R}.$$

*Correspondence: oznur.kulak@giresun.edu.tr
2010 *AMS Mathematics Subject Classification:* 43A15.

The set $IP([0,\infty])$ is the family of $p \in L^{\infty}([0,\infty])$ such that there exist the limits $p(0) = \lim_{x\to 0} p(x)$, $p(\infty) = \lim_{x\to\infty} p(x)$ and we have

$$|p(x) - p(0)| \le \frac{C}{\ln \frac{1}{|x|}}\ , \qquad |x| \le \frac{1}{2} \quad (C > 0)$$

and

$$|p(x) - p(\infty)| \le \frac{C}{\ln(e + |x|)}\ , \qquad |x| > 2 \ \ (C > 0)\,. \tag{1.1}$$

We also denote $IP_a([0,l]) = IP([0,l]) \cap P_a([0,l])$. If $l = \infty$, it is enough that the inequality (1.1) is satisfied [4].

Let $\Omega \subset X$. We denote $l = \mu(\Omega)$. Assume that $p, q \in P_0([0,l])$. The variable exponent Lorentz spaces $L^{p(.),q(.)}(\Omega,\mu)$ are defined as the set of all (equivalence classes) measurable functions f on X such that $J_{p,q}(f) < \infty$, where

$$J_{p,q}(f) = \int_0^l t^{\frac{q(t)}{p(t)}-1} \left(f^*(t)\right)^{q(t)} dt. \tag{1.2}$$

We use the notation

$$\|f\|^1_{L^{p(.),q(.)}(\Omega,\mu)} = \inf\left\{\lambda > 0 : J_{p,q}(\frac{f}{\lambda}) \le 1\right\} \ \ [2]\,.$$

Let $p \in IP_0([0,l])$ and $q \in IP_1([0,l])$. If $l = \infty$, then the equality (1.2) is equivalent to the following sum:

$$\int_0^1 t^{\frac{q(0)}{p(0)}-1} \left(f^*(t)\right)^{q(t)} dt + \int_1^\infty t^{\frac{q(\infty)}{p(\infty)}-1} \left(f^*(t)\right)^{q(t)} dt \ \ [2]\,.$$

If $l < \infty$, then the equality (1.2) is equivalent to the integral $\int_0^l t^{\frac{q(0)}{p(0)}-1} \left(f^*(t)\right)^{q(t)} dt$ [4]. The space $L^{p(.),q(.)}(\Omega,\mu)$ is a normed vector space with norm

$$\|f\|_{L^{p(.),q(.)}(\Omega,\mu)} = \inf\left\{\lambda > 0 : J_{p,q}(\frac{f}{\lambda}) \le 1\right\}$$

such that $J_{p,q}(f) = \int_0^l t^{\frac{q(t)}{p(t)}-1} \left(f^{**}(t)\right)^{q(t)} dt$ [4].

For $0 \le p \le q \le \infty$, the inclusion $l^p(X) \subset l^q(X)$ is known. In [13], the inclusion $L^p(\mu) \subset L^q(\mu)$ was characterized by all positive measures whenever $0 < p \le q \le \infty$. Then Romero [10] improved some results of [13]. Lastly, the more general inclusion $L^p(\mu) \subset L^q(\nu)$ was considered by [8], where μ and ν are two measures on (X,Σ). Moreover, in [5], Gürkanlı considered inclusion theorems of Lorentz spaces. Embeddings for discrete weighted Lebesgue spaces with variable exponents were studied by Nekvinda [9]. In [1], the inclusion $L^{p(.)}(\mu) \subset L^{q(.)}(\nu)$ was considered by Aydın and Gürkanlı. In [2], Bandaliev considered embeddings between variable exponent Lebesgue spaces with different measures. Also in this area, Diening et al. studied Lebesgue and Sobolev spaces with variable exponent [3] . Now, in the present paper, we investigate the existence of the inclusion $L^{p_1(.),q_1(.)}(X,\mu) \subset L^{p_2(.),q_2(.)}(X,\nu)$ under what conditions.

2. Main results

Let (X,Σ,μ) be a measure space. If two measures μ and ν are absolutely continuous with respect to each other ($\mu << \nu$ and $\nu << \mu$) then we denote this by $\mu \approx \nu$ [11] .

Lemma 1 *The inclusion $L^{p_1(.),q_1(.)}(X,\mu) \subset L^{p_2(.),q_2(.)}(X,\nu)$ holds in the sense of equivalence classes if and only if $\mu \approx \nu$ and $L^{p_1(.),q_1(.)}(X,\mu) \subset L^{p_2(.),q_2(.)}(X,\nu)$ in the sense of individual functions.*

Proof Assume that $L^{p_1(.),q_1(.)}(X,\mu) \subset L^{p_2(.),q_2(.)}(X,\nu)$ holds in the sense of equivalence classes. Let $f \in L^{p_1(.),q_1(.)}(X,\mu)$ be any individual function. That means $f \in L^{p_1(.),q_1(.)}(X,\mu)$ in the sense of equivalence classes. Therefore, we have $f \in L^{p_2(.),q_2(.)}(X,\nu)$ in the sense of equivalence classes from the assumption. Thus we obtain $f \in L^{p_2(.),q_2(.)}(X,\nu)$ in the sense of individual functions. Therefore, we find the inclusion $L^{p_1(.),q_1(.)}(X,\mu) \subset L^{p_2(.),q_2(.)}(X,\nu)$ in the sense of individual functions. Let $E \in S$ with $\mu(E) = 0$. Then since $\chi_E = 0$ μ-almost everywhere (a.e), we have

$$J_{p_1,q_1}(\chi_E) = \int_0^l t^{\frac{q_1(t)}{p_1(t)}-1} \left(\chi_E^*(t)\right)^{q_1(t)} dt = \int_0^l t^{\frac{q_1(t)}{p_1(t)}-1} \left(\chi_{[0,\mu(E))}(t)\right)^{q_1(t)} dt$$

$$= \int_0^{\mu(E)} t^{\frac{q_1(t)}{p_1(t)}-1} dt = 0$$

and we write $\chi_E \in L^{p_1(.),q_1(.)}(X,\mu)$. Therefore, χ_E is in the equivalence classes of $0 \in L^{p_1(.),q_1(.)}(X,\mu)$. Moreover, the equivalence classes of 0 (with respect to μ) are also an element of $L^{p_2(.),q_2(.)}(X,\nu)$. Thus χ_E is in the equivalence classes of $0 \in L^{p_2(.),q_2(.)}(X,\nu)$ with respect to ν. This implies $\nu(E) = 0$. Therefore, $\nu << \mu$. Similarly, $\mu << \nu$ is proved. The proof of the other side is clear.

