Turkish Journal of Mathematics
http://journals.tubitak.gov.tr/math/
Research Article

Turk J Math
(2016) 40: 647 – 656
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doi:10.3906/mat-1505-67

Further results on edge - odd graceful graphs

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Received: 16.05.2015 • **Accepted/Published Online:** 11.10.2015 • **Final Version:** 08.04.2016

Abstract: We present edge - odd graceful labeling for the following families of graphs: wheel W_n, for $n \equiv 1,\ 2$ and $3\ mod\ 4$; $C_n \bigodot \bar{K}_{2m-1}$; even helms; $P_n \bigodot \bar{K}_{2m}$ and $K_{2,s}$. Also we present two theorems of non edge - odd graceful graphs and an idea to label complete graphs.

Key words: Edge - odd graceful labeling, Cycles, Symmetrical tree, Corona

1. Introduction

Solairaju and Chithra [3] defined a graph G with q edges to be edge-odd graceful if there is a bijection f from the edges of the graph to $\{1,\ 3,\ 5,\ ...,\ 2q\ -1\}$ such that, when each vertex is assigned the sum of all the edges incident to it $mod\ 2q$, the resulting vertex labels are distinct.

They prove that the following graphs are odd-graceful: paths with at least 3 vertices; odd cycles; ladders $P_n \times P_2$ $(n \geq 3)$; stars with an even number of edges; and crowns $C_n \bigodot K_1$. In [4] they prove the following graphs have edge-odd graceful labelings: P_n $(n > 1)$ with a pendant edge attached to each vertex (combs); the graph obtained by appending $2n+1$ pendant edges to each endpoint of P_2 or P_3; and the graph obtained by subdividing each edge of the star $K_{1,2n}$.

In this paper we try to give some further results: we present an edge - odd graceful labeling to each of these families: wheel W_n for $n \equiv 1,\ 2$ and $3\ mod\ 4$; $C_n \odot \overline{K}_{2m-1}$; even helms; $P_n \odot \overline{K}_{2m}$ and $K_{2,s}$. We prove that the trees of odd number of vertices and odd degrees can't be an edge - odd graceful graphs. Also we prove that the cycle C_n is not an edge - odd graceful graph when n is even. Finally we give a simple way to label complete graphs which provides edge - odd graceful labelings to a good number of complete graphs K_n within the range $n \in \{4,\ 5,\ 6,\ \ldots,\ 99,\ 100\}$.

2. Main results

Definition 2.1 [1, 3, 4] *A graph $G(V(G),\ E(G))$ with p vertices and q edges is said to have an edge - odd graceful labeling if there exists a bijection f from $E(G)$ to $\{1,\ 3,\ 5,\ \ldots,\ 2q-1\}$ so that the induced mapping f^+ from $V(G)$ to $\{0,\ 1,\ 2,\ \ldots,\ 2q-1\}$, given by: $f^+(v) = \left(\sum_{u \in V(G)} f(vu)\right) mod(2q)$, is an injective.*

We call the graph that admits an edge - odd graceful labeling: an edge - odd graceful graph.

Example 2.2 *In Figure 1 we present an edge - odd graceful labeling of the Peterson graph.*

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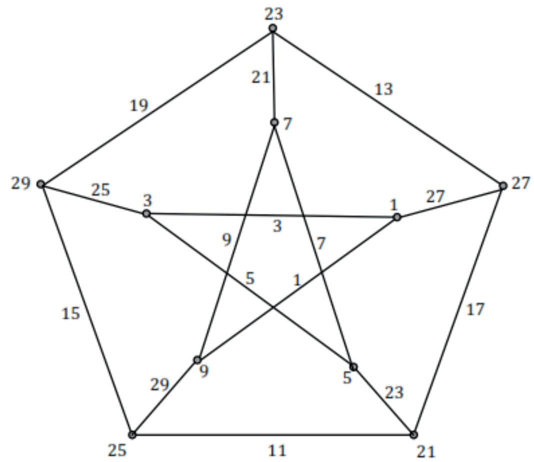

Figure 1. Peterson Graph

Theorem 2.3 *The wheel* W_n *is an edge - odd graceful graph when* n *is even and* $n \equiv 1$ *mod* 4.

