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Stability and data dependence results for the Jungck–Khan iterative scheme

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Abstract: The Jungck–Khan iterative scheme for a pair of nonself operators contains as a special case Jungck–Ishikawa and Jungck–Mann iterative schemes. In this paper, we establish improved results about convergence, stability, and data dependence for the Jungck–Khan iterative scheme.

Key words: Jungck–Khan iterative scheme, convergence, stability, weak w^2-stability, data dependency

1. Introduction

The case of nonself mappings is much more complicated than that of self ones and therefore it is not considered in many situations. Inspired by the work of Khan [7], here we tackle this problem in the context of two nonself operators.

Definition 1 *[5] Let X be a set and S, $T: X \to X$ be mappings.*

1. *A point x in X is called:*
   (i) coincidence point of S and T if $Sx = Tx$,
   (ii) common fixed point of S and T if $x = Sx = Tx$.
2. *If $w = Sx = Tx$ for some x in X, then w is called a point of coincidence of S and T.*
3. *A pair (S, T) is said to be:*
   (i) commuting if $TSx = STx$ for all $x \in X$,
   (ii) weakly compatible if they commute at their coincidence points, i.e. $STx = TSx$ whenever $Sx = Tx$.

Let X be a Banach space, Y be an arbitrary set, and $S, T: Y \to X$ be two nonself operators such that $T(Y) \subseteq S(Y)$.

Definition 2 *([15]) We say that the sequences $\{Sx_n\}_{n=0}^{\infty}$ and $\{Sy_n\}_{n=0}^{\infty}$ in X are $S-$equivalent if*

$$\lim_{n \longrightarrow \infty} \|Sx_n - Sy_n\| = 0.$$

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Definition 3 *Let $S, T : Y \to X$ be two nonself operators on an arbitrary set Y such that $T(Y) \subseteq S(Y)$, p be a coincidence point of S and T, and $\{Sx_n\}_{n=0}^{\infty} \subset X$ be an iterative sequence generated by the general algorithm of form*

$$\begin{cases} x_0 \in Y, \\ Sx_{n+1} = f(T, x_n), \; n \in \mathbb{N}, \end{cases}$$

where x_0 is an initial approximation and f is a function. Suppose that $\{Sx_n\}_{n=0}^{\infty}$ converges to p.

1. *([11]) Let $\{Sy_n\}_{n=0}^{\infty} \subset X$ be an arbitrary sequence. Then $\{Sx_n\}_{n=0}^{\infty}$ is said to be stable with respect to (S,T) if and only if $\lim_{n\longrightarrow\infty} \|Sy_{n+1} - f(T, y_n)\| = 0$ implies that $\lim_{n\longrightarrow\infty} Sy_n = p$.*
2. *([15], [16]) Let $\{Sy_n\}_{n=0}^{\infty} \subset X$ be an $S-$ equivalent sequence of $\{Sx_n\}_{n=0}^{\infty} \subset X$. Then $\{Sx_n\}_{n=0}^{\infty}$ is said to be weak w^2-stable with respect to (S,T) if and only if $\lim_{n\longrightarrow\infty} \|Sy_{n+1} - f(T, y_n)\| = 0$ implies that $\lim_{n\longrightarrow\infty} Sy_n = p$.*

Recently, Khan et al. [8] defined the Jungck–Khan iterative scheme as

$$\begin{cases} x_0 \in Y, \\ Sx_{n+1} = (1 - \alpha_n - \beta_n) Sx_n + \alpha_n Ty_n + \beta_n Tx_n, \\ Sy_n = (1 - b_n - c_n) Sx_n + b_n Tz_n + c_n Tx_n, \\ Sz_n = (1 - a_n) Sx_n + a_n Tx_n, \; n \in \mathbb{N}, \end{cases} \tag{1}$$

where $\{\alpha_n\}_{n=0}^{\infty}$, $\{\beta_n\}_{n=0}^{\infty}$, $\{a_n\}_{n=0}^{\infty}$, $\{b_n\}_{n=0}^{\infty}$, and $\{c_n\}_{n=0}^{\infty} \subset [0,1]$ are real sequences satisfying $\alpha_n + \beta_n$, $b_n + c_n \in [0,1]$ for all $n \in \mathbb{N}$.

The following definitions and lemmas will be needed in proving our main results.