Throughout, we assume that $p, q \in P_0([0,l])$ unless the contrary is stated. □

Lemma 2 ***a)*** *Let $\mu(X) = \infty$, $p, q \in IP_1([0,\infty])$, $q(\infty) > p(\infty)$ and $q(0) < p(0)$. If $(f_n)_{n\in\mathbb{N}}$ convergences to f in $L^{p(.),q(.)}(X,\mu)$ then $(f_n)_{n\in\mathbb{N}}$ convergences to f in measure.*

b) *Let $\mu(X) < \infty$ and $p, q \in IP_1([0,\infty])$. If $(f_n)_{n\in\mathbb{N}}$ convergences to f in $L^{p(.),q(.)}(X,\mu)$ then $(f_n)_{n\in\mathbb{N}}$ convergences to f in measure.*

Proof **a)** Assume that $(f_n)_{n\in\mathbb{N}}$ convergences to f in $L^{p(.),q(.)}(X,\mu)$. Then we write

$$J_{p,q}(f_n - f) \cong \int_0^1 t^{\frac{q(0)}{p(0)}-1} (f_n - f)^*(t)^{q(t)} dt + \int_1^\infty t^{\frac{q(\infty)}{p(\infty)}-1} (f_n - f)^*(t)^{q(t)} dt \to 0$$

for $n \to \infty$. Since $q(\infty) > p(\infty)$ and $q(0) < p(0)$, we have

$$\int_0^\infty (f_n - f)^* (t)^{q(t)} \, dt \le J_{p,q} (f_n - f) \to 0.$$

for $n \to \infty$. Then $(f_n - f)^*$ convergences to 0 in $L^{q(.)}([0,\infty])$. Thus we find that $(f_n - f)^*$ convergences to 0 in measure (with respect to measure on $[0,\infty]$) by [7]. Furthermore, since

$$\lambda_{(f_n - f)^*} (\varepsilon) = \mu \left(\left\{ t : (f_n - f)^* (t) > \varepsilon \right\} \right) = \mu \left(\{ x : (f_n - f)(x) > \varepsilon \} \right) = \lambda_{(f_n - f)} (\epsilon) \quad [6]$$

for all $\varepsilon > 0$, f_n converges to f in measure.

b) Assume that $(f_n)_{n\in\mathbb{N}}$ convergences to f in $L^{p(.),q(.)}(X,\mu)$. Then since $l = \mu(X) < \infty$,

$$J_{p,q} (f_n - f) \cong \int_0^l t^{\frac{q(0)}{p(0)} - 1} (f_n - f)^* (t)^{q(t)} \, dt \to 0 \tag{2.1}$$

holds for $n \to \infty$. In addition, $L^{p(.),q(.)}(X,\mu)$ is a Banach function space [4] and we have

$$\int_X (f_n - f)(x) \, dx \le C_X \| f_n - f \|_{L^{p(.),q(.)}(X,\mu)} . \tag{2.2}$$

Therefore by using (2.1) and (2.2), we obtain $(f_n)_{n\in\mathbb{N}}$ convergences to f in $L^1(X)$. Thus f_n converges to f in measure. $\square$

Theorem 1 ***a)*** *Let $p_i, q_i \in IP_1([0,\infty])$, $(i = 1,2)$, $\mu(X) = \infty$, $q_i(\infty) > p_i(\infty)$, and $q_i(0) < p_i(0)$, $(i = 1,2)$. Then the inclusion*

$$L^{p_1(.),q_1(.)}(X,\mu) \subset L^{p_2(.),q_2(.)}(X,\nu)$$

holds in the sense of equivalence classes if and only if $\mu \approx \nu$ and there exists $C > 0$ such that

$$\|f\|^1_{L^{p_2(.),q_2(.)}(X,\nu)} \le C \|f\|^1_{L^{p_1(.),q_1(.)}(X,\mu)}$$

for all $f \in L^{p_1(.),q_1(.)}(X,\mu)$.

b) *Let $p_i, q_i \in IP_1([0,l])$, $(i = 1,2)$ and $l = \mu(X) < \infty$. Then the inclusion*

$$L^{p_1(.),q_1(.)}(X,\mu) \subset L^{p_2(.),q_2(.)}(X,\nu)$$

holds in the sense of equivalence classes if and only if $\mu \approx \nu$ and there exists $C > 0$ such that

$$\|f\|^1_{L^{p_2(.),q_2(.)}(X,\nu)} \le C \|f\|^1_{L^{p_1(.),q_1(.)}(X,\mu)}$$

for all $f \in L^{p_1(.),q_1(.)}(X,\mu)$.

Proof a) Suppose that $L^{p_1(.),q_1(.)}(X,\mu) \subset L^{p_2(.),q_2(.)}(X,\nu)$ holds in the sense of equivalence classes. We define the unit operator $I(f) = f$ from $L^{p_1(.),q_1(.)}(X,\mu)$ into $L^{p_2(.),q_2(.)}(X,\nu)$. Now we show that I is closed. Let $(f_n)_{n\in\mathbb{N}}$ be a sequence such that $f_n \to f$ in $L^{p_1(.),q_1(.)}(X,\mu)$ and $I(f_n) = f_n \to g$ in $L^{p_2(.),q_2(.)}(X,\nu)$. Thus, by Lemma 2, $(f_n)_{n\in\mathbb{N}}$ convergences to f in measure (with respect to μ). Hence there exists subsequence $(f_{n_i})_{n_i\in\mathbb{N}} \subset (f_n)_{n\in\mathbb{N}}$ such that $(f_{n_i})_{n_i\in\mathbb{N}}$ pointwise converges to f, μ-almost everywhere (a.e.). Moreover, since $(f_n)_{n\in\mathbb{N}}$ convergences to g in $L^{p_2(.),q_2(.)}(X,\nu)$, it is easy to show that $(f_{n_i})_{n_i\in\mathbb{N}}$ convergences to g in $L^{p_2(.),q_2(.)}(X,\nu)$. Then $(f_{n_i})_{n_i\in\mathbb{N}}$ convergences to g in measure (with respect to ν). Thus we find a subsequence $\left(f_{n_{i_k}}\right)_{n_{i_k}\in\mathbb{N}} \subset (f_{n_i})_{n_i\in\mathbb{N}}$ such that $\left(f_{n_{i_k}}\right)_{n_{i_k}\in\mathbb{N}}$ convergences to g pointwise ν-a.e. Let M be a set of the points such that $\left(f_{n_{i_k}}\right)_{n_{i_k}\in\mathbb{N}}$ does not convergence to g pointwise. Hence $\nu(M) = 0$. From the assumption $L^{p_1(.),q_1(.)}(X,\mu) \subset L^{p_2(.),q_2(.)}(X,\nu)$ in the sense of equivalence classes and so we write $\mu \approx \nu$ by Lemma 1. Thus $\nu(M) = \mu(M) = 0$. Hence $\left(f_{n_{i_k}}\right)_{n_{i_k}\in\mathbb{N}}$ convergences to g pointwise μ-a.e. Consequently using the following inequality

$$|f(x) - g(x)| \leq \left|f(x) - f_{n_{i_k}}(x)\right| + \left|f_{n_{i_k}}(x) - g(x)\right|,$$

we have $f = g$ μ-a.e. and $f = g$ ν-a.e. That means I is closed. By the closed graph theorem, there exists $C > 0$ such that

$$\|f\|^1_{L^{p_2(.),q_2(.)}(X,\nu)} \leq C\,\|f\|^1_{L^{p_1(.),q_1(.)}(X,\mu)}.$$

The proof of the other direction is easy.

