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Point-wise slant submanifolds in almost contact geometry

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Abstract: In this paper, we introduce point-wise slant submanifolds of almost contact and almost contact 3-structure manifolds. We characterize them, give some examples, and obtain necessary and sufficient conditions for a point-wise slant submanifold of a 3-Sasakian manifold to be a slant submanifold. Moreover, we show that there exist no proper Sasakian point-wise 3-slant submanifolds.

Key words: Sasakian manifold, point-wise slant submanifold

1. Introduction

Chen [4] generalized the notion of totally real and holomorphic submanifolds in complex geometry by introducing slant submanifolds. In [11], Lotta studied slant submanifolds of contact manifolds that were the generalization of invariant and anti-invariant submanifolds. Since then, many authors have obtained important and interesting results about slant submanifolds of complex [13, 14, 16, 17] and almost contact [1, 3, 8, 12] manifolds.

On the other hand, Etayo [6] has extended this type of submanifold by defining quasi-slant submanifolds. In such submanifolds, at any given point, the slant angle is independent of the choice of any nonzero vector field of submanifold. Later, Chen and Garay [5] studied and characterized these submanifolds under the name point-wise slant submanifolds. Recently, Sahin showed [15] the existence of warped product point-wise semislant submanifolds of Kaehler manifolds, contrary to the semislant case. He and Lee [10] also investigated point-wise slant submersion from almost Hermitian manifolds.

The importance of slant submanifolds in almost contact geometry motivated us to define point-wise slant submanifolds of an almost contact and an almost contact 3-structure manifold. We characterized them and obtained necessary and sufficient conditions for a point-wise slant submanifold of a 3-Sasakian to have constant slant function.

This paper is organized as follows. In Section 2, we review some basic information about almost contact and 3-structure manifolds. In Section 3, we define and characterize point-wise slant submanifolds of almost contact and 3-structure manifolds. In Section 4, some properties of point-wise slant submanifolds of Sasakian and 3-Sasakian manifolds are studied. Moreover, it is shown that Sasakian point-wise 3-slant submanifolds are 3-slant.

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2. Almost contact metric 3-structure manifolds

Let $\overline{M}$ be a $(2m+1)$ dimensional manifold and ϕ, ξ, and η be a tensor field of type $(1,1)$, a vector field, and a 1-form on $\overline{M}$, respectively. If ϕ, ξ, and η satisfy

$$\eta(\xi) = 1$$

$$\phi^2(X) = -X + \eta(X)\xi \tag{1}$$

for any vector field X on $\overline{M}$, then $\overline{M}$ is said to have an almost contact structure (ϕ, ξ, η).
If g is a compatible Riemannian metric on $\overline{M}$ such that

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \tag{2}$$

then $(\overline{M}, \xi, \eta, \phi, g)$ is called an almost contact metric structure. Eqs. (1) and (2) imply $\phi\xi = 0$ and $\eta o \phi = 0$ [2].

A Sasakian manifold is an important type of this structure and is defined as follows. Let $\overline{\nabla}$ be the Levi-Civita connection of $\overline{M}$. An almost contact metric manifold $(\overline{M}, \xi, \eta, \phi, g)$ is called a Sasakian manifold, if $\forall X, Y \in T\overline{M}$

$$(\overline{\nabla}_X \phi)Y = g(X, Y)\xi - \eta(Y)X \text{ and } \overline{\nabla}_X \xi = -\phi X. \tag{3}$$

Definition 1 *[9] Let there exist three almost contact metric structures* $(\xi_i, \eta_i, \phi_i, g)$, $i = 1, 2, 3$, *on* $\overline{M}$ *such that*

$$\eta_i(\xi_j) = 0, \quad \phi_i\xi_j = -\phi_j\xi_i = \xi_k, \quad \eta_i o \phi_j = -\eta_j o \phi_i = \eta_k, \tag{4}$$

$$\phi_i o \phi_j - \eta_j \otimes \xi_i = -\phi_j o \phi_i + \eta_i \otimes \xi_j = \phi_k, \tag{5}$$

$$g(\phi_i X, \phi_i Y) = g(X, Y) - \eta_i(X)\eta_i(Y), \ \forall X, Y \in T\overline{M}, \tag{6}$$

where (i, j, k) *is a cyclic permutation of* $(1, 2, 3)$. *Then* $(\overline{M}, \xi_i, \eta_i, \phi_i, g)_{i\in\{1,2,3\}}$ *is said to be an almost contact metric 3-structure manifold.*