Definition 4 *([9])The pair of operators $S, T : Y \to X$ is contractive if there exist a real number $\delta \in [0,1)$ and a monotone increasing function $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ such that $\varphi(0) = 0$ and for all x, $y \in Y$, we have*

$$\|Tx - Ty\| \leq \delta \|Sx - Sy\| + \varphi(\|Sx - Tx\|). \tag{2}$$

Definition 5 *([1]) Let $T, \widetilde{T} : X \to X$ be two operators. We say that $\widetilde{T}$ is an approximate operator of T if for all $x \in X$ and for a fixed $\varepsilon > 0$, we have*

$$\left\| Tx - \widetilde{T}x \right\| \leq \varepsilon.$$

Lemma 1 *([17]) Let $\{\sigma_n\}_{n=0}^{\infty}$ and $\{\rho_n\}_{n=0}^{\infty}$ be nonnegative real sequences satisfying the following inequality:*

$$\sigma_{n+1} \leq (1 - \lambda_n) \sigma_n + \rho_n,$$

where $\lambda_n \in (0,1)$, for all $n \geq n_0$, $\sum_{n=1}^{\infty} \lambda_n = \infty$, and $\frac{\rho_n}{\lambda_n} \to 0$ as $n \to \infty$. Then $\lim_{n\to\infty} \sigma_n = 0$.

Lemma 2 *([14]) Let $\{\sigma_n\}_{n=0}^{\infty}$ be a nonnegative sequence of real numbers. Assume there exists $n_0 \in \mathbb{N}$, such that for all $n \geq n_0$ one has the inequality*

$$\sigma_{n+1} \leq (1 - \mu_n) \sigma_n + \mu_n \gamma_n,$$

where $\mu_n \in (0,1)$, *for all* $n \in \mathbb{N}$, $\sum\limits_{n=0}^{\infty} \mu_n = \infty$ *and* $\gamma_n \geq 0$, $\forall n \in \mathbb{N}$. *Then the following inequality holds:*

$$0 \leq \limsup_{n\to\infty} \sigma_n \leq \limsup_{n\to\infty} \gamma_n.$$

2. Convergence and stability results

For the sake of simplicity, we make the following assumptions in the rest of the paper: $S, T : Y \to X$ satisfies contractive condition (2), where $T(Y) \subseteq S(Y)$, $S(Y)$ is a complete subspace of X and $C(S,T)$ denotes the set of coincidence points of S and T.

Theorem 1 *Let* $\{Sx_n\}_{n=0}^{\infty}$ *be the Jungck–Khan iterative scheme (1) with* $\sum\limits_{n=0}^{\infty} \alpha_n = \infty$. *Suppose that there exists a* $z \in C(S,T)$ *such that* $Sz = Tz = p$ *(say). Then* $\{Sx_n\}_{n=0}^{\infty}$ *converges strongly to* p. *Moreover,* p *is the unique common fixed point of the pair* (S,T) *provided* $Y = X$, *and* S *and* T *are weakly compatible.*

Proof. It follows from (1) and (2) that

$$\|Sx_{n+1} - p\| \leq (1 - \alpha_n - \beta_n) \|Sx_n - p\| + \alpha_n \|Ty_n - p\| + \beta_n \|Tx_n - p\|, \tag{3}$$

$$\|Tx_n - p\| \leq \delta \|Sx_n - p\|, \tag{4}$$

$$\|Sz_n - p\| \leq [1 - a_n (1 - \delta)] \|Sx_n - p\|, \tag{5}$$

$$\|Tz_n - p\| \leq \delta [1 - a_n (1 - \delta)] \|Sx_n - p\|, \tag{6}$$

and

$$\|Ty_n - p\| \leq \delta \{(1 - b_n - c_n) + b_n\delta [1 - a_n (1 - \delta)] + c_n\delta\} \|Sx_n - p\|. \tag{7}$$

Combining (3)–(7), we get

$$\|Sx_{n+1} - p\| \leq \{1 - \alpha_n - \beta_n + \alpha_n\delta \{1 - b_n - c_n + b_n\delta [1 - a_n (1 - \delta)] + c_n\delta\} + \beta_n\delta\} \|Sx_n - p\|, \tag{8}$$

Since $1 - a_n (1 - \delta) \leq 1$ and $1 - (b_n + c_n)(1 - \delta) \leq 1$, (8) becomes

$$\begin{aligned} \|Sx_{n+1} - p\| &\leq \{1 - \alpha_n - \beta_n + \alpha_n\delta [1 - (b_n + c_n)(1 - \delta)] + \beta_n\delta\} \|Sx_n - p\| \\ &\leq [1 - (\alpha_n + \beta_n)(1 - \delta)] \|Sx_n - p\|. \end{aligned} \tag{9}$$