In this Theorem, (b) can be proved easily by using the technique of the proof in (a). □

Lemma 3 ***a)*** *If $\nu(E) \leq \mu(E)$ for all $E \in \Sigma$, then the inequality*

$$\|f\|^1_{L^{p(.),q(.)}(X,\nu)} \leq \|f\|^1_{L^{p(.),q(.)}(X,\mu)}$$

holds for all $f \in L^{p(.),q(.)}(X,\mu)$.

b) *Let $p \in IP_0([0,l])$, $1 \leq q < \infty$. If there exists $M > 0$ such that $\nu(E) \leq M\mu(E)$ for all $E \in \Sigma$, then the inequality*

$$\|f\|^1_{L^{p(.),q}(X,\nu)} \leq M\,\|f\|^1_{L^{p(.),q}(X,\mu)}$$

holds for all $f \in L^{p(.),q}(X,\mu)$.

Proof a) Let $\nu(E) \leq \mu(E)$ for all $E \in \Sigma$. From [5], we have $f^{*,\nu}(t) \leq f^{*,\mu}(t)$ ($f^{*,\nu}$ and $f^{*,\mu}$ are the rearrangements of f with respect to the measures ν and μ respectively) for all $t \geq 0$. This implies

$$\int_0^l t^{\frac{q(t)}{p(t)}-1}\left(f^{*,\nu}\right)(t)^{q(t)}\,dt \leq \int_0^l t^{\frac{q(t)}{p(t)}-1}\left(f^{*,\mu}\right)(t)^{q(t)}\,dt.$$

where $l = \mu(X)$. Thus we have

$$\|f\|^1_{L^{p(.),q(.)}(X,\nu)} \leq \|f\|^1_{L^{p(.),q(.)}(X,\mu)}.$$

b) Let $l=\mu(X)=\infty$. Assume that there exists $M>0$ such that $\nu(E)\leq M\mu(E)$ for all $E\in\Sigma$. If we take $k=M\mu$, then k is a measure. Then it is known that $f^{*,k}(t)=f^{*,\mu}\left(\frac{t}{M}\right)\geq f^{*,\nu}(t)$ by [5]. Therefore, if we set $\frac{t}{M}=u$, then

$$J_{p,q}^{\nu}(f)=\int_0^\infty t^{\frac{q}{p(t)}-1}f^{*,\nu}(t)^q\,dt\leq J_{p,q}^{k}(f)=\int_0^\infty t^{\frac{q}{p(t)}-1}f^{*,k}(t)^q\,dt$$

$$=\int_0^\infty t^{\frac{q}{p(t)}-1}f^{*,\mu}\left(\frac{t}{M}\right)^q dt$$

$$\cong\int_0^1 t^{\frac{q}{p(0)}-1}f^{*,\mu}\left(\frac{t}{M}\right)^q dt+\int_1^\infty t^{\frac{q}{p(\infty)}-1}f^{*,\mu}\left(\frac{t}{M}\right)^q dt$$

$$=M^{\frac{q}{p(0)}}\int_0^1 u^{\frac{q}{p(0)}-1}f^{*,\mu}(u)^q\,du+M^{\frac{q}{p(\infty)}}\int_1^\infty u^{\frac{q}{p(\infty)}-1}f^{*,\mu}(u)^q\,du$$

$$\lesssim M_0\int_0^\infty u^{\frac{q}{p(t)}-1}f^{*,\mu}(u)^q\,du=M_0J_{p,q}^{\mu}(f)$$

where $M_0=$maks$\left\{M^{\frac{q}{p(0)}},M^{\frac{q}{p(\infty)}}\right\}$. Thus we have $\|f\|^1_{L^{p(.),q}(X,\nu)}\leq M\|f\|^1_{L^{p(.),q}(X,\mu)}$.

Similarly the Lemma is proved for $l=\mu(X)<\infty$. □

Lemma 4 *Let $p,q\in IP_1([0,l])$, and $l=\mu(X)<\infty$.*

a) *If $\mu\approx\nu$ and there exists $M>0$ such that $\nu(E)\leq M\mu(E)$ for all $E\in\Sigma$ then the inclusion $L^1(X,\mu)\subset L^1(X,\nu)$ holds.*

b) *If the inclusion $L^1(X,\mu)\subset L^1(X,\nu)$ holds then the inclusion $L^{p(.),q(.)}(X,\mu)\subset L^{p(.),q(.)}(X,\nu)$ holds.*

Proof **a)** It is known by [5].

b) Take any $f\in L^{p(.),q(.)}(X,\mu)$ is given. Since $\chi_{[0,\infty]}t^{\frac{q(t)}{p(t)}-1}f^*(t)^{q(t)}\in L^1(\mu)$ and $L^1(\mu)\subset L^1(\nu)$. Thus we obtain $\chi_{[0,\infty]}t^{\frac{q(t)}{p(t)}-1}f^*(t)^{q(t)}\in L^1(\nu)$. That means $f\in L^{p(.),q(.)}(X,\nu)$. □

Theorem 2 *Let $p,q\in IP_1([0,l])$ and $l=\mu(X)<\infty$. Then the inclusion $L^{p(.),q(.)}(X,\mu)\subset L^{p(.),q(.)}(X,\nu)$ holds if and only if $\mu\approx\nu$ and there exists $M>0$ such that $\nu(E)\leq M\mu(E)$ for all $E\in\Sigma$.*

