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# Some topological properties of the spaces of almost null and almost convergent double sequences

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**Abstract:** Let $\mathcal{C}_{f_0}$ and $\mathcal{C}_f$ denote the spaces of almost null and almost convergent double sequences, respectively. We show that $\mathcal{C}_{f_0}$ and $\mathcal{C}_f$ are BDK-spaces, barreled and bornological, but they are not monotone and so not solid. Additionally, we establish that both of the spaces $\mathcal{C}_{f_0}$ and $\mathcal{C}_f$ include the space $\mathcal{BS}$ of bounded double series.

**Key words:** Double sequence, Pringsheim convergence, almost convergence

## 1. Introduction

We denote the set of all complex valued double sequences by $\Omega$, which is a vector space with coordinatewise addition and scalar multiplication. Any vector subspace of $\Omega$ is called *a double sequence space*. A double sequence $x = (x_{mn})$ of complex numbers is said to be *bounded* if $\|x\|_\infty = \sup_{m,n\in\mathbb{N}} |x_{mn}| < \infty$, where $\mathbb{N} = \{0, 1, 2, \ldots\}$. The space of all bounded double sequences is denoted by $\mathcal{M}_u$, which is a Banach space with the norm $\|\cdot\|_\infty$. Consider the sequence $x = (x_{mn}) \in \Omega$. If for every $\varepsilon > 0$ there exists $n_0 = n_0(\varepsilon) \in \mathbb{N}$ and $l \in \mathbb{C}$ such that $|x_{mn} - l| < \varepsilon$ for all $m, n > n_0$, then we say that the double sequence $x$ is *convergent* in *Pringsheim's sense* to the limit $l$ and write $p - \lim_{m,n\to\infty} x_{mn} = l$, where $\mathbb{C}$ denotes the complex field. By $\mathcal{C}_p$, we denote the space of all convergent double sequences in Pringsheim's sense. It is well known that there are such sequences in the space $\mathcal{C}_p$ but not in the space $\mathcal{M}_u$. Indeed, if we define the sequence $x = (x_{mn})$ by

$$x_{mn} := \begin{cases} m & , \quad m \in \mathbb{N},\ n = 0, \\ n & , \quad n \in \mathbb{N},\ m = 0, \\ 0 & , \quad m, n \in \mathbb{N} \setminus \{0\}, \end{cases}$$

for all $m, n \in \mathbb{N}$, then it is trivial that $x \in \mathcal{C}_p \setminus \mathcal{M}_u$, since $p - \lim_{m,n\to\infty} x_{mn} = 0$ but $\|x\|_\infty = \infty$. Therefore, we can consider the space $\mathcal{C}_{bp}$ of the double sequences that are both convergent in Pringsheim's sense and bounded, i.e. $\mathcal{C}_{bp} = \mathcal{C}_p \cap \mathcal{M}_u$. A sequence in the space $\mathcal{C}_p$ is said to be *regularly convergent* if it is a single convergent sequence with respect to each index and denote the space of all such sequences by $\mathcal{C}_r$. Also by $\mathcal{C}_{bp0}$ and $\mathcal{C}_{r0}$, we denote the spaces of all double sequences converging to 0 contained in the sequence spaces $\mathcal{C}_{bp}$ and $\mathcal{C}_r$, respectively. Móricz [8] proved that $\mathcal{C}_{bp}$, $\mathcal{C}_{bp0}$, $\mathcal{C}_r$, and $\mathcal{C}_{r0}$ are Banach spaces with the norm $\|\cdot\|_\infty$.