In such manifolds we have

$$g(\phi_i X, Y) = -g(X, \phi_i Y). \tag{7}$$

Moreover, if

$$(\overline{\nabla}_X \phi_i)Y = g(X, Y)\xi_i - \eta_i(Y)X, \ \forall X, Y \in T\overline{M} \text{ and } \overline{\nabla}_X \xi_i = -\phi_i X, \tag{8}$$

then $(\overline{M}, \xi_i, \eta_i, \phi_i, g)_{i\in\{1,2,3\}}$ is called a 3-Sasakian manifold. It is well known that 3-Sasakian manifolds are Einstein manifolds [7] and so are important manifolds in mathematical physics.

Let M be an immersed submanifold of almost contact metric 3-structure manifold $(\overline{M}, \xi_i, \eta_i, \phi_i, g)_{i\in\{1,2,3\}}$. We denote its Levi-Civita connection and normal bundle by ∇ and $(TM)^{\perp}$, respectively. The Gauss and Weingarten formulas are given by

$$\overline{\nabla}_X Y = \nabla_X Y + B(X,Y) \quad \text{and} \quad \overline{\nabla}_X V = D_X V - A_V X, \tag{9}$$

for $X, Y \in TM$ and $V \in (TM)^{\perp}$, where D, B, and A are the connection in the normal bundle, the second fundamental form, and the shape operator, respectively.
Moreover, for any $X \in TM$ and $V \in (TM)^{\perp}$ we decompose the $\phi_i X$ and $\phi_i V$ as the following equations:

$$\phi_i X = T_i X + N_i X \text{ and } \phi_i V = t_i V + n_i V, \tag{10}$$

where T_i and t_i are tangential components of ϕ_i, and N_i and n_i are normal components of ϕ_i. If M is a submanifold of an almost contact metric manifold $(\overline{M}, \xi, \eta, \phi, g)$, in such a way, the decomposition of ϕ to tangential components T and t, and normal components N and n implies

$$\phi X = TX + NX \text{ and } \phi V = tV + nV. \tag{11}$$

3. Point-wise slant submanifolds of almost contact manifolds

Let M be a submanifold of an almost contact metric manifold $(\overline{M}, \xi, \eta, \phi, g)$. Then M is said to be a slant submanifold if the angle between ϕX and T_pM is constant at any point $p \in M$ and for any X linearly independent of ξ [3, 11].

The author and Malek introduced 3-slant submanifolds of an almost contact metric 3-structure manifold $(\overline{M}, \xi_i, \eta_i, \phi_i, g)_{i\in\{1,2,3\}}$ [12]. On these submanifolds for all $i = 1, 2, 3$, at any point $p \in M$ the angle between $\phi_i X$ and T_pM is constant for each $X \in T_pM$ linearly independent of ξ_i. In both previous definitions the angle is independent of the choice of p and X. Now we introduce the notion of point-wise slant submanifolds of almost contact manifolds by following the approach of [5] in almost Hermitian manifolds.

Definition 2 *Let M be a submanifold of an almost contact metric manifold $\overline{M}$. We say that M is a point-wise slant submanifold with slant angle $\Theta_p(X)$ if at any point $p \in M$ the Wirtinger angle between ϕX and T_pM is constant for each nonzero $X \in T_pM$ linearly independent of ξ. It means that the function $\Theta_p(X)$ does not depend on the choice of X.*

Definition 3 *Let M be a submanifold of an almost contact metric 3-structure manifold $(\overline{M}, \xi_i, \eta_i, \phi_i, g)_{i\in\{1,2,3\}}$. M is a point-wise 3-slant submanifold if at any point $p \in M$ and for each nonzero $X \in T_pM$ linearly independent of ξ_i, the Wirtinger angle between $\phi_i X$ and T_pM is constant for all $i \in \{1, 2, 3\}$. In fact, the angle $\Theta_p(X)$ between $\phi_i X$ and $T_j X$ only depends on the choice of p and it is independent of the choice of X and i, j.*

