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On a question about almost prime ideals

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Abstract: In this paper, by giving an example we answer positively the question "Does there exist a P-primary ideal I in a Noetherian domain R such that $PI = I^2$, but I is not almost prime?", asked by S. M. Bhatwadekar and P. K. Sharma. We also investigated conditions under which the answer to the above mentioned question is negative.

Key words: Almost prime ideal, primary ideal, Noetherian domain

1. Introduction

Throughout, R will be a commutative ring with identity. In [2], Bhatwadekar and Sharma defined an almost prime ideal, as a proper ideal I of an integral domain R as follows: if for $a, b \in R$, with $ab \in I - I^2$, then either $a \in I$ or $b \in I$. Then they studied some properties of almost prime ideals of integral domains and proved that in Noetherian domains an almost prime ideal is primary [2, Corollary 2.10]. They also proved that in regular domains almost prime ideals are precisely the prime ideals [2, Theorem 2.15]. Then Anderson and Bataineh [1, Theorem 22] characterized the Noetherian rings in which every proper ideal is a product of almost prime ideals.

2. Examples and results

In [2], Bhatwadekar and Sharma posed the following question.

Question: Does there exist a P-primary ideal I in a Noetherian domain R such that $PI = I^2$, but I is not almost prime?

In the next example, we show that the answer to this question is YES.

Note that all equalities and memberships in the following examples can be checked by Macaulay2 (Macaulay2 can be run online for free at
http://habanero.math.cornell.edu:3690/).

Example 1 *Let* $S = \mathbb{Q}[x, y, z, w]$, $J = (x^2 - zw, z^2 - yw, y^3 - xw, w^3 - xy^2z)$, $I = (x, y, z, w^2)$, *and* $m = (x, y, z, w)$. *By an easy check with Macaulay2, we find out that* J *is a prime ideal of* S *and since* $m^2 \subseteq I$, *we have* I *is an* m*-primary ideal of* S. *Thus,* $R = \frac{S}{J}$ *is a Noetherian domain and* $\overline{I}$ *is an* $\overline{m}$*-primary ideal of* R, *where* $\overline{I} = \frac{I+J}{J}$ *and* $\overline{m} = \frac{m}{J}$. *Now since (by an easy check with Macaulay2)* $I^2 + J = Im + J$, *we*

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have $\overline{I}^2 = \overline{I}\overline{m}$*, and since* $w^2 \in I + J$*,* $w^2 \notin I^2 + J$ *and* $w \notin I + J$*, we have* $(w + J)(w + J) = w^2 + J \in \overline{I} - \overline{I}^2$*, but* $w + J \notin \overline{I}$*. This shows that* $\overline{I}$ *is not almost prime.*

Remark: In the above example since $w^2 + J \in \overline{m}^2 - \overline{I}^2$, we have $\overline{I}^2 \neq \overline{m}^2$.
Note that if we omit the word domain in the above question and define almost prime ideal for any commutative ring as mentioned in the definition, there is an easier example for this question, as follows:

Example 2 *Let* K *be a field and* $R = \frac{K[[x]]}{(x^3)}$*; thus* R *is a Noetherian ring but it is not a domain. Now if* $I = \frac{(x^2)}{(x^3)}$ *and* $P = \frac{(x)}{(x^3)}$*, then* I *is* P*-primary (note that* P *is a maximal ideal of* R *and* $P^2 \subseteq I$*). It is easy to show that* $I^2 = IP \neq P^2$*, but* $(x + (x^3))(x + (x^3)) = x^2 + (x^3) \in I - I^2$ *and* $x + (x^3) \notin I$*. This shows that* I *is not an almost prime ideal.*

Now we state some conditions so that the answer to the above question is negative.

Proposition 3 *Let* I *be a* P*-primary ideal of* R *such that* $P^2 = I^2$*. Then* I *is almost prime.*

Proof Let for $a, b \in R$, $ab \in I - I^2$, $a \notin I$, and $b \notin I$. Since $a \notin I$ and I is P-primary, we must have $b \in P$. Similarly $a \in P$. Thus $ab \in P^2 = I^2$, which is a contradiction. □

We now show that $P^2 = I^2$ is actually a sufficient condition in Proposition 3, but it is not necessary.