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The concept of almost convergence for single sequences was introduced by Lorentz [7] and for double sequences by Móricz and Rhoades [9]. A double sequence $x=(x_{kl})$ of complex numbers is said to be *almost convergent* to a generalized limit $\alpha$ if

$$p-\lim_{q,r\to\infty}\sup_{s,t>0}\left|\frac{1}{(q+1)(r+1)}\sum_{k=s}^{s+q}\sum_{l=t}^{t+r}x_{kl}-\alpha\right|=0.$$

In this case, $\alpha$ is called the $f_2$-limit of $x$. We denote the space of all almost convergent double sequences by $\mathcal{C}_f$. Note that, in contrast to the single sequences, a $p-$convergent double sequence need not be almost convergent. However, every bounded convergent double sequence is almost convergent, and every almost convergent double sequence is also bounded, i.e. $\mathcal{C}_{bp}\subset\mathcal{C}_f\subset\mathcal{M}_u$ and each inclusion is proper [11]. Referring to [9], we introduce the space $\mathcal{C}_{f_0}$ of all almost null double sequences by

$$\mathcal{C}_{f_0}:=\left\{x=(x_{kl})\in\Omega:p-\lim_{q,r\to\infty}\sup_{s,t>0}\left|\frac{1}{(q+1)(r+1)}\sum_{k=0}^{q}\sum_{l=0}^{r}x_{k+s,l+t}\right|=0\right\}.$$

Throughout this work, we consider the mapping $\sigma$ of the set of positive integers into itself having no finite orbits, that is $\sigma^p(k)\neq k$ for all positive integers $k\geq 0$ and $p\geq 1$, where $\sigma^p(k)$ is $p$th iterate of $\sigma$ at $k$. Thus, a $\sigma-$mean extends the limit functional on $c$ in the sense that $\phi(x)=\lim x$ for all $x\in c$, the space of convergent sequences [10].

**Definition 1.1** *[6] A bounded double sequence $x=(x_{kl})$ of real numbers is said to be $\sigma-$convergent to a limit $l$ if*

$$p-\lim_{q,r}\tau_{qrst}(x)=l \quad \textit{uniformly in} \quad s,t\in\mathbb{N},$$

*where*

$$\tau_{qrst}(x)=\frac{1}{(q+1)(r+1)}\sum_{k=0}^{q}\sum_{l=0}^{r}x_{\sigma^k(s),\sigma^l(t)}. \tag{1.1}$$

*In this case, we write $\sigma_2-\lim x=l$. By $V_\sigma^2$, we denote the set of all bounded $\sigma-$convergent double sequences. Clearly, $\mathcal{C}_{bp}\subset V_\sigma^2$.*

*One can see that in contrast to the case for single sequences, a convergent double sequence need not be $\sigma-$convergent. However, every bounded convergent double sequence is $\sigma-$convergent. In the case $\sigma(k)=k+1$, $\sigma-$convergence of double sequences is reduced to almost convergence.*

**Definition 1.2** *[15] A topological vector space $\lambda$ over $\mathbb{R}$ or $\mathbb{C}$ is called locally convex if it is a Hausdorff space such that every neighborhood of any $x\in\lambda$ contains a convex neighborhood of $x$.*

**Definition 1.3** *[17] A locally convex double sequence space $\lambda$ is called a DK-space if all of the seminorms $r_{kl}:\lambda\to\mathbb{R}$, $x=(x_{kl})\mapsto|x_{kl}|$ for all $k,l\in\mathbb{N}$ are continuous. A DK-space with a Fréchet topology is called an FDK-space. A normed FDK-space is called a BDK-space.*

**Definition 1.4** *[17, p. 36] A double sequence space $\lambda$ is said to be monotone if $xu = (x_{kl}u_{kl}) \in \lambda$ for every $x = (x_{kl}) \in \lambda$ and $u = (u_{kl}) \in \{0,1\}^{\mathbb{N}\times\mathbb{N}}$.*

**Definition 1.5** *[4, p. 153] A double sequence space $\lambda$ is said to be solid if and only if*

$$\widetilde{\lambda} := \{(u_{kl}) \in \Omega : \exists (x_{kl}) \in \lambda \text{ such that } |u_{kl}| \leq |x_{kl}| \text{ for all } k,l \in \mathbb{N}\} \subset \lambda.$$