Proof $\Longrightarrow$ By Theorem 1, there exists $M>0$ such that

$$\|f\|^1_{L^{p(.),q(.)}(X,\nu)}\leq M\|f\|^1_{L^{p(.),q(.)}(X,\mu)}\tag{2.3}$$

for all $f \in L^{p(.),q(.)}(X,\mu)$. Moreover,

$$J_{p,q}^{\nu}(f) \cong \int_0^l t^{\frac{q(0)}{p(0)}-1}\left((\chi_E)^{*,\nu}(t)\right)^{q(t)} dt = \int_0^{\nu(E)} t^{\frac{q(0)}{p(0)}-1} dt = \frac{p(0)}{q(0)}\nu(E)^{\frac{q(0)}{p(0)}}$$

holds. Similarly, we have

$$J_{p,q}^{\mu}(f) \cong \frac{p(0)}{q(0)}\mu(E)^{\frac{q(0)}{p(0)}}.$$

Therefore, we write

$$\|\chi_E\|_{L^{p(.),q(.)}(X,\nu)}^1 \cong \frac{p(0)}{q(0)}\nu(E)^{\frac{q(0)}{p(0)}} \tag{2.4}$$

and

$$\|\chi_E\|_{L^{p(.),q(.)}(X,\mu)}^1 \cong \frac{p(0)}{q(0)}\mu(E)^{\frac{q(0)}{p(0)}}. \tag{2.5}$$

Thus, from (2.3), (2.4), and (2.5), we have

$$\nu(E) \le M\mu(E).$$

$\Longleftarrow$From Lemma 4, the proof is clear. □

Theorem 3 *Let $p_i, q_i \in IP_1([0,l])$, $(i=1,2)$, $l=\mu(X)<\infty$, and $q_1(0)p_2(0) > q_2(0)p_1(0)$. If $L^{p_1(.),q_1(.)}(X,\mu) \subset L^{p_2(.),q_2(.)}(X,\mu)$ then there exists a constant $m \ge 0$ such that $\mu(E) \ge m$ for every μ-nonnull set $E \in \Sigma$.*

Proof By Theorem 1, there exists $C>0$ such that

$$\|f\|_{L^{p_2(.),q_2(.)}(X,\mu)}^1 \le C\|f\|_{L^{p_1(.),q_1(.)}(X,\mu)}^1$$

for all $f \in L^{p_1(.),q_1(.)}(X,\mu)$. Let $E \in \Sigma$ be a μ-nonnull set. Since $\mu(E)<\infty$, we have

$$J_{p_1,q_1}(\chi_E) \cong \int_0^l t^{\frac{q_1(0)}{p_1(0)}-1}\left(\chi_E^*(t)\right)^{q_1(t)} dt = \int_0^{\mu(E)} t^{\frac{q_1(0)}{p_1(0)}-1} dt = \frac{p_1(0)}{q_1(0)}\mu(E)^{\frac{q_1(0)}{p_1(0)}}.$$

and so $\|\chi_E\|_{L^{p_1(.),q_1(.)}(X,\nu)}^1 \cong \frac{p_1(0)}{q_1(0)}\mu(E)^{\frac{q_1(0)}{p_1(0)}}$ holds. Similarly, we have $\|\chi_E\|_{L^{p_2(.),q_2(.)}(X,\nu)}^1 \cong \frac{p_2(0)}{q_2(0)}\mu(E)^{\frac{q_2(0)}{p_2(0)}}$. Then we write

$$\frac{p_2(0)}{q_2(0)}\mu(E)^{\frac{q_2(0)}{p_2(0)}} \le C\frac{p_1(0)}{q_1(0)}\mu(E)^{\frac{q_1(0)}{p_1(0)}}$$

$$\frac{1}{C}\frac{q_1(0)p_2(0)}{p_1(0)q_2(0)} \le \mu(E)^{\frac{q_1(0)}{p_1(0)}-\frac{q_2(0)}{p_2(0)}}.$$

If we set $m = \left(\frac{1}{C}\frac{q_1(0)p_2(0)}{p_1(0)q_2(0)}\right)^{\frac{p_1(0)p_2(0)}{q_1(0)p_2(0)-q_2(0)p_1(0)}}$, we obtain $\mu(E) \ge m$ for every μ-nonnull set $E \in \Sigma$. □

Theorem 4 *Let* $p,q \in IP_1([0,l])$ *and* $l=\mu(X)<\infty$.

a) *If* $q_2(.) \leq q_1(.)$, $q_1(0) \leq p_1(0)$, *and* $q_2(0) \geq p_2(0)$ *then the inclusion*

$$L^{p_1(.),q_1(.)}(X,\mu) \subset L^{p_2(.),q_2(.)}(X,\mu)$$

holds.

b) *If* $q(.) \leq p(.)$ *and* $q(0)=p(0)$, *then the inclusion*

$$L^{p(.),p(.)}(X,\mu) \subset L^{p(.),q(.)}(X,\mu)$$

holds.

c) *If* $p(.) \leq q(.)$ *and* $q(0)=p(0)$, *then the inclusion*

$$L^{p(.),q(.)}(X,\mu) \subset L^{p(.),p(.)}(X,\mu)$$

holds.

d) *If* $q(0) \geq p(0)$ *then the inclusion*

$$L^{q(.),q(.)}(X,\mu) \subset L^{p(.),q(.)}(X,\mu)$$

holds.

e) *If* $q(0) \leq p(0)$, *then the inclusion*

$$L^{p(.),q(.)}(X,\mu) \subset L^{q(.),q(.)}(X,\mu)$$

holds.

Proof **a)** Take any $f \in L^{p_1(.),q_1(.)}(X,\mu)$. Then we have

$$\infty > J_{p_1,q_1}(f) \cong \int_0^l t^{\frac{q_1(0)}{p_1(0)}-1}(f^*(t))^{q_1(t)}\,dt \geq l^{\frac{q_1(0)}{p_1(0)}-1}\int_0^l (f^*(t))^{q_1(t)}\,dt.$$

Therefore, we obtain $f^* \in L^{q_1(.)}([0,l])$. Moreover, since $q_2(.) \leq q_1(.)$, we write $L^{q_1(.)}([0,l]) \subset L^{q_2(.)}([0,l])$ from [7]. That means $f^* \in L^{q_2(.)}([0,l])$. From this result, we have

$$J_{p_2,q_2}(f) \cong \int_0^l t^{\frac{q_2(0)}{p_2(0)}-1}(f^*(t))^{q_2(t)}\,dt \leq l^{\frac{q_2(0)}{p_2(0)}-1}\int_0^l (f^*(t))^{q_2(t)}\,dt < \infty.$$

Thus we find that $f \in L^{p_2(.),q_2(.)}(X,\mu)$.

b) Take any $f \in L^{p(.),p(.)}(X,\mu)$. That means $f^* \in L^{p(.)}([0,l])$. Again since $q(.) \leq p(.)$, we know that $L^{p(.)}([0,l]) \subset L^{q(.)}([0,l])$ from [7]. Therefore, we have

$$J_{p,q}(f) \cong \int_0^l t^{\frac{q(0)}{p(0)}-1}(f^*(t))^{q(t)}\,dt = \int_0^l (f^*(t))^{q(t)}\,dt < \infty.$$