Since $\alpha_k \leq \alpha_k + \beta_k$ for all $k \in \mathbb{N}$, therefore we get

$$\sum_{n=0}^{k} \alpha_n \leq \sum_{n=0}^{k} (\alpha_n + \beta_n),$$

which implies when $k \to \infty$,

$$\sum_{n=0}^{\infty} \alpha_n \leq \sum_{n=0}^{\infty} (\alpha_n + \beta_n).$$

Thus assumption $\sum_{n=0}^{\infty} \alpha_n = \infty$ implies $\sum_{n=0}^{\infty} (\alpha_n + \beta_n) = \infty$. Now it can be seen easily that inequality (9) fulfills all the conditions of Lemma 1. An application of Lemma 1 to (9) gives $\lim_{n\to\infty} \|Sx_n - p\| = 0$.

Now we prove p is a unique common fixed point of S and T, when $Y = X$.

Assume there exists another coincidence point q of the pair (S, T). Then there exists $z^* \in X$ such that $Sz^* = Tz^* = q$. However,

$$0 \le \|p - q\| \le \|Tz - Tz^*\| \le \delta \|Sz - Sz^*\| + \varphi (\|Sz - Tz\|) = \delta \|p - q\|,$$

which implies $p = q$ as $\delta \in [0, 1)$. Since S and T are weakly compatible and $Sz = Tz = p$, so $Tp = TTz = TSz = STz$ and hence $Tp = Sp$. Therefore, Tp is a point of coincidence of S, T and as the point of coincidence is unique so $Tp = p$. Thus $Tp = Sp = p$ and therefore p is unique common fixed point of S and T.

We now prove that Jungk–Khan iterative scheme (1) is weak w^2-stable with respect to (S, T).

Theorem 2 *Let $\{Sx_n\}_{n=0}^{\infty}$ be the Jungck–Khan iterative scheme (1) with $\sum_{n=0}^{\infty} \alpha_n = \infty$. Suppose that there exists a $z \in C(S, T)$ such that $Sz = Tz = p$ (say) and $\{Sx_n\}_{n=0}^{\infty}$ converges strongly to p. Let $\{Su_n\}_{n=0}^{\infty} \subset X$ be an $S-$ equivalent sequence of $\{Sx_n\}_{n=0}^{\infty} \subset X$. Set*

$$\begin{cases} \varepsilon_n = \|Su_{n+1} - (1 - \alpha_n - \beta_n) Su_n - \alpha_n Tv_n - \beta_n Tu_n\|, \\ Sv_n = (1 - b_n - c_n) Su_n + b_n Tw_n + c_n Tu_n, \\ Sw_n = (1 - a_n) Su_n + a_n Tu_n, \text{ for all } n \in \mathbb{N}, \end{cases} \tag{10}$$

Then $\{Sx_n\}_{n=0}^{\infty}$ is weak w^2-stable with respect to (S, T).

Proof. The sequence $\{Sx_n\}_{n=0}^{\infty}$ will be weak w^2-stable with respect to (S, T) if $\lim_{n\to\infty} Su_n = p$. Let $\lim_{n\to\infty} \varepsilon_n = 0$.

It follows from (1), (2), and (10) that

$$\begin{aligned} \|Su_{n+1} - p\| &\le \|Su_{n+1} - Sx_{n+1}\| + \|Sx_{n+1} - p\| \\ &\le \|Su_{n+1} - (1 - \alpha_n - \beta_n) Su_n - \alpha_n Tv_n - \beta_n Tu_n\| \\ &\quad + (1 - \alpha_n - \beta_n) \|Sx_n - Su_n\| + \alpha_n \|Ty_n - Tv_n\| \\ &\quad + \beta_n \|Tx_n - Tu_n\| + \|Sx_{n+1} - p\|, \end{aligned} \tag{11}$$

$$\|Tx_n - Tu_n\| \le \delta \|Sx_n - Su_n\| + \varphi (\|Sx_n - Tx_n\|), \tag{12}$$

$$\|Ty_n - Tv_n\| \le \delta \|Sy_n - Sv_n\| + \varphi (\|Sy_n - Ty_n\|), \tag{13}$$

$$\begin{aligned} \|Sy_n - Sv_n\| &\le (1 - b_n - c_n) \|Sx_n - Su_n\| \\ &\quad + b_n \|Tz_n - Tw_n\| + c_n \|Tx_n - Tu_n\|, \end{aligned} \tag{14}$$

$$\|Tz_n - Tw_n\| \le \delta \|Sz_n - Sw_n\| + \varphi (\|Sz_n - Tz_n\|), \tag{15}$$

$$\|Sz_n - Sw_n\| \leq (1 - a_n)\|Sx_n - Su_n\| + a_n \|Tx_n - Tu_n\|, \tag{16}$$