Proof Let the wheel W_n be given as indicated in *Figure 2*:

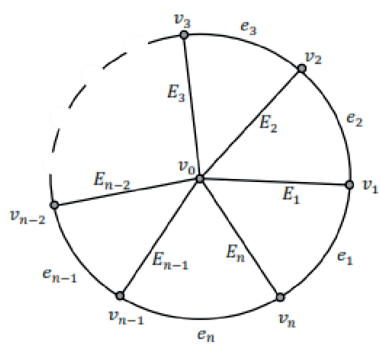

Figure 2.

We define the labeling function f as follows:

$$f(e_i) = 4n - 2i + 1; i = 1, 2, \dots, n$$

$$f(E_i) = 2i - 1; i = 1, 2, 3, \dots, n.$$

We realize that:

$f^+(v_i) = (f(e_i) + f(E_i) + f(e_{i+1}))\ mod\,(4n) = (4n + f(e_{i+1}))\,mod\,(4n) = f(e_{i+1})\,; i = 1, 2, 3, \dots, n-1$, and

$f^+(v_n) = (f(e_n) + f(E_n) + f(e_1))\ mod\,(4n) = (4n + f(e_1))\,mod\,(4n) = f(e_1)\,.$

$$f^+(v_0) = \sum_{i=1}^{n} f(E_i) mod\,(4n) = \sum_{i=1}^{n} (2i-1) mod\,(4n) = n^2\ mod\,(4n)$$

$$\Longrightarrow f^{+}(v_0) \equiv \begin{cases} 0 \; mod\,(4n)\,, \; n \equiv 0 \; mod \; 4 \\ 2n \; mod \; (4n)\,, \; n \equiv 2 \; mod \; 4 \\ n \; mod \; (4n)\,, \; n \equiv 1 \; mod \; 4 \end{cases}$$

We can see that in these three cases there is no repetition in the vertex labels which completes the proof. □

Example 2.4 *In Figure 3 we present the labeling of* W_5 *and* W_6 *according to Theorem 2.3 .*

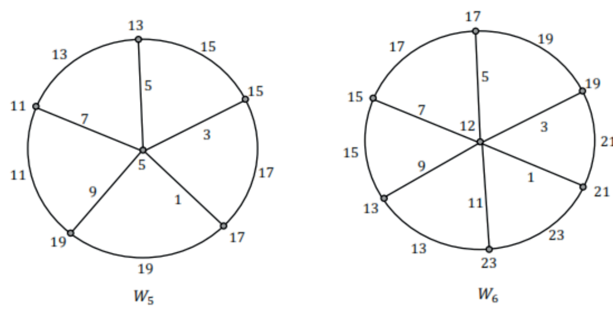

Figure 3.

Theorem 2.5 *The corona [2]* $C_n \odot \overline{K}_{2m-1}$ *is an edge - odd graceful graph.*

Proof Let the vertex and the edge symbols be given as in *Figure 4*:

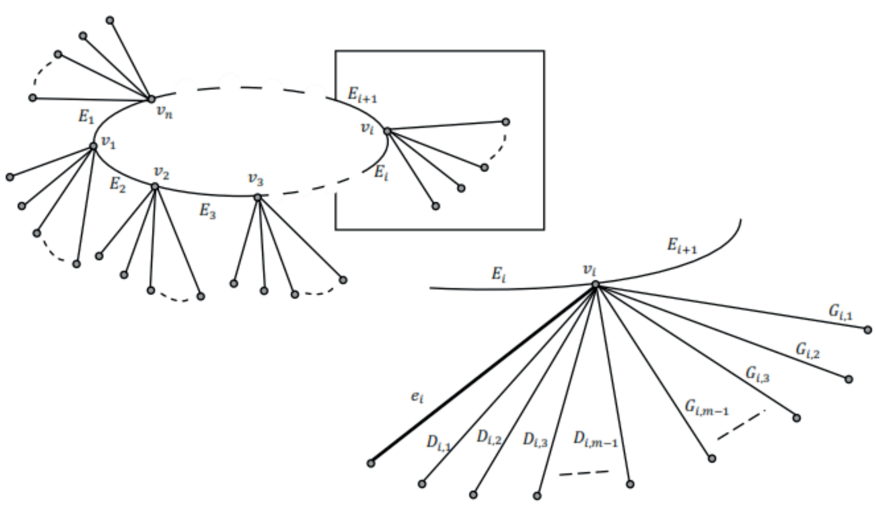

Figure 4.