Thus we obtain $f \in L^{p(.),q(.)}(X,\mu)$.

c) This hypothesis is proved easily using the technique in (b).

d) Take any $f \in L^{q(.),q(.)}(X,\mu)$. Assume that $l = \mu(X) \geq 1$. Then since $J_{q,q}(f) \cong \int_0^l (f^*(t))^{q(t)}\,dt < \infty$, and since $q(0) \geq p(0)$, we have

$$J_{p,q}(f) \cong \int_0^1 t^{\frac{q(0)}{p(0)}-1}(f^*(t))^{q(t)}\,dt + \int_1^l t^{\frac{q(0)}{p(0)}-1}(f^*(t))^{q(t)}\,dt$$

$$\leq \int_0^1 (f^*(t))^{q(t)}\,dt + l^{\frac{q(0)}{p(0)}-1}\int_1^l (f^*(t))^{q(t)}\,dt < \infty$$

Therefore, $f \in L^{p(.),q(.)}(X,\mu)$. Now let $l < 1$ and so we have

$$J_{p,q}(f) \cong \int_0^l t^{\frac{q(0)}{p(0)}-1}(f^*(t))^{q(t)}\,dt < \int_0^l (f^*(t))^{q(t)}\,dt < \infty.$$

Thus similarly $f \in L^{p(.),q(.)}(X,\mu)$.

e) Take any $f \in L^{p(.),q(.)}(X,\mu)$. Assume that $l = \mu(X) \geq 1$. Then since $J_{p,q}(f) \cong \int_0^l t^{\frac{q(0)}{p(0)}-1}(f^*(t))^{q(t)}\,dt < \infty$, we have

$$\int_0^1 t^{\frac{q(0)}{p(0)}-1}(f^*(t))^{q(t)}\,dt \leq J_{p,q}(f) < \infty$$

and

$$\int_1^l t^{\frac{q(0)}{p(0)}-1}(f^*(t))^{q(t)}\,dt \leq J_{p,q}(f) < \infty.$$

In addition, since $q(0) \leq p(0)$, we have

$$J_{q,q}(f) \cong \int_0^l (f^*(t))^{q(t)}\,dt \leq \int_0^1 t^{\frac{q(0)}{p(0)}-1}(f^*(t))^{q(t)}\,dt + \int_1^l t^{\frac{q(0)}{p(0)}-1}t^{-\left(\frac{q(0)}{p(0)}-1\right)}(f^*(t))^{q(t)}\,dt$$

$$\leq \int_0^1 t^{\frac{q(0)}{p(0)}-1}(f^*(t))^{q(t)}\,dt + l^{-\left(\frac{q(0)}{p(0)}-1\right)}\int_1^l t^{\frac{q(0)}{p(0)}-1}(f^*(t))^{q(t)}\,dt < \infty$$

Therefore, $f \in L^{q(.),q(.)}(X,\mu)$. Now let $l < 1$ and we have

$$J_{q,q}(f) \cong \int_0^l (f^*(t))^{q(t)}\,dt < \int_0^l t^{\frac{q(0)}{p(0)}-1}(f^*(t))^{q(t)}\,dt < \infty.$$

Thus similarly $f \in L^{q(.),q(.)}(X,\mu)$. □

Theorem 5 *If* $\mu(X) < \infty$, $1 \le q_2(.) \le q_1(.)$, $\frac{1}{p_1(.)} + \frac{1}{q_2(.)} = 1$ *and* $\frac{1}{p_2(.)} + \frac{1}{q_1(.)} = 1$ *then the inclusion*

$$L^{p_1(.),q_1(.)}(X,\mu) \subset L^{p_2(.),q_2(.)}(X,\mu)$$

holds.

Proof Take any $f \in L^{p_1(.),q_1(.)}(X,\mu)$. Then we have

$$t^{\left(\frac{1}{p_2(t)} - \frac{1}{q_2(t)}\right)} f^*(t) = t^{\left(\frac{1}{p_2(t)} - \frac{1}{q_2(t)}\right)} f^*(t)\, t t^{-1}$$

$$= t^{\left(\frac{1}{p_2(t)} - \frac{1}{q_2(t)}\right)} f^*(t)\, t^{\left(\frac{1}{p_1(.)} + \frac{1}{q_2(.)}\right)} t^{-\left(\frac{1}{p_2(.)} + \frac{1}{q_1(.)}\right)} = t^{\left(\frac{1}{p_1(t)} - \frac{1}{q_1(t)}\right)} f^*(t).$$

Therefore, write $t^{\left(\frac{1}{p_2(t)} - \frac{1}{q_2(t)}\right)} f^*(t) \in L^{q_1(.)}([0,l]) \subset L^{q_2(.)}([0,l])$. That means $f \in L^{p_2(.),q_2(.)}(X,\mu)$. □

Theorem 6 *Let* $p_i, q_i \in IP_1([0,l]), (i = 1,2), \mu(X) < \infty$, $q_2(.) \ge q_1(.), q_2(0) \ge p_2(0)$, *and* $q_1(0) \le p_1(0)$. *If there exists a constant* $m > 0$ *such that* $\mu(E) \ge m$ *for every* μ*-nonnull set* $E \in \Sigma$ *then the inclusion*

$$L^{p_1(.),q_1(.)}(X,\mu) \subset L^{p_2(.),q_2(.)}(X,\mu)$$

holds.

Proof Take any $f \in L^{p_1(.),q_1(.)}(X,\mu)$. Define that the set $E_n = \{x \in X : |f(x)| > n\}$ for every $n \in \mathbb{N}$. Since $q_1(0) \le p_1(0)$, we write $L^{p_1(.),q_1(.)}(X,\mu) \subset L^{q_1(.),q_1(.)}(X,\mu)$ from Theorem 4. Therefore, there exists $C > 0$ such that

$$\|f\|^1_{L^{q_1(.),q_1(.)}(X,\mu)} \le C\|f\|^1_{L^{p_1(.),q_1(.)}(X,\mu)} \tag{2.6}$$

for all $f \in L^{p_1(.),q_1(.)}(X,\mu)$. On the other hand, since $|f(x)| > n > 1$ for all $x \in E_n$, we have $|f^*(t)| > 1$ for all $t \in [0,\mu(E_n)]$. Thus if we set $\frac{t}{2} = u$, then we have

$$n^{q_{1-}}\mu(E_n) \le \int\limits_{E_n} |f|^{q_{1-}}\, d\mu = \int\limits_X |f\chi_{E_n}|^{q_{1-}}\, d\mu = \int\limits_0^\infty \left|(f\chi_{E_n})^*\right|(t)^{q_{1-}}\, dt$$