Combining (11)–(16), we get

$$\begin{aligned}\|Su_{n+1} - p\| \quad &\leq \quad \varepsilon_n + \{1 - \alpha_n - \beta_n + \alpha_n\delta(1 - b_n - c_n) \\ &\quad + \alpha_n b_n \delta^2 [1 - a_n(1-\delta)] + \alpha_n c_n \delta^2 + \beta_n \delta\} \|Sx_n - Su_n\| \\ &\quad + \{\beta_n + \alpha_n a_n b_n \delta^2 + \alpha_n \delta c_n\} \varphi(\|Sx_n - Tx_n\|) \\ &\quad + \alpha_n \varphi(\|Sy_n - Ty_n\|) + \alpha_n \delta b_n \varphi(\|Sz_n - Tz_n\|) + \|Sx_{n+1} - p\|. \end{aligned} \tag{17}$$

Since $1 - a_n(1-\delta) \leq 1$ and $1 - (b_n + c_n)(1-\delta) \leq 1$, (17) becomes

$$\begin{aligned}\|Su_{n+1} - p\| \quad &\leq \quad \varepsilon_n + [1 - (\alpha_n + \beta_n)(1-\delta)] \|Sx_n - Su_n\| \\ &\quad + [\beta_n + \alpha_n \delta(a_n b_n \delta + c_n)] \varphi(\|Sx_n - Tx_n\|) \\ &\quad + \alpha_n \varphi(\|Sy_n - Ty_n\|) + \alpha_n b_n \delta \varphi(\|Sz_n - Tz_n\|) + \|Sx_{n+1} - p\|. \end{aligned} \tag{18}$$

Now we have

$$\|Sx_n - Tx_n\| \leq (1+\delta)\|Sx_n - p\|,$$

$$\|Sy_n - Ty_n\| \leq (1+\delta)[1 - (b_n + c_n)(1-\delta)]\|Sx_n - p\|,$$

$$\|Sz_n - Tz_n\| \leq (1+\delta)[1 - a_n(1-\delta)]\|Sx_n - p\|.$$

It follows from the assumption $\lim_{n\to\infty}\|Sx_n - p\| = 0$ that

$$\lim_{n\to\infty}\|Sx_n - Tx_n\| = \lim_{n\to\infty}\|Sy_n - Ty_n\| = \lim_{n\to\infty}\|Sz_n - Tz_n\| = 0.$$

As φ is continuous, so we have

$$\lim_{n\to\infty}\varphi(\|Sx_n - Tx_n\|) = \lim_{n\to\infty}\varphi(\|Sy_n - Ty_n\|) = \lim_{n\to\infty}\varphi(\|Sz_n - Tz_n\|) = 0. \tag{19}$$

Since $\{Su_n\}_{n=0}^{\infty}$, $\{Sx_n\}_{n=0}^{\infty} \subset X$ are $S-$equivalent sequences, therefore we have

$$\lim_{n\to\infty}\|Sx_n - Su_n\| = 0. \tag{20}$$

Now taking the limit on both sides of (18) and then using $\lim_{n\to\infty}\|Sx_n - p\| = 0$, (19), and (20) lead to $\lim_{n\to\infty}\|Su_{n+1} - p\| = 0$. Thus $\{Sx_n\}_{n=0}^{\infty}$ is weak w^2-stable with respect to (S,T).

Example 1 *Let $X = [0,1]$ be endowed with the usual metric. Define two operators T , $S : [0,1] \to [0,1]$ by $Tx = \frac{x}{4}$ and $Sx = x$ with a coincide point $p = 0$. It is clear that $T([0,1]) \subseteq S([0,1])$, and $S([0,1]) = [0,1]$ is a complete subspace of $[0,1]$. Now we show that the pair (S,T) satisfies condition (2) with $\delta = \frac{1}{4}$. To do this, define φ by $\varphi(t) = \frac{t}{4}$. Now φ is increasing, continuous, and $\varphi(0) = 0$. Therefore, for all $x, y \in [0,1]$, we have*

$$|Tx - Ty| = \left|\frac{x}{4} - \frac{y}{4}\right| \leq \frac{1}{4}\left|x - \frac{x}{4}\right| + \frac{1}{4}|x - y|,$$

or equivalently

$$0 \leq \left| x - \frac{x}{4} \right|,$$

which holds for all $x \in [0,1]$. *Thus the pair* (S,T) *satisfies condition (2).*