*Moreover, $\lambda$ is monotone whenever $\lambda$ is solid.*

**Definition 1.6** *[15] Let $\lambda$ be a vector space over the field $\mathbb{C}$, and let $A, B$ be subsets of $\lambda$. We say that $A$ absorbs $B$ if there exists $\alpha_0 \in \mathbb{C}$ such that $B \subset \alpha A$ whenever $|\alpha| \geq |\alpha_0|$. A subset $C$ of $\lambda$ is circled if $\alpha C \subset C$ whenever $|\alpha| \leq 1$.*

Some authors, for example Wilansky [16] and Boos [5], used the term "balanced" instead of the term "circled".

**Definition 1.7** *[15] A locally convex space $\lambda$ is bornological if every circled, convex subset $A \subset \lambda$ that absorbs every bounded set in $\lambda$ is a neighborhood of $0$.*

**Definition 1.8** *[5] Let $\lambda$ be a locally convex space. Then a subset is called barrel if it is absolutely convex, absorbing, and closed in $\lambda$. Moreover, $\lambda$ is called a barreled space if each barrel is a neighborhood of zero.*

**Lemma 1.9** *[15] Every Banach space and every Fréchet space is a barreled space.*

**Lemma 1.10** *[15] Every Fréchet space and hence every Banach space is a bornological.*

**Lemma 1.11** *[5, Theorem 6.3.12, p. 284] Let $(X, p)$ be a seminormed space and $q$ be a seminorm on $X$. Then the following statements are equivalent:*

*(a) $q$ is continuous.*

*(b) $q$ is continuous at zero.*

*(c) There exists $M > 0$ such that $q(x) \leq Mp(x)$ for all $x \in X$.*

Altay and Başar [1] introduced the space $\mathcal{BS}$ of bounded series as follows:

$$\mathcal{BS} := \left\{ x = (x_{kl}) \in \Omega : \|x\|_{\mathcal{BS}} = \sup_{m,n\in\mathbb{N}} \left| \sum_{k,l=0}^{m,n} x_{kl} \right| < \infty \right\}.$$

They also showed that $\mathcal{BS}$ is a Banach space with the norm $\|\cdot\|_{\mathcal{BS}}$.

The reader can refer to Başar [2], and Mursaleen and Mohiuddine [14] for relevant terminology and required details on the spaces of double sequences and related topics.

## 2. Main Results

In this section, we show that $\mathcal{C}_{f_0}$ and $\mathcal{C}_f$ are BDK-spaces, barreled and bornological. Moreover, we prove that they are not monotone and so not solid. Finally, we give two inclusion relations between the spaces $\mathcal{C}_{f_0}$, $\mathcal{C}_f$ and $\mathcal{BS}$.

**Remark 2.1** *$\mathcal{C}_{f_0}$ and $\mathcal{C}_f$ are Banach spaces with the supremum norm.*

**Proof** Mursaleen and Mohiuddine [13] have proved that $V_\sigma^2$ is a Banach space normed by

$$\|x\| = \sup_{q,r,s,t\in\mathbb{N}} |\tau_{qrst}(x)|,$$

where $\tau_{qrst}(x)$ is defined by (1.1). Therefore, as a particular case, $\mathcal{C}_f$ is a Banach space with the norm

$$\|x\|_{\mathcal{C}_f} = \sup_{q,r,s,t\in\mathbb{N}} \left| \frac{1}{(q+1)(r+1)} \sum_{k=0}^{q}\sum_{l=0}^{r} x_{k+s,l+t} \right|. \tag{2.1}$$

One can see by a similar way that $\mathcal{C}_{f_0}$ endowed with the norm $\|\cdot\|_{\mathcal{C}_f}$ is a Banach space.