We define the edge labeling function f:

$$f(E_i) = 2(i-1)m + 1, \;\; f(e_i) = 2q - (2(i-1)m + 1)\,, \;\; i = 1, \; 2, \; 3, \; \ldots, \; n.$$

$$f(D_{i,j}) = 2q - (2(i-1)m + 2j + 1)\,, \;\; j = 1, \; 2, \; 3, \; \ldots, m-1$$

$$f\left(G_{i,j}\right)=2\left(i-1\right)m+2j+1,\ j=1,\ 2,\ 3,\ \ldots,m-1.$$

We realize the following: $f\left(D_{i,j}\right)+f\left(G_{i,j}\right)\equiv 0\ mod\left(2q\right),\ j=1,\ 2,\ 3,\ \ldots,\ m-1$.

Also: $f\left(E_i\right)+f\left(e_i\right)\equiv 0\ mod\ (2q)$, so verifying the vertex labelings we get that:

$f^+\left(v_i\right)=\left(\sum_{j=1}^{m-1}f\left(D_{i,j}\right)+\sum_{j=1}^{m-1}f\left(G_{i,j}\right)+f\left(E_i\right)+f\left(e_i\right)+f\left(E_{i+1}\right)\right)mod(2q)=f\left(E_{i+1}\right),\ i=1,2,\ldots,\ n-1$, and similarly we get: $f^+\left(v_n\right)=f\left(E_1\right)$.

Each pendant vertex takes the labels of its incident edge. Overall all vertex labels are edge labels so there are no repeated vertex labels, which completes the proof. □

Example 2.6 *In Figure 5: we give* $C_3\odot K_5$ *labeled according to Theorem 2.5.*

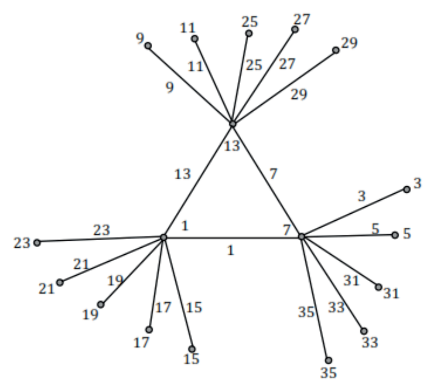

Figure 5. edge - odd graceful labeling of $C_3\odot K_5$

Theorem 2.7 *The helm* H_n *[1] (i.e. the graph obtained from a wheel by attaching a pendant edge at each vertex of the* n*-cycle) is an edge - odd graceful for even* n.

Proof Let the helm H_n be given as indicated in *Figure 6*:

We define now the labeling f as follows:

$$f\left(E_i\right)=2i-1,\ i=1,\ 2,\ 3,\ \ldots,\ n$$

$$f\left(D_i\right)=6n-\left(2i-1\right),\ i=1,\ 2,\ 3,\ \ldots,\ n$$

$$f\left(G_i\right)=4n-\left(2i-1\right),\ i=1,\ 2,\ 3,\ \ldots,\ n$$

Let us now verify f^+ for the vertices:

$$f^+\left(v_i\right)=\left(f\left(G_i\right)+f\left(D_i\right)+f\left(E_{i-1}\right)+f\left(D_{i-1}\right)\right)\ mod\ \left(6n\right)=$$

$$=4n-4i+2\ mod\left(6n\right)=4n-4i+2,\ i=2,\ 3,\ \ldots,\ n.$$

Similarly we find: $f^+\left(v_1\right)=\left(f\left(G_1\right)+f\left(D_1\right)+f\left(E_n\right)+f\left(D_n\right)\right)\ mod\left(6n\right)=4n-2$.

$$f\left(v_0\right)=\sum_{i=1}^{n}f\left(E_i\right)mod\left(6n\right)=\sum_{i=1}^{n}\left(2i-1\right)mod\left(6n\right)=n^2\ mod\left(6n\right)$$