Let $\{Sx_n\}_{n=0}^{\infty}$ *be the sequence defined by Jungck–Khan iterative scheme (1) with* $\alpha_n = \beta_n = a_n = b_n = c_n = \frac{1}{n+2}$ *and* $x_0 \in [0,1]$. *Then we have*

$$z_n = Sz_n = \left(1 - \frac{1}{n+2}\right) x_n + \frac{1}{n+2}\frac{x_n}{4} = \left(1 - \frac{3}{4(n+2)}\right) x_n, \tag{21}$$

$$y_n = Sy_n = \left(1 - \frac{2}{n+2}\right) x_n + \frac{1}{n+2}\frac{z_n}{4} + \frac{1}{n+2}\frac{x_n}{4}, \tag{22}$$

$$x_{n+1} = Sx_{n+1} = \left(1 - \frac{2}{n+2}\right) x_n + \frac{1}{n+2}\frac{y_n}{4} + \frac{1}{n+2}\frac{x_n}{4}, \ \forall n \in \mathbb{N}. \tag{23}$$

Combining (21)–(23), we get that

$$x_{n+1} = Sx_{n+1} = \left(1 - \frac{3}{2}\left(\frac{1}{n+2} + \frac{1}{4(n+2)^2} + \frac{1}{32(n+2)^3}\right)\right) x_n, \ \forall n \in \mathbb{N}. \tag{24}$$

Let $t_n = \frac{3}{2}\left(\frac{1}{n+2} + \frac{1}{4(n+2)^2} + \frac{1}{32(n+2)^3}\right)$. *It is easy to see that* $t_n \in (0,1)$ *for all* $n \in \mathbb{N}$ *and* $\sum\limits_{n=0}^{\infty} t_n = \infty$. *Hence an application of Lemma 1 to (24) leads to* $\lim_{n\to\infty} x_n = 0 = S(0) = T(0)$.

To show that Jungck–Khan iterative scheme (1) is weak w^2-*stable with respect to* (S,T), *we use the sequence* $\{Sy_n\}$ *defined by* $Sy_n = \frac{1}{n+3}$. *It is clear that the sequence* $\{Sy_n\}$ *is an approximate of* $\{Sx_n\}$. *Then*

$$\begin{aligned}
\varepsilon_n &= |Sy_{n+1} - f(T, y_n)| \\
&= \left| y_{n+1} - \left(1 - \frac{3}{2}\left(\frac{1}{n+2} + \frac{1}{4(n+2)^2} + \frac{1}{32(n+2)^3}\right)\right) y_n \right| \\
&= \left| \frac{1}{n+4} - \left(1 - \frac{3}{2}\left(\frac{1}{n+2} + \frac{1}{4(n+2)^2} + \frac{1}{32(n+2)^3}\right)\right) \frac{1}{n+3} \right| \\
&= \frac{32n^3 + 408n^2 + 1299n + 1228}{64(n+3)(n+4)(n+2)^3}.
\end{aligned}$$

Clearly, $\lim_{n\to\infty} \varepsilon_n = 0$. *Therefore, Jungck–Khan iterative scheme (1) is weak* w^2-*stable with respect to* (S,T).

3. Data dependency

The study of data dependence of fixed points in a normed space setting has become a new trend (see [2–4,6,8,10,12–14] and references therein). For data dependency of fixed points, the reader is referred to the book by Berinde [1].

Definition 6 *Let* (S,T), $\left(\widetilde{S},\widetilde{T}\right):Y\to X$ *be nonself operator pairs on an arbitrary set* Y *such that* $T(Y)\subseteq S(Y)$ *and* $\widetilde{T}(Y)\subseteq\widetilde{S}(Y)$. *We say that the pair* $\left(\widetilde{S},\widetilde{T}\right)$ *is an approximate operator pair of* (S,T) *if for all* $x\in X$ *and for fixed* $\varepsilon_1>0$ *and* $\varepsilon_2>0$, *we have*

$$\left\|Tx-\widetilde{T}x\right\|\le\varepsilon_1,\ \left\|Sx-\widetilde{S}x\right\|\le\varepsilon_2.$$