$$\le \int\limits_0^\infty |f^*|\left(\frac{t}{2}\right)^{q_{1-}} \chi_{[0,\mu(E_n)]}\left(\frac{t}{2}\right) dt = 2\int\limits_0^\infty |f^*|(u)^{q_{1-}}\, \chi_{[0,\mu(E_n)]}(u)\, du$$

$$= 2\int\limits_0^{\mu(E_n)} |f^*|(u)^{q_{1-}}\, du \le 2\int\limits_0^{\mu(E_n)} |f^*|(u)^{q_1(u)}\, du \le 2\int\limits_0^l |f^*|(u)^{q_1(u)}\, du$$

for every $n \in \mathbb{N}$. Then using the inequality (2.6), we obtain

$$n^{q_{1-}}\mu(E_n) \le 2C\int\limits_0^l t^{\frac{q_1(0)}{p_1(0)}-1} |f^*|(u)^{q_1(u)}\, du < \infty \tag{2.7}$$

for every $n \in \mathbb{N}$. By the hypothesis, either $\mu(E_n) = 0$ or $\mu(E_n) \geq m$. Since the sequence (E_n) is nonincreasing and $\bigcap_{n=1}^{\infty} E_n = \varnothing$, we have $\mu(E_n) \to 0$. Thus there exists $n_0 \in \mathbb{N}$ such that $|f(x)| \leq n_0$, $\mu - a.e.$ for all $x \in X$, and so we write $|f^*(t)| \leq n_0$, $\mu - a.e.$ for all $t \in [0, l]$. Therefore, we have

$$\int_0^l |f^*|(t)^{q_2(t)}\, dt = \int_0^l |f^*|(t)^{q_2(t)-q_1(t)} |f^*|(t)^{q_1(t)}\, dt \leq \int_0^l n_0^{q_2(t)-q_1(t)} |f^*|(t)^{q_1(t)}\, dt.$$

Therefore, we write

$$\int_0^l |f^*|(t)^{q_2(t)}\, dt < n_0^{q_2^+ - q_{1-}} \int_0^l |f^*|(t)^{q_1(t)}\, dt < \infty. \tag{2.8}$$

Thus we have $f \in L^{q_2(.),q_2(.)}(X,\mu)$. That means $L^{p_1(.),q_1(.)}(X,\mu) \subset L^{q_2(.),q_2(.)}(X,\mu)$. Similarly, using the inequalities (2.8), we write $L^{q_1(.),q_1(.)}(X,\mu) \subset L^{q_2(.),q_2(.)}(X,\mu)$. Lastly using $q_2(0) \geq p_2(0)$, we obtain

$$L^{p_1(.),q_1(.)}(X,\mu) \subset L^{q_1(.),q_1(.)}(X,\mu) \subset L^{q_2(.),q_2(.)}(X,\mu) \subset L^{p_2(.),q_2(.)}(X,\mu)$$

from Theorem 2.4. □

Lemma 5 ***Hölder inequality for variable exponent Lorentz spaces:***

Let $1 \leq q(.) \leq q^+ < \infty$, $\frac{1}{p(t)} + \frac{1}{p'(t)} = 1$, *and* $\frac{1}{q(t)} + \frac{1}{q'(t)} = 1$. *If* $f \in L^{p(.),q(.)}(X,\mu)$ *and* $g \in L^{p'(.),q'(.)}(X,\mu)$ *then* $fg \in L^1(X,\mu)$ *and there exists* $C > 0$ *such that*

$$\int_X |f(x)\, g(x)|\, d\mu(x) \leq C \|f\|_{L^{p(.),q(.)}(X,\mu)} \|g\|_{L^{p'(.),q'(.)}(X,\mu)}.$$

Proof Let $f \in L^{p(.),q(.)}(X,\mu)$ and $g \in L^{p'(.),q'(.)}(X,\mu)$. Then we set $\frac{t}{2} = u$

$$\int_X |f(x)\, g(x)|\, d\mu(x) = \int_0^\infty \left|(fg)^*(t)\right| dt \leq \int_0^\infty f^*\left(\frac{t}{2}\right) g^*\left(\frac{t}{2}\right) dt$$

$$= 2\int_0^\infty f^*(u)\, g^*(u)\, dt = 2\int_0^\infty t^{1-1} f^*(u)\, g^*(u)\, dt = 2\int_0^\infty t^{\left(\frac{1}{p(t)}+\frac{1}{p'(t)}\right)} t^{-\left(\frac{1}{q(t)}+\frac{1}{q'(t)}\right)} f^*(u)\, g^*(u)\, dt.$$

$$= 2\int_0^\infty t^{\left(\frac{1}{p(t)}-\frac{1}{q(t)}\right)} f^*(u)\, t^{\left(\frac{1}{p'(t)}-\frac{1}{q'(t)}\right)} g^*(u)\, dt$$

Furthermore, by using the Hölder inequality for variable exponent Lebesgue spaces in [7, 12], the inequalities $f^* \leq f^{**}$ and $g^* \leq g^{**}$, there exists $C_1 > 0$ such that

$$\int_X |f(x)\, g(x)|\, d\mu(x) \leq 2\int_0^\infty t^{\left(\frac{1}{p(t)}-\frac{1}{q(t)}\right)} f^*(u)\, t^{\left(\frac{1}{p'(t)}-\frac{1}{q'(t)}\right)} g^*(u)\, dt$$

$$\leq 2C_1 \left\| t^{\left(\frac{1}{p(t)}-\frac{1}{q(t)}\right)} f^* \right\|_{L^{q(.)}([0,\infty))} \left\| t^{\left(\frac{1}{p'(t)}-\frac{1}{q'(t)}\right)} g^* \right\|_{L^{q(.)}([0,\infty))} = C \left\| f \right\|_{L^{p(.),q(.)}(X,\mu)} \left\| h \right\|_{L^{p'(.),q'(.)}(X,\mu)}$$

where $C = 2C_1$. □

Theorem 7 ***a)*** *Let* $1 \leq q(.) \leq q^+ < \infty$. *If* $\frac{1}{p(t)} + \frac{1}{p'(t)} = 1$ *and* $\frac{1}{q(t)} + \frac{1}{q'(t)} = 1$ *then the inclusion*

$$L^{p(.),q(.)}(X,\mu) . L^{p'(.),q'(.)}(X,\mu) \subset L^1(X,\mu)$$

holds.

b) *Let* $p \in IP_0([0,l]), q \in IP_1([0,l])$, $q(t) = \{ \begin{smallmatrix} c, & x\in\Omega \\ d, & x\notin\Omega \end{smallmatrix}$, *and* $c,d \geq 1$ *such that* $\Omega \subset [0,\infty)$. *If* $q(0) > p(0)$ *and* $q(\infty) \geq p(\infty)$ *then*

$$L^1(X,\mu) \subset L^{p(.),q(.)}(X,\mu) . L^{p'(.),q'(.)}(X,\mu)$$

holds such that $X = [0,\infty]$ *and* $\mu(x) = dx$.