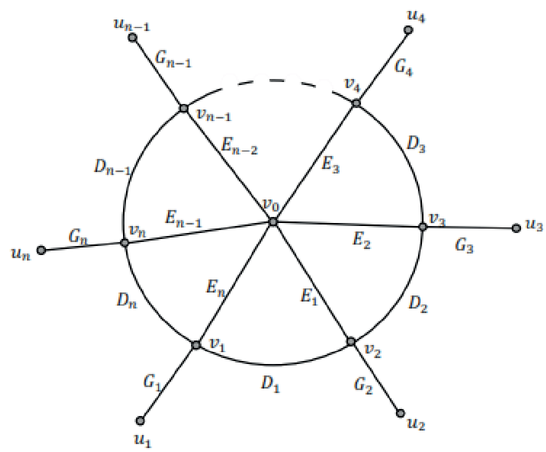

Figure 6.

$$\Longrightarrow f(v_0) = \begin{cases} 0 \ mod\,(6n), \ n \equiv 0 \ mod \ 6 \\ 2n \ mod(6n), \ n \equiv 2 \ mod \ 6 \\ 4n \ mod\,(6n), n \equiv 4 \ mod \ 6 \end{cases}$$

In all cases $f(v_0)$ is not congruent to $f^+(v_i)$ modulo $(6n)$, $\forall i = 1, 2, 3, \ldots, n$ which completes the proof. □

Example 2.8 *The graph H_6 labeled according to Theorem 2.7 is presented in Figure 7.*

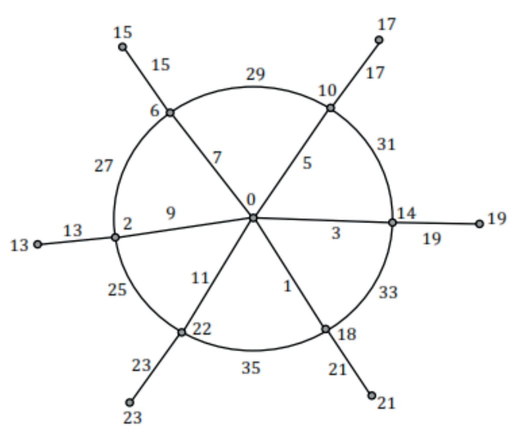

Figure 7.

Theorem 2.9 *The graph $P_n \odot \overline{K}_{2m}$, $n \geq 3$ is an edge - odd graceful graph.*

Proof Let the graph $P_n \odot \overline{K}_{2m}$ be given as indicated in *Figure 8*.

With $q = 2nm + n - 1$, we define the edge labeling function f:

$$f(D_{i,j}) = 2q - (2(i-1)m + 2j - 1), \quad j = 1, 2, 3, \ldots, m, \ i = 1, 2, 3, \ldots, n$$

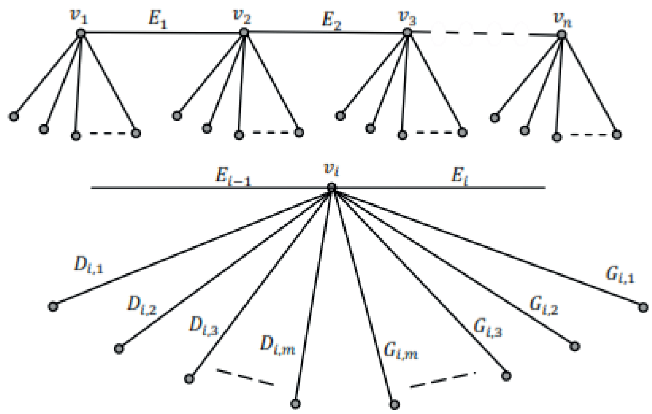

Figure 8.

$$f(G_{i,j}) = 2(i-1)m + 2j - 1, \quad j = 1,\ 2,\ 3,\ \ldots, m, i = 1,\ 2,\ 3,\ \ldots,\ n$$

$$= f(E_i) = 2nm + 2i - 1,\ i = 1,\ 2,\ 3,\ \ldots,\ n-1$$

We consider the labeling of vertices, so we find:

$f^+(v_1) = \left(\sum_{j=1}^{m} f(D_{1,j}) + \sum_{j=1}^{m} f(G_{1,j}) + f(E_1)\right)\ mod(2q) = f(E_1)$