Example 4 *Let* $S = \mathbb{Q}[x, y, z, w]$*,* $J = (x^2 - zw, z^2 - yw, y^3 - xw, w^3 - xy^2z)$*,* $K = (x, y, z)$*, and* $m = (x, y, z, w)$*. As in example 1,* J *is a prime ideal of* S*. Thus* $R = \frac{S}{J}$ *is a Noetherian domain and* $\overline{K}$ *is an* $\overline{m}$*-primary ideal of* R *(*$\overline{m}^2 \subseteq \overline{K}$*). Now since (by an easy check with Macaulay2)* $K^2 + J = Km + J \neq m^2 + J$*, we have* $\overline{K}^2 = \overline{K}\overline{m} \neq \overline{m}^2$*. It is easily seen that* $w^3 + J \in \overline{K}^2 \subseteq \overline{K}$ *and* $w^2 + J \notin \overline{K}$*. Now we show that* $\overline{K}$ *is an almost prime ideal of* R*. Let* $f, g \in S$ *and* $(f + J)(g + J) \in \overline{K} - \overline{K}^2$*. We can write* f *and* g *in the form* $f = xf_1 + yf_2 + w^3 f_3 + r_2 w^2 + r_1 w + r_0$ *and* $g = xg_1 + yg_2 + w^3 g_3 + s_2 w^2 + s_1 w + s_0$*, where* $r_i, s_j \in \mathbb{Q}$ *and* $f_i, g_j \in \mathbb{Q}[x, y, z, w]$ *(note that* f_i*,* g_j*,*r_i*, and* s_j *are not unique). Since* $(f + J)(g + J) \in \overline{K}$*, we have* $fg \in K + J \subseteq m$*, and so* $r_0 s_0 = 0$*; hence* $r_0 = 0$ *or* $s_0 = 0$*. Let* $r_0 = 0$ *and so* $f = xf_1 + yf_2 + w^3 f_3 + r_2 w^2 + r_1 w$ *and* $g = xg_1 + yg_2 + w^3 g_3 + s_2 w^2 + s_1 w + s_0$*. Thus* $fg = (xf_1 + yf_2 + w^3 f_3 + r_2 w^2 + r_1 w)(xg_1 + yg_2 + w^3 g_3 + s_2 w^2 + s_1 w + s_0) = xh_1 + yh_2 + w^3 h_3 + r_1 s_0 w + (r_1 s_1 + r_2 s_0) w^2 \in K + J$*, where* $h_i \in \mathbb{Q}[x, y, z, w]$*. Therefore,* $r_1 s_0 w + (r_1 s_1 + r_2 s_0) w^2 \in K + J$*; hence* $w(r_1 s_0 + (r_1 s_1 + r_2 s_0) w) \in K + J$*. Now since* $K + J$ *is an* m*-primary ideal of* $\mathbb{Q}[x, y, z, w]$ *and* $w \notin K$*, we must have* $r_1 s_0 + (r_1 s_1 + r_2 s_0) w \in m$*. Thus* $r_1 s_0 = 0$ *and so* $s_0 = 0$ *or* $r_1 = 0$*. Hence we have the following cases:*
Case 1: *If* $s_0 = 0$*, then we have* $f = xf_1 + yf_2 + w^3 f_3 + r_2 w^2 + r_1 w$ *and* $g = xg_1 + yg_2 + w^3 g_3 + s_2 w^2 + s_1 w$*. Thus* $fg = (xf_1 + yf_2 + w^3 f_3 + r_2 w^2 + r_1 w)(xg_1 + yg_2 + w^3 g_3 + s_2 w^2 + s_1 w) = xh_1 + yh_2 + w^3 h_3 + r_1 s_1 w^2 \in K + J$*, where* $h_i \in \mathbb{Q}[x, y, z, w]$*. Therefore,* $r_1 s_1 w^2 \in K + J$*, and since* $w^2 \notin K + J$*, we must have* $r_1 s_1 = 0$*. Thus we have the following subcases:*
Subcase 1.1: *If* $r_1 = 0$ *we have* $f = xf_1 + yf_2 + w^3 f_3 + r_2 w^2$ *and* $g = xg_1 + yg_2 + w^3 g_3 + s_2 w^2 + s_1 w$ *and thus* $fg = (xf_1 + yf_2 + w^3 f_3 + r_2 w^2)(xg_1 + yg_2 + w^3 g_3 + s_2 w^2 + s_1 w) \in K^2 + J$*; this gives* $fg + J \in \overline{K}^2$*, a contradiction.*
Subcase 1.2: *If* $s_1 = 0$*, a contradiction, as in the above subcase.*