Theorem 3 *Let* $\left(\widetilde{S},\widetilde{T}\right):Y\to X$ *be an approximate operator pair of the pair* $(S,T):Y\to X$ *satisfying contractive condition (2). Suppose that* $\widetilde{S}(Y)$ *is a complete subspace of* X. *Let* $z\in C(S,T)$ *and* $\widetilde{z}\in C(\widetilde{S},\widetilde{T})$ *be the coincidence points of* S , T *and* $\widetilde{S}$, $\widetilde{T}$ *respectively, that is,* $Sz=Tz=p$ *and* $\widetilde{S}z=\widetilde{T}z=\widetilde{p}$. *Let* $\{Sx_n\}_{n=0}^{\infty}$ *be the Jungck–Khan iterative scheme (1) with* $\sum_{n=0}^{\infty}\alpha_n=\infty$ *and* $\left\{\widetilde{S}\widetilde{x}_n\right\}_{n=0}^{\infty}$ *a sequence defined by*

$$\begin{cases}\widetilde{x_0}\in X,\\ \widetilde{S}\widetilde{x}_{n+1}=(1-\alpha_n-\beta_n)\,\widetilde{S}\widetilde{x}_n+\alpha_n\widetilde{T}\widetilde{y}_n+\beta_n\widetilde{T}\widetilde{x}_n,\\ \widetilde{S}\widetilde{y}_n=(1-b_n-c_n)\,\widetilde{S}\widetilde{x}_n+b_n\widetilde{T}\widetilde{z}_n+c_n\widetilde{T}\widetilde{x}_n,\\ \widetilde{S}\widetilde{z}_n=(1-a_n)\,\widetilde{S}\widetilde{x}_n+a_n\widetilde{T}\widetilde{x}_n,\ n\in\mathbb{N}.\end{cases}\tag{25}$$

Assume that $\{Sx_n\}_{n=0}^{\infty}$ *and* $\left\{\widetilde{S}\widetilde{x}_n\right\}_{n=0}^{\infty}$ *converge to* p *and* $\widetilde{p}$, *respectively. Then we have*

$$\|p-\widetilde{p}\|\le\frac{8\varepsilon}{1-\delta},$$

where $\varepsilon=\max\{\varepsilon_1,\varepsilon_2\}$.

Proof. Using the same arguments as in the proof of ([8], Theorem 4.1), we have

$$\begin{aligned}\left\|Sx_{n+1}-\widetilde{S}\widetilde{x}_{n+1}\right\| &\le (1-\alpha_n-\beta_n)\left\|Sx_n-\widetilde{S}\widetilde{x}_n\right\|\\ &\quad+\alpha_n\left\|Ty_n-\widetilde{T}\widetilde{y}_n\right\|+\beta_n\left\|Tx_n-\widetilde{T}\widetilde{x}_n\right\|,\end{aligned}\tag{26}$$

$$\left\|Ty_n-\widetilde{T}\widetilde{y}_n\right\|\le\delta\left\|Sy_n-S\widetilde{y}_n\right\|+\varphi\left(\|Sy_n-Ty_n\|\right)+\varepsilon_1,\tag{27}$$

$$\left\|Tx_n-\widetilde{T}\widetilde{x}_n\right\|\le\delta\left\|Sx_n-\widetilde{S}\widetilde{x}_n\right\|+\varphi\left(\|Sx_n-Tx_n\|\right)+\delta\varepsilon_2+\varepsilon_1,\tag{28}$$

$$\begin{aligned}\|Sy_n-S\widetilde{y}_n\| &\le (1-b_n-c_n)\left\|Sx_n-\widetilde{S}\widetilde{x}_n\right\|\\ &\quad+b_n\left\|Tz_n-\widetilde{T}\widetilde{z}_n\right\|+c_n\left\|Tx_n-\widetilde{T}\widetilde{x}_n\right\|+\varepsilon_2,\end{aligned}\tag{29}$$

$$\left\|Tz_n-\widetilde{T}\widetilde{z}_n\right\|\le\delta\left\|Sz_n-S\widetilde{z}_n\right\|+\varphi\left(\|Sz_n-Tz_n\|\right)+\varepsilon_1,\tag{30}$$

$$\begin{aligned}\|Sz_n - S\widetilde{z}_n\| &\leq [1 - a_n(1-\delta)]\left\|Sx_n - \widetilde{S}\widetilde{x}_n\right\| \\ &\quad + a_n\varphi(\|Sx_n - Tx_n\|) + a_n(\delta\varepsilon_2 + \varepsilon_1) + \varepsilon_2. \end{aligned} \tag{31}$$