$f^+(v_n) = \left(\sum_{j=1}^{m} f(D_{n,j}) + \sum_{j=1}^{m} f(G_{n,j}) + f(E_{n-1})\right)\ mod(2q) = f(E_{n-1})$

$f^+(v_i) = \left(\sum_{j=1}^{m} f(D_{i,j}) + \sum_{j=1}^{m} f(G_{i,j}) + f(E_{i-1}) + f(E_i)\right) mod\,(2q) =$

$= (f(E_{i-1}) + f(E_i))\, mod\,(2q) = (4nm + 4i - 4)\, mod(2q),\ i = 2, 3, \ldots,\ n-1$

hence $f^+(v_i) = \begin{cases} 4nm + 4i - 4,\ i = 2, 3, \ldots,\ \frac{n+1}{2} \\ 4\left(i - \frac{n+1}{2}\right),\ i = \frac{n+3}{2}, \frac{n+5}{2}, \ldots, n-1 \end{cases}$, when n is odd

or it follows that: $f^+(v_i) = \begin{cases} 4nm + 4i - 4,\ i = 2, 3, \ldots,\ \frac{n}{2} \\ 4\left(i - \frac{n}{2}\right) - 2,\ i = \frac{n+2}{2}, \frac{n+4}{2}, \ldots, n-1 \end{cases}$, when n is even.

In both cases all these labels are even and different numbers. □

Regarding the pendant vertices we can see that each of them takes the label of its edge, so they are all odd and different numbers.

Example 2.10 *In Figure 9: we give $P_4 \odot K_4$ labeled according to Theorem 2.9.*

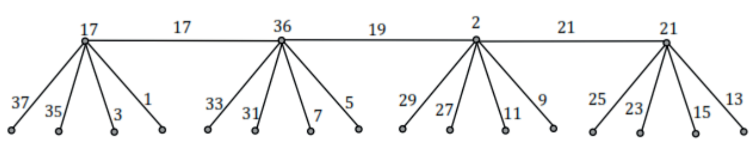

Figure 9. edge - odd graceful labeling of $P_4 \odot K_4$

Theorem 2.11 *The complete bipartite graph $K_{2,s}$ is an edge - odd graceful graph when s is odd.*

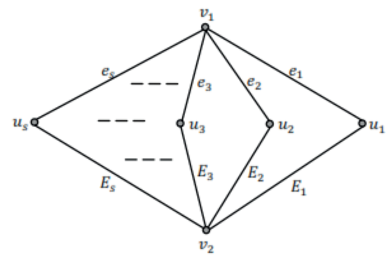

Figure 10.

Proof Let the complete bipartite graph $K_{2,s}$ be given as indicated in *Figure 10*.

We define the labeling function f as follows:

$$f(e_i) = 2i - 1,\ i = 1,\ 2,\ 3,\ \dots,\ s$$

$$f(E_i) = 2s + 2i - 1,\ = 1,\ 2,\ 3,\ \dots,\ s$$

Considering the vertex labels we find:

$$f(u_i) = (2s + 4i - 2)\ mod\,(4s) = \begin{cases} 2s + 4i - 2,\ i = 1,\ 2,\ \dots,\ \frac{s-1}{2} \\ 4\left(i - \frac{s+1}{2}\right),\ i = \frac{s+1}{2},\ \frac{s+3}{2}, \dots,\ s \end{cases}$$

$$f(v_1) = \left(\sum_{i=1}^{s} 2i - 1\right) mod\ \ (4s) = s^2\ mod\,(4s) = \begin{cases} s,\ s \equiv 1\ mod\ 4 \\ 3s,\ s \equiv 3\ mod\ 4 \end{cases}$$

$$f(v_2) = \left(\sum_{i=1}^{s} 2s + 2i - 1\right) mod\ \ (4s) = 3s^2\ mod\,(4s) = \begin{cases} 3s,\ s \equiv 1\ mod\ 4 \\ s,\ s \equiv 3\ mod\ 4 \end{cases}$$

Hence there are no repeated vertex labels. □

Example 2.12 *Figure 11 presents $K_{2,3}$ and $K_{2,5}$ labeled according to Theorem 2.11*