On these submanifolds $\Theta(X)$ can be considered a function called a slant function. If at a point $p \in M$, $\Theta_p = 0$ or $N_i = 0$ (resp. $\Theta_p = \frac{\pi}{2}$ or $T_i = 0$) then p is called invariant point (resp. anti-invariant point). M is an invariant submanifold if $\Theta_p = 0$ and an anti-invariant submanifold if $\Theta_p = \frac{\pi}{2}$ for any $p \in M$. Otherwise, M

is a proper point-wise 3-slant submanifold.
As trivial examples, the slant and 3-slant submanifolds are point-wise slant submanifolds and their slant angles are constant on all points of the submanifolds. In the next nontrivial examples we show the existence of point-wise slant and 3-slant submanifolds.

Example 1 *Consider the following cosymplectic structure on* $\overline{M}=\mathbb{R}^5$ *:*

$$\eta = dt, \xi = \partial t, g = \sum_{i=1}^{2}(dx_i \otimes dx_i + dy_i \otimes dy_i) + dt \otimes dt,$$

$$\phi(x_1, x_2, y_1, y_2, t) = (-y_1, -y_2, x_1, x_2, 0).$$

Let $M(u,v) = (u, u, v\cos f, v\sin f, t)$ *, where* f *is a real value function on* $\overline{M}$ *. Then* M *is a point-wise slant submanifold with slant function* $\Theta = \cos^{-1}(\frac{\cos f+\sin f}{\sqrt{2}})$ *.*

Example 2 *Let* $\overline{M}=\mathbb{R}^{11}$ *and* $g=\sum_{i=1}^{11} dx_i \otimes dx_i$ *. Let*

$$\phi_1((x_i)_{i=\overline{1,11}}) = (-x_3, x_4, x_1, -x_2, -x_7, x_8, x_5, -x_6, 0, -x_{11}, x_{10}),$$

$$\phi_2((x_i)_{i=\overline{1,11}}) = (-x_4, -x_3, x_2, x_1, -x_8, -x_7, x_6, x_5, x_{11}, 0, -x_9),$$

$$\phi_3((x_i)_{i=\overline{1,11}}) = (-x_2, x_1, -x_4, x_3, -x_6, x_5, -x_8, x_7, -x_{10}, x_9, 0),$$

$$\xi_1 = \partial x_9, \xi_2 = \partial x_{10}, \xi_3 = \partial x_{11} \text{ and } \eta_1 = dx_9, \eta_2 = dx_{10}, \eta_3 = dx_{11}.$$

It can be verified that $(\overline{M}, \xi_i, \eta_i, \phi_i, g)_{i\in\{1,2,3\}}$ *is an almost contact metric 3-structure manifold. Now we consider a submanifold* M *of* $(\overline{M}, \xi_i, \eta_i, \phi_i, g)_{i\in\{1,2,3\}}$ *given by the following equations:*

$$x_1 = v\sin f,\ x_2 = x_3 = x_4 = x_9 = x_{10} = x_{11} = 0,$$

$$x_5 = x_6 = x_7 = ku\sin f,\ x_8 = v\cos f,$$

for $k \in \mathbb{R}^+$ *and* $f:\mathbb{R}^{11}\to\mathbb{R}$ *. By some computations, one can see that* M *is a point-wise 3-slant submanifold of* $\overline{M}$ *with slant function* $\Theta = \cos^{-1}(\frac{\cos f}{k\sqrt{3}})$ *.*

Theorem 1 *Let* M *be a submanifold of almost contact metric 3-structure* $(\overline{M}, \xi_i, \eta_i, \phi_i, g)$ *such that* ξ_i *'s are normal to* M *for* $i=1,2,3$ *. Then* M *is a point-wise 3-slant submanifold if and only if there exists a real function* Θ *on* M *such that*

$$T_iT_jX = -cos^2\Theta X,\ \forall X \in TM,\ \forall i,j \in \{1,2,3\}. \tag{12}$$

Proof Let M be a point-wise 3-slant submanifold and Θ be the angle between $\phi_i X$ and T_pM. Then from (6) and (7) we have

$$cos\Theta = \frac{g(\phi_i X, T_j X)}{|\phi_i X||T_j X|} = -\frac{g(X, \phi_i T_j X)}{|X||T_j X|} = -\frac{g(X, T_i T_j X)}{|X||T_j X|} \tag{13}$$