Combining (26)–(31), we get

$$\begin{aligned}\left\|Sx_{n+1} - \widetilde{S}\widetilde{x}_{n+1}\right\| &\leq \{1 - \alpha_n - \beta_n + \beta_n\delta + \alpha_n\delta(1 - b_n - c_n) \\ &\quad + \alpha_n\delta^2(c_n + b_n[1 - a_n(1-\delta)])\}\left\|Sx_n - \widetilde{S}\widetilde{x}_n\right\| \\ &\quad + \alpha_n\delta b_n\varphi(\|Sz_n - Tz_n\|) + \alpha_n\varphi(\|Sy_n - Ty_n\|) \\ &\quad + \left[\alpha_n\delta^2 b_n a_n + \alpha_n\delta c_n + \beta_n\right]\varphi(\|Sx_n - Tx_n\|) \\ &\quad + \left[\alpha_n\delta^2 b_n a_n + \alpha_n\delta b_n + \alpha_n\delta c_n + \alpha_n + \beta_n\right](\delta\varepsilon_2 + \varepsilon_1). \end{aligned} \tag{32}$$

As $\alpha_n,\ \beta_n,\ a_n,\ b_n,\ c_n,\ \alpha_n + \beta_n,\ b_n + c_n \in [0,1]$ for all $n \in \mathbb{N}$, and $\delta \in [0,1)$, so we have

$$\begin{cases} 1 - a_n(1-\delta) < 1, \\ 1 - (b_n + c_n)(1-\delta) < 1, \\ \alpha_n \leq \alpha_n + \beta_n, \\ \beta_n \leq \alpha_n + \beta_n, \\ \left[\delta^2 b_n a_n + \delta b_n + \delta c_n + 1\right](\delta\varepsilon_2 + \varepsilon_1) < 4(\varepsilon_2 + \varepsilon_1). \end{cases} \tag{33}$$

An application of inequalities in (33) to (32) gives

$$\begin{aligned}\left\|Sx_{n+1} - \widetilde{S}\widetilde{x}_{n+1}\right\| &\leq [1 - (\alpha_n + \beta_n)(1-\delta)]\left\|Sx_n - \widetilde{S}\widetilde{x}_n\right\| \\ &\quad + (\alpha_n + \beta_n)\{\delta b_n\varphi(\|Sz_n - Tz_n\|) + \varphi(\|Sy_n - Ty_n\|) \\ &\quad + \left[\delta^2 b_n a_n + \delta c_n + 1\right]\varphi(\|Sx_n - Tx_n\|) + 4(\varepsilon_2 + \varepsilon_1)\}. \end{aligned} \tag{34}$$

Define

$$\begin{aligned}\sigma_n &= \left\|Sx_n - \widetilde{S}\widetilde{x}_n\right\|, \\ \mu_n &= (\alpha_n + \beta_n)(1-\delta) \in (0,1), \\ \gamma_n &= \frac{\begin{bmatrix} \delta b_n\varphi(\|Sz_n - Tz_n\|) + \varphi(\|Sy_n - Ty_n\|) \\ + \left[\delta^2 b_n a_n + \delta c_n + 1\right]\varphi(\|Sx_n - Tx_n\|) + 4(\varepsilon_2 + \varepsilon_1) \end{bmatrix}}{1-\delta} \end{aligned}$$

Thus, (34) becomes

$$\sigma_{n+1} \leq (1 - \mu_n)\sigma_n + \mu_n\gamma_n. \tag{35}$$

As in the proof of Theorem 1, the assumption $\sum_{n=0}^{\infty}\alpha_n = \infty$ implies $\sum_{n=0}^{\infty}(\alpha_n + \beta_n) = \infty$. It is easy to check that $\sigma_n,\ \mu_n$, and γ_n satisfy all the conditions of Lemma 2. Also as in the proof of Theorem 2, we have

$$\lim_{n\to\infty}\varphi(\|Sx_n - Tx_n\|) = \lim_{n\to\infty}\varphi(\|Sy_n - Ty_n\|) = \lim_{n\to\infty}\varphi(\|Sz_n - Tz_n\|) = 0.$$

Hence an application of Lemma 2 to (35) leads to

$$\|p-\widetilde{p}\| \leq \frac{8\varepsilon}{1-\delta},$$

where $\varepsilon = \max\{\varepsilon_1, \varepsilon_2\}$.