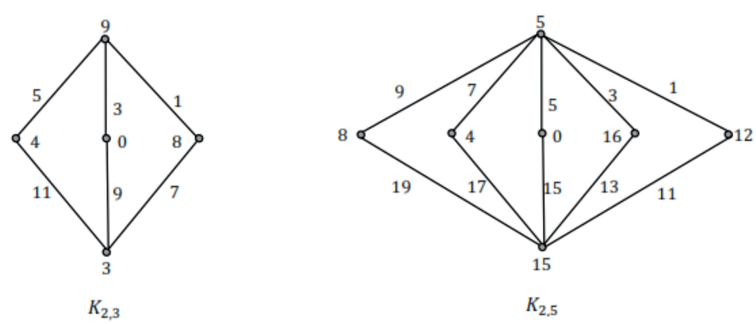

Figure 11.

3. Two theorems of non edge - odd graceful graphs

Theorem 3.1 *The tree in which all vertices have odd degrees is not edge - odd graceful.*

Proof When a vertex in a given tree has an odd degree then it must have an odd label, since its label is an odd summation of odd numbers modulo $2q$, but among the possible odd numbers of the set: $\{0,\ 1,\ 2,\ 3,\ \ldots,\ 2q-1\}$ we have only q different odd numbers, hence they will not be enough to cover the labels of $q+1$ vertices of this tree without repetition. □

Example 3.2 *The two trees shown in Figure 12 are not edge - odd graceful.*

Figure 12.

Theorem 3.3 *The cycle C_n is not an edge - odd graceful graph when n is even.*

Proof Suppose that an even cycle C_n admits an edge - odd graceful labeling, then the edge labels are the elements of the set: $\{1,\ 3,\ 5,\ \ldots,\ 2n-1\}$, so by the definition of the edge - odd graceful labeling we can deduce that the vertex labels must be exactly $0,\ 2,\ 4,\ \ldots, 2n-2\ (mod\ 2n)$. Now the summation of all vertex labels could be calculated twice:

1) $\sum_{v\in V(C_n)} f^*(v) = \left(2\sum_{e\in V(C_n)} f(e)\right) \equiv 2n^2 \equiv 0\ (mod\ 2n)$, (*since every edge label is calculated twice when we pass through all vertices*)

2) $\sum_{v\in V(C_n)} f^*(v) = \left(\sum_{i=0}^{2n-2} i\right) \equiv n(n-1) \equiv n\ (mod\ 2n)$

The difference in answers in 1) and 2) leads to a contradiction. □

4. Some edge - odd graceful complete graphs

Introductory Example 4.1 *Let A_4, A_5 and A_6 be the following given matrices.*

$$A_4 = \begin{pmatrix} v_1 & 1 & 3 & 5 \\ v_2 & 1 & 7 & 9 \\ v_3 & 3 & 7 & 11 \\ v_4 & 5 & 9 & 11 \end{pmatrix}, A_5 = \begin{pmatrix} v_1 & 1 & 3 & 5 & 7 \\ v_2 & 1 & 9 & 11 & 13 \\ v_3 & 3 & 9 & 15 & 17 \\ v_4 & 5 & 11 & 15 & 19 \\ v_5 & 7 & 13 & 17 & 19 \end{pmatrix},$$

$$A_6 = \begin{pmatrix} v_1 & 1 & 3 & 5 & 7 & 9 \\ v_2 & 1 & 11 & 13 & 15 & 17 \\ v_3 & 3 & 11 & 19 & 21 & 23 \\ v_4 & 5 & 13 & 19 & 25 & 27 \\ v_5 & 7 & 15 & 21 & 25 & 29 \\ v_6 & 9 & 17 & 23 & 27 & 29 \end{pmatrix}.$$

We will define a labeling of the complete graph K_n "g^+" according to the matrix A_n, $n = 4,\ 5$ and 6, as follows:

The label of the vertex v_i "$g^+(v_i)$", $i = 1,\ 2,\ \ldots,\ n$, is the summation of the numbers in its row, where these numbers are given as the labels of the edges that are incident to v_i. With realizing that: $g^+(v_1) = \sum_{j=1}^{n-1} 2j - 1 = (n-1)^2$ and

$$g^+(v_i) = g^+(v_{i-1}) + (n-i)(2n-2i) + 2(i-2),\ i = 2,\ 3,\ \ldots, n.$$

We make some calculations and discuss each one of these defined labelings as follows:

1) For A_4 and K_4 we see that: $g^+(v_3) = g^+(v_1) + 12 \Longrightarrow g^+(v_3) \equiv g^+(v_1)\, mod\,(2q = 12)$, so the defined labeling in this case is not edge - odd graceful labeling.