This completes the proof. □

**Corollary 2.2** *By Lemma 1.9 and Remark 2.1, $\mathcal{C}_{f_0}$ and $\mathcal{C}_f$ are barreled spaces.*

**Corollary 2.3** *By Lemma 1.10 and Remark 2.1, $\mathcal{C}_{f_0}$ and $\mathcal{C}_f$ are bornological spaces.*

**Theorem 2.4** *$\mathcal{C}_{f_0}$ and $\mathcal{C}_f$ endowed with the norm $\|\cdot\|_{\mathcal{C}_f}$ defined by (2.1) are BDK-spaces.*

**Proof** To avoid the repetition of similar statements, we give the proof only for the space $\mathcal{C}_f$. Since every norm (normed space) is a seminorm (seminormed space), we say that $\mathcal{C}_f$ is a seminormed space with the seminorm (2.1). Furthermore, we define new seminorms in the space $\mathcal{C}_f$ by $r_{kl} : \mathcal{C}_f \to \mathbb{R}$, $x = (x_{kl}) \mapsto |x_{kl}|$ for all $k, l \in \mathbb{N}$. Now we shall show that each one is continuous. Using Lemma 1.11, by the property of supremum, we easily find a $M > 0$ for all $x \in \mathcal{C}_f$ such that $r_{kl}(x) = |x_{kl}| \leq M\|x\|_{\mathcal{C}_f}$ for all $k, l \in \mathbb{N}$. Therefore, the seminorms $r_{kl}$ are continuous for each $k, l \in \mathbb{N}$, that is, the space $\mathcal{C}_f$ is a DK-space. Moreover, since it is a Banach space by Remark 2.1, it has Fréchet topology. Therefore, it is a BDK-space with the norm (2.1).

This step concludes the proof. □

**Theorem 2.5** *The spaces $\mathcal{C}_{f_0}$ and $\mathcal{C}_f$ are not monotone.*

**Proof** Let us consider the double sequence $x = (x_{kl})$ defined by $x_{kl} = (-1)^{k+l}$ for all $k, l \in \mathbb{N}$. Thus, we have

$$\begin{aligned} \frac{1}{(q+1)(r+1)} \left| \sum_{k=s}^{s+q}\sum_{l=t}^{t+r} x_{kl} \right| &= \frac{1}{(q+1)(r+1)} \left| \sum_{k=s}^{s+q}\sum_{l=t}^{t+r} (-1)^{k+l} \right| \\ &= \frac{1}{(q+1)(r+1)} \left| (-1)^{s+t} \frac{[1+(-1)^q][1+(-1)^r]}{4} \right|. \end{aligned} \tag{2.2}$$

If we take supremum over $s,t\in\{1,2,\dots\}$ in the relation (2.2) and next apply $p-$limit as $q,r\to\infty$, then we have $x\in\mathcal{C}_{f_0}$. Now we take a double sequence $y=(y_{kl})\in\{0,1\}^{\mathbb{N}\times\mathbb{N}}$ defined for all $k,l\in\mathbb{N}$ by

$$y_{kl}:=\begin{cases}(-1)^{k+l} & , \quad k+l \text{ is even}\\ 0 & , \quad \text{otherwise}\end{cases}$$

and let $z=(z_{kl})=(x_{kl}y_{kl})$. Hence, when $q,r,s$, and $t$ are even we have

$$\begin{aligned}\frac{1}{(q+1)(r+1)}\left|\sum_{k=s}^{s+q}\sum_{l=t}^{t+r}z_{kl}\right| &= \frac{1}{(q+1)(r+1)}\left|\sum_{k=s}^{s+q}\sum_{l=t}^{t+r}x_{kl}y_{kl}\right|\\ &= \frac{1}{(q+1)(r+1)}\left[\frac{r+2}{2}+\frac{q(r+1)}{2}\right],\end{aligned}$$

which yields that

$$p-\lim_{q,r\to\infty}\sup_{s,t>0}\frac{1}{(q+1)(r+1)}\left|\sum_{k=s}^{s+q}\sum_{l=t}^{t+r}z_{kl}\right|=\frac{1}{2}.$$

Hence, $z\notin\mathcal{C}_{f_0}$. Furthermore, we obtain the same result in other cases of $q,r,s$, and $t$.