Proof **a)** Let $f \in L^{p(.),q(.)}(X,\mu)$ and $h \in L^{p'(.),q'(.)}(X,\mu)$. From Lemma 5, there exists $C > 0$ such that

$$\int_X |f(x) h(x)| \, d\mu(x) \leq C \|f\|_{L^{p(.),q(.)}(X,\mu)} \|h\|_{L^{p'(.),q'(.)}(X,\mu)}.$$

Thus we obtain $f.h \in L^1(X,\mu)$. That means $L^{p(.),q(.)}(X,\mu) . L^{p'(.),q'(.)}(X,\mu) \subset L^1(X,\mu)$.

b) Take any $g \in L^1(X,\mu)$. Define that

$$A_1 = \{x : 0 < |g(x)| < \infty\},$$

$$A_2 = \{x : |g(x)| = 0\},$$

and

$$(A_1 \cup A_2)^c = \{x : |g(x)| = \infty\}$$

such that $A_1 \cup A_2 \cup (A_1 \cup A_2)^c = [0,\infty]$. Now define that the functions

$$f(x) = \{ \begin{smallmatrix} |g(x)|, & x\in A_1 \\ 0 & x\in A_2 \\ \infty, & x \in (A_1 \cup A_2)^c \end{smallmatrix}$$

and

$$h(x) = \{ \begin{smallmatrix} \frac{|g(x)|}{|g(x)|^{q(x)}}, & x\in A_1 \\ 0 & x\in A_2 \\ \infty, & x \in (A_1 \cup A_2)^c \end{smallmatrix}.$$

We also have $|g| = |fh|$. Since $g \in L^1(X,\mu)$, we know that $|g(x)| < \infty$ $(a.e.)$. Thus we have $\mu((A_1 \cup A_2)^c)$=0. On the other hand, we have

$$J_{p,q}(f) \cong \int_0^1 t^{\frac{q(0)}{p(0)}-1} (f^*(t))^{q(t)} \, dt + \int_1^\infty t^{\frac{q(\infty)}{p(\infty)}-1} (f^*(t))^{q(t)} \, dt$$

$$\leq \int_0^1 (f^*(t))^{q(t)}\,dt + \int_1^\infty (f^*(t))^{q(t)}\,dt$$

$$= \int_\Omega \left(g^*(t)^{\frac{1}{c}}\right)^c dt + \int_{\mathbb{R}^+\backslash\Omega} \left(g^*(t)^{\frac{1}{d}}\right)^d dt$$

$$= \int_X (g(x))^{q(x)}\,dx < \infty$$

and similarly $J_{p',q'}(h) < \infty$. Thus we obtain $f \in L^{p(.),q(.)}(X,\mu)$ and $h \in L^{p'(.),q'(.)}(X,\mu)$. Therefore, from the inequality $|g| = |fh|$, we find that $g \in L^{p(.),q(.)}(X,\mu).L^{p'(.),q'(.)}(X,\mu)$. Thus the proof is completed. □

Theorem 8 *Let $p_i \in IP_0([0,l]), q_i \in IP_1([0,l]), (i=1,2), l = \mu(X) < \infty$. If $L^{p_1(.),q_1(.)}(X,\mu) \subset L^{p_2(.),q_2(.)}(X,\mu)$ such that $q_{2-} < q_1^+ \frac{p_1(0)q_2(0)}{q_1(0)p_2(0)}$ then any collection of disjoint measurable sets of positive measure is a finite element.*

Proof Assume that (E_n) is a sequence of disjoint measurable sets such that $\mu(E_n) \neq 0$ for infinite one n. Thus, since $\overset{\infty}{\underset{n=1}{\cup}} E_n \subset X$, we have $\mu\left(\overset{\infty}{\underset{n=1}{\cup}} E_n\right) < \infty$. Therefore, we write $\sum_{n=1}^{\infty} \mu(E_n) < \infty$. That means $\lim_{n\to\infty} \mu(E_n) = 0$. Then there exists subsequence $(E_{n_k})_{n_k}$ such that $E = \overset{\infty}{\underset{k=1}{\cup}} E_{n_k}$, $E_{n_k} \cap E_{n_j}$ $(k \neq j)$, $\mu(E) < \infty$, and $\mu(E_{n_k}) = 2^{-kq_1^+ \frac{p_1(0)}{q_1(0)}} \mu(E)$ for every $k \in \mathbb{N}$. Define that

$$f(x) = \sum_{k=1}^{\infty} \left(2^k k^{-2}\right) \chi_{E_{nk}}(x).$$

Then since $2q_1^+ > 1$

$$J_{p_1,q_1}(f) \cong \int_0^{\mu(E)} t^{\frac{q_1(0)}{p_1(0)}-1} (f^*(t))^{q_1(t)}\,dt = \sum_{i=1}^{\infty} \int_0^{\mu\left(E_{n_i}\right)} t^{\frac{q_1(0)}{p_1(0)}-1} (f^*(t))^{q_1(t)}\,dt$$

$$= \sum_{i=1}^{\infty} \int_0^{\mu\left(E_{n_i}\right)} t^{\frac{q_1(0)}{p_1(0)}-1} \left(\left(\sum_{k=1}^{\infty} \left(2^k k^{-2}\right) \chi_{E_{nk}}\right)^*(t)\right)^{q_1(t)} dt$$

$$\leq \int_0^{\mu\left(E_{n_1}\right)} t^{\frac{q_1(0)}{p_1(0)}-1} 2^{q_1^+} dt + \int_0^{\mu\left(E_{n_2}\right)} \left(2^2 2^{-2}\right)^{q_1^+} t^{\frac{q_1(0)}{p_1(0)}-1} dt + \int_0^{\mu\left(E_{n_3}\right)} \left(2^3 3^{-2}\right)^{q_1^+} t^{\frac{q_1(0)}{p_1(0)}-1} dt + ...$$

$$+ \int_0^{\mu\left(E_{n_m}\right)} \left(2^m m^{-2}\right)^{q_1^+} t^{\frac{q_1(0)}{p_1(0)}-1} dt + ...$$

$$= \frac{p_1(0)}{q_1(0)} \left\{ 2^{q_1^+} \mu(E_{n_1})^{\frac{q_1(0)}{p_1(0)}} + \left(2^2 2^{-2}\right)^{q_1^+} \mu(E_{n_2})^{\frac{q_1(0)}{p_1(0)}} + \left(2^3 3^{-2}\right)^{q_1^+} \mu(E_{n_3})^{\frac{q_1(0)}{p_1(0)}} \right\}$$