2) For A_5 and K_5 we see that: $g^+(v_5) = g^+(v_1) + 40 \Longrightarrow g^+(v_5) \equiv g^+(v_1)\, mod\,(2q = 20)$, also the defined labeling in this case is not edge - odd graceful labeling.

3) For A_6 and K_6 we see there are no identical vertex labels modulo $2q = 30$, i.e. the labeling g^+ represents an edge - odd graceful labeling of K_6.

Note 4.2 *In the following proposition we are going to generalize what is given in the previous example.*

Proposition 4.3 *Let g be a labeling of the complete graph K_n:*

$g : E(K_n) \longrightarrow \{1,\ 3,\ 5,\ \ldots,\ 2q-1 = n(n-1)-1\}$, for which the vertices have their incident labeled edges according to the following matrix:

$$\begin{pmatrix} v_1 & 1 & 3 & 5 & 7 & \ldots & 2n-5 & 2n-3 \\ v_2 & 1 & 2n-1 & 2n+1 & 2n+3 & \ldots & 4n-9 & 4n-7 \\ v_3 & 3 & 2n-1 & 4n-5 & 4n-3 & \ldots & 6n-15 & 6n-13 \\ v_4 & 5 & 2n+1 & 4n-5 & 6n-5 & \ldots & 8n-23 & 8n-21 \\ v_5 & 7 & 2n+3 & 4n-3 & 6n-5 & & 10n-33 & 10n-31 \\ \vdots & \vdots & \vdots & \vdots & \vdots & :::: & \vdots & \vdots \\ v_{n-1} & 2n-5 & 4n-9 & 6n-15 & 8n-23 & \ldots & 2n^2-(n^2+n+5) & 2n^2-(n^2+n+1) \\ v_n & 2n-3 & 4n-7 & 6n-13 & 8n-21 & \ldots & 2n^2-(n^2+n+3) & 2n^2-(n^2+n+1) \end{pmatrix}$$

The numbers in the row in front of the vertex v_i are the $n-1$ labels of the edges that are incident to it, $i = 1,\ 2, \ldots,\ n$. Let now g^+ be the mapping which is defined as follows: $g^+(v_i)$ is equal to the summation over all labels in its row.

We can see the following facts:

$$g^+(v_1) = \sum_{j=1}^{n-1} 2j - 1 = (n-1)^2$$

$$g^{+}(v_i) = g^{+}(v_{i-1}) + (n-i)(2n-2i) + 2(i-2),\ i = 2,\ 3,\ \ldots, n.$$

We will test now whether or not $2q = n(n-1)$ divides:

1) $(n-i)(2n-2i) + 2(i-2),\ i = 2,\ 3,\ \ldots, n$

2) $\sum_{j=2}^{i}(n-j)(2n-2j) + 2(j-2),\ i = 3,\ \ldots, n$.

If $n(n-1)$ does not divide these numbers, then we will not find different vertices with the same label "*as in case 3) in Introductory Example 4.1*" and then the previous labeling will be an edge - odd graceful labeling, and $g^{+} \equiv f^{+}$.

We do these tests using *Mathematica* until we reach K_{100}, so we get that: K_n is edge - odd graceful graph for $n =$

$n =$	6	9	13	15	16	18	19	21	22	27
$n =$	28	30	31	33	34	36	37	39	40	42
$n =$	43	45	46	48	49	51	52	54	55	57
$n =$	58	60	61	64	66	67	69	70	72	73
$n =$	75	76	78	79	81	82	84	85	90	91
$n =$	93	94	96	97	99	100	*	*	*	*

Note 4.4 *Proposition 4.3 suggests that we can find complete graphs of some orders which are edge - odd graceful, which means that there is no upper bound for the number of edges for these orders.*

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