Case 2: *If* $r_1 = 0$*, then* $f = xf_1 + yf_2 + w^3f_3 + r_2w^2$ *and* $g = xg_1 + yg_2 + w^3g_3 + s_2w^2 + s_1w + s_0$*. Therefore,* $fg = (xf_1 + yf_2 + w^3f_3 + r_2w^2)(xg_1 + yg_2 + w^3g_3 + s_2w^2 + s_1w + s_0) = xh_1 + yh_2 + w^3h_3 + r_2s_0w^2 \in K + J$*. Thus* $r_2s_0w^2 \in K + J$*, and since* $w^2 \notin K + J$*,* $r_2s_0 = 0$*. We have the following subcases:*
Subcase 2.1: *If* $r_2 = 0$*, then we have* $f = xf_1 + yf_2 + w^3f_3 \in K$*, which is the desired conclusion.*
Subcase 2.2: *If* $s_0 = 0$*, then* $f = xf_1 + yf_2 + w^3f_3 + r_2w^2$ *and* $g = xg_1 + yg_2 + w^3g_3 + s_2w^2 + s_1w$*. Therefore,* $fg = (xf_1 + yf_2 + w^3f_3 + r_2w^2)(xg_1 + yg_2 + w^3g_3 + s_2w^2 + s_1w) \in K^2 + J$*. This gives* $fg + J \in \overline{K}^2$*, a contradiction.*
Therefore, $\overline{K}$ *is an almost prime ideal of* R*.*

Bhatwadekar and Sharma in [2, Corollary 2.8] proved that for an ideal I in a quasi-local domain (R, m) with $m^2 \subseteq I \subseteq m$, then $I^2 = m^2$ if and only if I is almost prime. The following proposition is a similar result.

Proposition 5 *Let* I *be a* P*-primary ideal in an integral domain* R *such that* $I^2 = IP$*. If* I *is generated by two elements, then* I *is almost prime if and only if* $I^2 = P^2$*.*

Proof ($\Leftarrow$) It follows from Proposition 3.
($\Rightarrow$) Suppose that $I^2 = IP$ and I is generated by two elements. By [3, page 96, Proposition 13], $P^2 \subseteq I$. Now let there exist $a, b \in P$, such that $ab \notin I^2$. Since $P^2 \subseteq I$, we have $ab \in I - I^2$ and hence $a \in I$ or $b \in I$. Without loss of generality suppose that $a \in I$ and so $ab \in IP = I^2$, which is a contradiction. Therefore, $I^2 = P^2$. □

It is clear that if the ideal I in the above question is a cancellation ideal, then the answer to the question is NO. In a special case, if the cancellation law for ideals holds, the answer is NO, for example, in *Dedekind domains*, *Prüfer domains*, and *PIDs*.
The following proposition shows that if we want to consider the above question in a Noetherian domain ring R with a local property we can assume that R is local.

Proposition 6 *Let* I *be a* P*-primary ideal in an integral domain* R*, such that* $I^2 = IP$*. The following are equivalent:*
1) I *is an almost prime ideal of* R*;*
2) I_P *is an almost prime ideal of* R_P*, for every prime ideal* P *containing* I*;*
3) I_m *is an almost prime ideal of* R_m*, for every maximal ideal* m *containing* I*.*

Proof ($1 \Rightarrow 2$) This is proved by Bhatwadekar and Sharma [2, Lemma 2.13].
($2 \Rightarrow 3$) It is clear.
($3 \Rightarrow 1$) Let I_m be an almost prime ideal for all maximal ideal m containing I and for $a, b \in R$, $ab \in I - I^2$. Since $ab \notin I^2$, $(I^2 : ab) = \{r \in R \mid rab \in I^2\} \neq R$. Thus there exists a maximal ideal m of R such that $(I^2 : ab) \subseteq m$. Since $ab \in I$, we have $I \subseteq (I^2 : ab)$ and so $I \subseteq m$; therefore, I_m is an almost prime ideal of R_m. If $\frac{a}{1}\frac{b}{1} \in I_m^2$, then there exists $u \in R - m$ such that $uab \in I^2$. Thus $u \in (I^2 : ab) \subseteq m$, which is a contradiction. Hence $\frac{a}{1}\frac{b}{1} \notin I_m^2$ and therefore $\frac{a}{1}\frac{b}{1} \in I_m - I_m^2$. Hence by assumption $\frac{a}{1} \in I_m$ or $\frac{b}{1} \in I_m$. Without loss of generality let $\frac{a}{1} \in I_m$; thus there exists $u \in R - m$ such that $ua \in I$. Since I is a P-primary ideal, we have $u \in P$ or $a \in I$. Now since $I^2 = IP$, $P \subseteq (I^2 : I) \subseteq (I^2 : ab) \subseteq m$. This gives $u \notin P$, and so $a \in I$. Therefore, I is almost prime. □

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