Define the sequence $v=(v_j)$ by

$$v_j:=\begin{cases}1 & , \quad j=2^{2k},\dots,2^{2k+1}-1,\\ 0 & , \quad j=2^{2k+1},\dots,2^{2k+2}-1,\end{cases}\quad (k=0,1,2,\dots)$$

for all $j\in\{1,2,3,\dots\}$. Now, following Mursaleen and Mohiuddine [13], we construct the double sequence $u=(u_{kl})\notin\mathcal{C}_f$ all of whose rows are the sequence $v$, that is,

$$(u_{kl})=\begin{pmatrix}v\\v\\v\\\vdots\end{pmatrix}=\begin{pmatrix}1&0&0&1&1&1&1&0&0&0&0&0&0&0&0&1&\cdots\\1&0&0&1&1&1&1&0&0&0&0&0&0&0&0&1&\cdots\\1&0&0&1&1&1&1&0&0&0&0&0&0&0&0&1&\cdots\\\vdots&\vdots&\vdots&\vdots&\vdots&\vdots&\vdots&\vdots&\vdots&\vdots&\vdots&\vdots&\vdots&\vdots&\vdots&\vdots&\ldots\end{pmatrix}$$

and take the double sequence $x=(x_{kl})$ with $x_{kl}=1$ for all $k,l\in\mathbb{N}$. Therefore, we have that

$$\frac{1}{(q+1)(r+1)}\left|\sum_{k=s}^{s+q}\sum_{l=t}^{t+r}x_{kl}\right|=\frac{|(s+q-s+1)(t+r-t+1)|}{(q+1)(r+1)}=1. \tag{2.3}$$

If we take supremum over $s,t\in\{1,2,\dots\}$ in the relation (2.3) and nextly apply $p-$limit as $q,r\to\infty$, then we obtain that

$$p-\lim_{q,r\to\infty}\sup_{s,t>0}\frac{1}{(q+1)(r+1)}\left|\sum_{k=s}^{s+q}\sum_{l=t}^{t+r}x_{kl}\right|=1,$$

that is, $x\in\mathcal{C}_f$. Obviously, the sequence $u$ is in $\{0,1\}^{\mathbb{N}\times\mathbb{N}}$ but the multiply sequence $z=ux=u$ is not in $\mathcal{C}_f$. Thus, the space $\mathcal{C}_f$ is not monotone.

This completes the proof. □

**Corollary 2.6** *Since the spaces* $\mathcal{C}_{f_0}$ *and* $\mathcal{C}_f$ *are not monotone, they are not solid.*

**Theorem 2.7** *The inclusion* $\mathcal{BS} \subset \mathcal{C}_{f_0}$ *holds.*

**Proof** Let $x = (x_{kl}) \in \mathcal{BS}$. Then $M := \sup_{s,t\in\mathbb{N}} \left|\sum_{k,l=0}^{s,t} x_{kl}\right| < \infty$. Therefore, we get for all $q, r, s, t \in \mathbb{N}$ that

$$\begin{aligned}
\left|\sum_{k=s}^{s+q}\sum_{l=t}^{t+r} x_{kl}\right| &= \left|\sum_{k=0}^{s+q}\sum_{l=0}^{t+r} x_{kl} - \sum_{k=0}^{s-1}\sum_{l=0}^{t+r} x_{kl} - \sum_{k=s}^{s+q}\sum_{l=0}^{t-1} x_{kl}\right| \\
&\leq \left|\sum_{k=0}^{s+q}\sum_{l=0}^{t+r} x_{kl}\right| + \left|\sum_{k=0}^{s-1}\sum_{l=0}^{t+r} x_{kl}\right| + \left|\sum_{k=s}^{s+q}\sum_{l=0}^{t-1} x_{kl}\right| \\
&\leq 2M + \left|\sum_{k=s}^{s+q}\sum_{l=0}^{t-1} x_{kl}\right| \\
&= 2M + \left|\sum_{k=0}^{s+q}\sum_{l=0}^{t-1} x_{kl} - \sum_{k=0}^{s}\sum_{l=0}^{t-1} x_{kl}\right| \\
&\leq 2M + \left|\sum_{k=0}^{s+q}\sum_{l=0}^{t-1} x_{kl}\right| + \left|\sum_{k=0}^{s}\sum_{l=0}^{t-1} x_{kl}\right| \\
&\leq 4M,
\end{aligned}$$