On the other hand,

$$cos\Theta = \frac{|T_j X|}{|X|}, \tag{14}$$

and so (13) and (14) imply

$$cos^2\Theta = -\frac{g(X, T_i T_j X)}{|X|^2}, \tag{15}$$

and it follows (12). Conversely, we suppose that α and β are the angles $\widehat{\phi_i X, T_i X}$ and $\widehat{\phi_i X, T_j X}$, respectively, in the point $p \in M$. Thus, $cos\alpha = \frac{|T_i X|}{|X|}$ and $cos\beta = \frac{|T_j X|}{|X|}$. Moreover,

$$cos\alpha = \frac{g(\phi_i X, T_i X)}{|\phi_i X||T_i X|} = -\frac{g(X, T_i T_i X)}{|X||T_i X|} = -\frac{g(X, T_i T_i X)}{|X|^2 cos\alpha}, \tag{16}$$

$$cos\beta = \frac{g(\phi_i X, T_j X)}{|\phi_i X||T_j X|} = -\frac{g(X, T_i T_j X)}{|X||T_j X|} = -\frac{g(X, T_i T_j X)}{|X|^2 cos\beta}. \tag{17}$$

In the account of (12), (16) and (17) imply that the angles are equal and do not depend on the choice of X. This means that M is a point-wise 3-slant submanifold. $\square$

The proof of the following proposition is the same as Theorem 1.

Proposition 1 *Let M be a point-wise 3-slant submanifold of almost contact metric 3-structure $(\overline{M}, \xi_i, \eta_i, \phi_i, g)$ with slant function Θ. Then for all $X \in TM \setminus <\xi_i>$*

$$T_i T_j X = -cos^2\Theta X, \ \forall i, j \in \{1, 2, 3\}. \tag{18}$$

By using (7) and Proposition 1 immediately we have the following proposition.

Proposition 2 *Let M be a point-wise 3-slant submanifold of almost contact metric 3-structure $(\overline{M}, \xi_i, \eta_i, \phi_i, g)$ with slant function Θ. Then $\forall X, Y \in TM \setminus <\xi_i>$ and $\forall i, j \in \{1, 2, 3\}$*

$$g(T_i Y, T_j X) = cos^2\Theta g(Y, X), \tag{19}$$

$$g(N_i Y, N_j X) = sin^2\Theta g(Y, X). \tag{20}$$

Moreover, when the structure of $\overline{M}$ is almost contact metric, Proposition 1 can be stated as follows.

Proposition 3 *Let M be a point-wise slant submanifold of almost contact metric manifold $(\overline{M}, \xi, \eta, \phi, g)$ with slant function Θ. Then*

$$T^2 X = -cos^2\Theta X, \ \forall X \in TM \setminus <\xi>. \tag{21}$$

4. Point-wise slant submanifolds of 3-Sasakian manifolds

Lemma 1 *Let M be a point-wise 3-slant submanifold of $(\overline{M},\xi_i,\eta_i,\phi_i,g)_{i\in\{1,2,3\}}$ with slant function Θ. Then, for any unit vector field $X\in TM\setminus<\xi_1,\xi_2,\xi_3>$, we have*

$$T_iX=cos\Theta Z, \tag{22}$$

where Z is a unit vector field in TM and orthogonal to X.

Proof For any unit vector field $X\in TM\setminus<\xi_1,\xi_2,\xi_3>$, we have $|T_iX|=cos\Theta|\phi_iX|=cos\Theta|X|=cos\Theta$. Now let $Z=\frac{T_iX}{|T_iX|}$ be the unit vector field in the direction of T_iX. Then $T_iX=cos\Theta Z$. Moreover, since $g(\phi_iX,X)=0$ and $g(\phi_iX,X)=g(T_iX+N_iX,X)=g(T_iX,X)$, we conclude that Z is orthogonal to X. □

Theorem 2 *Let M be a point-wise 3-slant submanifold of a 3-Sasakian manifold $(\overline{M},\xi_i,\eta_i,\phi_i,g)_{i\in\{1,2,3\}}$. Then, the slant function Θ is constant if and only if $A_{N_iX}T_iX=A_{N_iT_iX}X$.*