Remark 1 *In this revisit of [8], we have:*

1. *Proved Theorem 1 in a slightly different way than Theorem 2.1;*
2. *Established ([8], Theorem 4.1) without the condition $\beta_n \leq \alpha_n$ for all $n \in \mathbb{N}$ in Theorem 3.*

Remark 2 *In the definition of stability, the sequence $\{Sy_n\}_{n\in\mathbb{N}}$ is taken as an arbitrary sequence, say $Sy_n = \frac{n}{n+1}$. Now using $Sy_n = \frac{n}{n+1}$ in place of $Sy_n = \frac{1}{n+3}$ in Example 1, we obtain*

$$\begin{aligned}
\varepsilon_n &= |Sy_{n+1} - f(T, y_n)| \\
&= \left| y_{n+1} - \left(1 - \frac{3}{2}\left(\frac{1}{n+2} + \frac{1}{4(n+2)^2} + \frac{1}{32(n+2)^3}\right)\right) y_n \right| \\
&= \left| \frac{n+1}{n+2} - \left(1 - \frac{3}{2}\left(\frac{1}{n+2} + \frac{1}{4(n+2)^2} + \frac{1}{32(n+2)^3}\right)\right) \frac{n}{n+1} \right|,
\end{aligned}$$

which implies $\varepsilon_n \to 0$ as $n \to \infty$. However, $\lim_{n\to\infty} Sy_n = \lim_{n\to\infty} \frac{n}{n+1} = 1$. Therefore, $\lim_{n\to\infty} \varepsilon_n = 0$ does not imply $\lim_{n\to\infty} Sy_n = 0$ for an arbitrary sequence $\{Sy_n\}_{n\in\mathbb{N}}$. Thus the Jungck–Khan iterative scheme (1) is not stable.

Here we have improved the stability result in [8] for weakly w^2-stability. The new result is supported by a numerical example.

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References

[1] Berinde V. Iterative Approximation of Fixed Points. Berlin, Germany: Springer, 2007.

[2] Chugh R, Kumar V. Data dependence of Noor and SP iterative schemes when dealing with quasi-contractive operators. Int J Comput Appl 2011; 40: 41-46.

[3] Gürsoy F. Applications of normal S-iterative method to a nonlinear integral equation. The Scientific World Journal 2014; 2014: 5 pages.

[4] Gürsoy F, Karakaya V, Rhoades BE. Data dependence results of new multi-step and S-iterative schemes for contractive-like operators. Fixed Point Theory A 2013; 76: 1-12.

[5] Jungck G, Hussain N. Compatible maps and invariant approximations. J Math Anal Appl 2007; 325: 1003-1012.

[6] Karakaya V, Gürsoy F, Doğan K, Ertürk M. Data dependence results for multistep and CR iterative schemes in the class of contractive-like operators. Abstr Appl Anal 2013; 2013: 1-7.

[7] Khan AR. On modified Noor iterations for asymptotically nonexpansive mappings. Bull Belg Math Soc Simon Stevin 2010; 17: 127-140.

[8] Khan AR, Kumar V, Hussain N. Analytical and numerical treatment of Jungck-type iterative schemes. Appl Math Comput 2014; 231: 521-535.

[9] Olatinwo MO. Some stability and strong convergence results for the Jungck-Ishikawa iteration process. Creative Math. & Inf 2008; 17: 33-42.

[10] Öztürk Çeliker F. Convergence analysis for a modified SP iterative method. The Scientific World Journal 2014; 2014: 5 pages.

[11] Singh SL, Bhatnagar C, Mishra SN. Stability of Jungck-type iterative procedures. Int J Math Math Sci 2005; 19: 3035-3043.

[12] Şoltuz SM. Data dependence for Mann iteration. Octogon Mathematical Magazine 2001; 9: 825-828.

[13] Şoltuz SM. Data dependence for Ishikawa iteration. Lecturas Matemáticas 2004; 25: 149-155.

[14] Şoltuz SM, Grosan T. Data dependence for Ishikawa iteration when dealing with contractive like operators. Fixed Point Theory A 2008; 2008: 1-7.

[15] Timiş I. On the weak stability of Picard iteration for some contractive type mappings and coincidence theorems. Int J Comput Appl 2012; 37: 9-13.

[16] Timiş I, Berinde V. Weak stability of iterative procedures for some coincidence theorems. Creative Math & Inf 2010; 19: 85-95.

[17] Weng X. Fixed point iteration for local strictly pseudocontractive mapping. Proc Amer Math Soc 1991; 113: 727-731.