and so

$$\frac{1}{(q+1)(r+1)}\left|\sum_{k=s}^{s+q}\sum_{l=t}^{t+r} x_{kl}\right| \leq \frac{4M}{(q+1)(r+1)}. \tag{2.4}$$

If we take supremum over $s, t \in \{1, 2, \dots\}$ in the relation (2.4) and next apply $p-$limit as $q, r \to \infty$, then we have

$$p-\lim_{q,r\to\infty}\sup_{s,t>0}\frac{1}{(q+1)(r+1)}\left|\sum_{k=s}^{s+q}\sum_{l=t}^{t+r} x_{kl}\right| = 0,$$

that is, $x \in \mathcal{C}_{f_0}$. Hence, the inclusion $\mathcal{BS} \subset \mathcal{C}_{f_0}$ holds. □

**Theorem 2.8** *The inclusion* $\mathcal{BS} \subset \mathcal{C}_f$ *strictly holds.*

**Proof** Since $\mathcal{C}_{f_0} \subset \mathcal{C}_f$ holds, the inclusion $\mathcal{BS} \subset \mathcal{C}_f$ also holds by Theorem 2.7. If we take $x = (x_{kl})$ with $x_{kl} = 1$ for all $k, l \in \mathbb{N}$, then we easily see that $x \notin \mathcal{BS}$ but $x \in \mathcal{C}_f$ from Theorem 2.5, as desired. □

## 3. Conclusion

Let $f_0$ and $f$ denote the spaces of almost null and almost convergent sequences. In 2010, Mursaleen [12] investigated certain properties of the space $f$ and essentially proved that the space $f$ is a nonseparable closed subspace of $(\ell_\infty, \|\cdot\|_\infty)$. Later, the spaces $\widehat{f_0}$ and $\widehat{f}$ of single sequences consisting of all sequences $x = (x_k)$

such that $(rx_k + sx_{k-1})$ belongs to the spaces $f_0$ and $f$ were recently introduced by Başar and Kirişçi [3], and some algebraic and topological properties of the spaces $f$, $\widehat{f}_0$, and $\widehat{f}$ were studied, where $r, s \in \mathbb{R} \setminus \{0\}$.

It is natural to expect the extension of the corresponding results obtained in [12] and [3] for the spaces $f$ and $\widehat{f}$ of single sequences to the spaces of double sequences. For this, we firstly need some properties of the spaces $\mathcal{C}_{f_0}$ and $\mathcal{C}_f$. In order to work on the domain of some four-dimensional triangle matrices in the spaces $\mathcal{C}_{f_0}$ and $\mathcal{C}_f$, as a beginning, we have showed that $\mathcal{C}_{f_0}$ and $\mathcal{C}_f$ are BDK-spaces, barreled and bornological, but are not monotone and solid. We also proved that both of the spaces $\mathcal{C}_{f_0}$ and $\mathcal{C}_f$ include the space $\mathcal{BS}$ of bounded double series. Of course, it is worth mentioning here that the investigation of the separability and the existence of fundamental set of the spaces $\mathcal{C}_{f_0}$ and $\mathcal{C}_f$, and also the investigation of the results for double sequences corresponding to those of Mursaleen [12] remain open.

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