Proof Since $\overline{M}$ is a 3-Sasakian manifold, from (8) and the Gauss formula, for any unit vector field $X\in TM\setminus<\xi_1,\xi_2,\xi_3>$ and $Y\in TM$, we have

$$\begin{aligned} g(X,Y)\xi_i=(\overline{\nabla}_Y\phi_i)X=\nabla_YT_iX+B(T_iX,Y)+D_YN_iX-A_{N_iX}Y-\\ T_i\nabla_YX-t_iB(X,Y)-N_i\nabla_YX-n_iB(X,Y). \end{aligned} \tag{23}$$

By taking the tangential part of (23), we get

$$g(X,Y)\xi_i=\nabla_YT_iX-A_{N_iX}Y-T_i\nabla_YX-t_iB(X,Y). \tag{24}$$

By using (22), Eq. (24) implies

$$\begin{aligned} g(X,Y)\xi_i=Y\cos\Theta Z+\cos\Theta\nabla_YZ-A_{N_iX}Y-T_i\nabla_YX-t_iB(X,Y)=\\ -\sin\Theta Y(\Theta)Z+\cos\Theta\nabla_YZ-A_{N_iX}Y-T_i\nabla_YX-t_iB(X,Y). \end{aligned} \tag{25}$$

We apply $g(Z,.)$ to (25). Since

$$g(Z,\nabla_YZ)=\frac{1}{2}\nabla_Yg(Z,Z)=0,$$

and

$$g(Z,T_i\nabla_YX)=-g(T_iZ,\nabla_YX)=\cos^2\Theta\frac{1}{2}\nabla_Yg(X,X)=0,$$

we get

$$0=-\sin\Theta Y(\Theta)-g(Z,A_{N_iX}Y)-g(Z,t_iB(X,Y)). \tag{26}$$

Thus, Θ is constant if and only if

$$-g(Z,A_{N_iX}Y)-g(Z,t_iB(X,Y))=0,$$

or

$$g(Y,A_{N_iX}Z)=g(N_iZ,B(X,Y))=g(Y,A_{N_iZ}X).$$

Therefore, the slant function Θ is constant if and only if $A_{N_iX}Z=A_{N_iZ}X$. □

Using the approach of the proof of Theorem 2, for a point-wise slant submanifold of a Sasakian manifold, implies the following theorem.

Theorem 3 *Let M be a point-wise slant submanifold of a Sasakian manifold $(\overline{M}, \xi_i, \eta_i, \phi_i, g)$. Then the slant function Θ is constant if and only if $A_{NX}Z = A_{NZ}X$.*

As an analogue of a Kaehlerian slant submanifold of quaternion manifolds, Sasakian 3-slant submanifolds of 3-structure manifolds have been defined in [12]. Let M be a point-wise 3-slant submanifold of an almost contact metric 3-structure manifold $(\overline{M}, \xi_i, \eta_i, \phi_i, g)$ which the vector structures are in TM. The submanifold M is called a Sasakian point-wise 3-slant submanifold if

$$(\nabla_Y T_i)X = g(X,Y)\xi_i - \eta_i(X)Y, \ \forall X, Y \in TM. \tag{27}$$

Now we show that these submanifolds cannot be a proper point-wise slant submanifold.

Theorem 4 *Any Sasakian point-wise 3-slant submanifolds are 3-slant submanifolds.*

Proof Let M be a Sasakian point-wise 3-slant submanifold of $(\overline{M}, \xi_i, \eta_i, \phi_i, g)$ and X be a unit vector field in $TM \backslash < \xi_i >$. From (22) and (27) for any $Y \in TM$, we have

$$\begin{aligned} g(X,Y)\xi_i &= (\nabla_Y T_i)X = \nabla_Y T_i X - T_i(\nabla_Y X) \\ &= \nabla_Y cos\Theta Z - T_i(\nabla_Y X) \\ &= Y(cos\Theta)Z + cos\Theta\nabla_Y Z - T_i(\nabla_Y X) \\ &= sin\Theta Y(\Theta)Z + cos\Theta\nabla_Y Z - T_i(\nabla_Y X). \end{aligned} \tag{28}$$

Since Z is orthogonal to X and ξ_i, by applying $g(Z,.)$ to (28), we obtain

$$0 = sin\Theta Y(\Theta), \tag{29}$$

which means Θ is constant. $\square$

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