$$+ \frac{p_1(0)}{q_1(0)} \sum_{k=4}^{\infty} \left(2^k k^{-2}\right)^{q_1^+} \mu(E_{n_k})^{\frac{q_1(0)}{p_1(0)}}$$

$$= \frac{p_1(0)}{q_1(0)} \left\{ 2^{q_1^+} \mu(E_{n_1})^{\frac{q_1(0)}{p_1(0)}} + \mu(E_{n_2})^{\frac{q_1(0)}{p_1(0)}} + \left(\frac{8}{9}\right)^{q_1^+} \mu(E_{n_3})^{\frac{q_1(0)}{p_1(0)}} \right\}$$

$$+ \frac{p_1(0)}{q_1(0)} \mu(E)^{\frac{q_1(0)}{p_1(0)}} \sum_{k=4}^{\infty} \left(2^k k^{-2}\right)^{q_1^+} 2^{-kq_1^+} \mu(E)^{\frac{q_1(0)}{p_1(0)}}$$

$$= \frac{p_1(0)}{q_1(0)} \left\{ 2^{q_1^+} \mu(E_{n_1})^{\frac{q_1(0)}{p_1(0)}} + \mu(E_{n_2})^{\frac{q_1(0)}{p_1(0)}} + \left(\frac{8}{9}\right)^{q_1^+} \mu(E_{n_3})^{\frac{q_1(0)}{p_1(0)}} \right\}$$

$$+ \frac{p_1(0)}{q_1(0)} \mu(E)^{\frac{q_1(0)}{p_1(0)}} \sum_{k=4}^{\infty} k^{-2q_1^+} < \infty$$

holds. Thus, we have $f \in L^{p_1(.),q_1(.)}(X,\mu)$. On the other hand, we find that

$$J_{p_2,q_2}(f) \cong \int_0^{\mu(E)} t^{\frac{q_2(0)}{p_2(0)}-1} \left(f^*(t)\right)^{q_2(t)} dt = \sum_{k=1}^{\infty} \int_0^{\mu\left(E_{n_k}\right)} t^{\frac{q_2(0)}{p_2(0)}-1} \left(f^*(t)\right)^{q_2(t)} dt$$

$$\geq \sum_{k=4}^{\infty} \int_0^{\mu\left(E_{n_k}\right)} t^{\frac{q_2(0)}{p_2(0)}-1} \left(\sum_{k=1}^{\infty} \left(2^k k^{-2}\right) \chi_{E_{nk}} \right)^* (t)^{q_2(t)} dt$$

$$= \sum_{k=4}^{\infty} \int_0^{\mu\left(E_{n_k}\right)} t^{\frac{q_2(0)}{p_2(0)}-1} \left(2^k k^{-2}\right)^{q_2(t)} dt, \quad \left(2^k k^{-2} > 1 \text{ for } k \geq 4\right)$$

$$\geq \sum_{k=4}^{\infty} \int_0^{\mu\left(E_{n_k}\right)} t^{\frac{q_2(0)}{p_2(0)}-1} \left(2^k k^{-2}\right)^{q_{2-}} dt = \frac{p_2(0)}{q_2(0)} \sum_{k=4}^{\infty} \left(2^k k^{-2}\right)^{q_{2-}} \mu(E_{n_k})^{\frac{q_2(0)}{p_2(0)}}$$

$$= \frac{p_2(0)}{q_2(0)} \mu(E)^{\frac{q_2(0)}{p_2(0)}} \sum_{k=4}^{\infty} \left(2^k k^{-2}\right)^{q_{2-}} 2^{-kq_1^+ \left(\frac{p_1(0)q_2(0)}{q_1(0)p_2(0)}\right)}$$

$$= \frac{p_2(0)}{q_2(0)} \mu(E)^{\frac{q_2(0)}{p_2(0)}} \sum_{k=4}^{\infty} k^{-2q_{2-}} 2^{k\left((q_{2-}) - q_1^+ \left(\frac{p_1(0)q_2(0)}{q_1(0)p_2(0)}\right)\right)}.$$

If we say that $b_k = k^{-2q_{2-}} 2^{k\left((q_{2-}) - q_1^+ \left(\frac{p_1(0)q_2(0)}{q_1(0)p_2(0)}\right)\right)}$, then we have $\underset{k\to\infty}{lim} b_k = 0$. Thus, we find that $f \notin L^{p_2(.),q_2(.)}(X,\mu)$. However, from the assumption, we must obtain $f \in L^{p_2(.),q_2(.)}(X,\mu)$. Therefore, we find that any collection of disjoint measurable sets of positive measure is a finite element. □

References

[1] Aydın İ, Gürkanlı AT. The inclusion $L^{p(x)}(\mu) \subset L^{q(x)}(\nu)$. Int J Appl Math 2009; 22: 1031-1040.

[2] Bandaliev RA. Embedding between variable exponent Lebesgue spaces with measures. Azerbaijan Journal of Mathematics 2012; 2: 119-126.

[3] Diening L, Harjulehto P, Hastö P, Ruzicka M. Lebesgue and Sobolev Spaces with Variable Exponent. Berlin, Heidelberg, Germany: Springer-Verlag, 2011.

[4] Ephremidze L, Kokilashvili V, Samko S. Fractional, maximal and singular operators in variable exponent Lorentz spaces. Fract Calc Appl Anal 2008; 11: 1-14.

[5] Gürkanlı AT. On the inclusion of some Lorentz spaces. J Math Kyoto Univ 2004; 44: 441-450.

[6] Hunt RA. On $L(p,q)$ spaces. Extrait de L'Enseignement Mathematique T., XII, fasc 1966; 4: 249-276.

[7] Kovacik O, Rakosnik J. On spaces $L^{p(x)}$ and $W^{k,p(x)}$. Czech Math J 1991; 41: 592-618.

[8] Miammee AG. The inclusion $L^p(\mu) \subset L^q(\nu)$. Amer Math Monthly 1991; 98: 342-345.

[9] Nekvinda A. Embeddings between discrete weighted Lebesgue spaces with variable exponent. Math Ineq Appl 2007; 10: 165-172.

[10] Romero JL. When $L^p(\mu)$ contained in $L^q(\nu)$. Amer Math Monthly 1983; 90: 203-206.

[11] Rudin W. Fourier Analysis on Groups. New York, NY, USA: Interscience, 1962.

[12] Samko SG. Convolution type operators in $L^{p(x)}$. Integral Transform 1998; Spec Funct 7: 123-144.

[13] Subramian B. On the inclusion $L^p(\mu) \subset L^q(\nu)$. Amer Math Monthly 1978; 85: 479-481.