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Generalizations of 2-absorbing primary ideals of commutative rings

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Abstract: Let R be a commutative ring with $1 \neq 0$ and $S(R)$ be the set of all ideals of R. In this paper, we extend the concept of 2-absorbing primary ideals to the context of ϕ-2-absorbing primary ideals. Let $\phi : S(R) \to S(R) \cup \emptyset$ be a function. A proper ideal I of R is said to be a ϕ-2-absorbing primary ideal of R if whenever $a, b, c \in R$ with $abc \in I - \phi(I)$ implies $ab \in I$ or $ac \in \sqrt{I}$ or $bc \in \sqrt{I}$. A number of results concerning ϕ-2-absorbing primary ideals are given.

Key words: Primary ideal, weakly primary ideal, prime ideal, weakly prime ideal, 2-absorbing ideal, n-absorbing ideal, weakly 2-absorbing ideal, 2-absorbing primary ideal, weakly 2-absorbing primary ideal, ϕ-prime ideal, ϕ-2-primary ideal, ϕ-2-absorbing ideal

1. Introduction

Throughout this paper R denotes a commutative ring with $1 \neq 0$ and the set of all ideals of R is denoted by $S(R)$. An ideal I of R is said to be proper if $I \neq R$. Let I be a proper ideal of R. Then $\sqrt{I} = \{r \in R : r^k \in I,$ for some $k \in \mathbb{N}\}$ denotes the radical ideal of R. Note that $\sqrt{0}$ is the set (ideal) of all nilpotent elements of R.

Generalizations of prime ideals to the context of ϕ-prime ideals are studied extensively in [1,12]. Various generalizations of prime (primary) ideals are also studied in [2–10,13,14].

Recall that a proper ideal I of R is called a 2*-absorbing ideal* of R as in [5] if whenever $abc \in I$ for some $a, b, c \in R$, then $ab \in I$ or $bc \in I$ or $ac \in I$. A proper ideal I of R is called a *weakly prime* ideal of R as in [2] if whenever $0 \neq ab \in I$ for some $a, b \in I$, then $a \in I$ or $b \in I$. A proper ideal I of R is called a *weakly primary* ideal of R as in [4] if whenever $0 \neq ab \in I$ for some $a, b \in I$, then $a \in I$ or $b \in \sqrt{I}$. Recall from [7] that a proper ideal of R is said to be a 2*-absorbing primary ideal* of R if whenever $a, b, c \in R$ with $abc \in I$, then $ab \in I$ or $ac \in \sqrt{I}$ or $bc \in \sqrt{I}$. Moreover, recall from [8] that a proper ideal I of R is said to be a *weakly 2-absorbing primary ideal* of R if whenever $a, b, c \in R$ with $0 \neq abc \in I$ implies $ab \in I$ or $ac \in \sqrt{I}$ or $bc \in \sqrt{I}$. Recall that a proper ideal I of R is called a ϕ*-2-absorbing ideal* of R as in [12] if whenever $a, b, c \in R$ with $abc \in I - \phi(I)$ implies $ab \in I$ or $ac \in I$ or $bc \in I$. A proper ideal I of R is called a ϕ*-prime ideal* of

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R as in [1] if whenever $a, b \in R$ with $ab \in I - \phi(I)$ implies $a \in I$ or $b \in I$. A proper ideal I of R is called a *ϕ-primary ideal* of R as in [10] if whenever $a, b \in R$ with $ab \in I - \phi(I)$ implies $a \in I$ or $b \in \sqrt{I}$. We show that ϕ-2-absorbing primary ideals enjoy analogues of many of the properties of (weakly) 2-absorbing primary ideals.

In this paper, we extend the concept of 2-absorbing primary ideal to the context of ϕ-2-absorbing primary ideal. Let $\phi : S(R) \to S(R) \cup \emptyset$ be a function. A proper ideal I of R is said to be a *ϕ-2-absorbing primary ideal* of R if whenever $a, b, c \in R$ with $abc \in I - \phi(I)$ implies $ab \in I$ or $ac \in \sqrt{I}$ or $bc \in \sqrt{I}$. Let I be a proper ideal of R and suppose that I is a ϕ-2-absorbing primary ideal of R. Then

1. If $\phi(J) = \emptyset$ for every $J \in S(R)$, then we say that $\phi = \phi_\emptyset$ and I is called a *$\phi_\emptyset$-2-absorbing primary ideal* of R, and hence I is a 2-absorbing primary ideal of R.

2. If $\phi(J) = 0$ for every $J \in S(R)$, then we say that $\phi = \phi_0$ and I is called a *ϕ_0-2-absorbing primary ideal* of R, and thus I is a weakly 2-absorbing primary ideal of R.

3. If $\phi(J) = J$ for every $J \in S(R)$, then we say that $\phi = \phi_1$ and I is called a *ϕ_1-2-absorbing primary ideal* of R.

4. If $n \geq 2$ and $\phi(J) = J^n$ for every $J \in S(R)$, then we say that $\phi = \phi_n$ and I is called a *ϕ_n-2-absorbing primary ideal* of R. In particular, if $n = 2$ and $\phi(J) = J^2$ for every $J \in S(R)$, then we say that I is an *almost-2-absorbing primary ideal* of R.

5. If $\phi(J) = \cap_{n=1}^{\infty} J^n$ for every $J \in S(R)$, then we say that $\phi = \phi_\omega$ and I is called a *ϕ_ω-2-absorbing primary ideal* of R.

Since $I - \phi(I) = I - (I \cap \phi(I))$, without loss of generality, we may assume that $\phi(I) \subseteq I$. Given two functions $\psi_1,\ \psi_2 : S(R) \to S(R) \cup \emptyset$, we say $\psi_1 \leq \psi_2$ if $\psi_1(J) \subseteq \psi_2(J)$ for each $J \in S(R)$. Hence it can be easily seen that $\phi_\emptyset \leq \phi_0 \leq \phi_\omega \leq \cdots \leq \phi_{n+1} \leq \phi_n \leq \cdots \leq \phi_2 \leq \phi_1$.

Throughout this paper, as it is noted earlier, if $\phi : S(R) \to S(R) \cup \emptyset$ is a function, then we always assume that $\phi(I) \subseteq I$.

Among many results in this paper, it is shown (Theorem 2.3) that a proper ideal I of R is a ϕ-2-absorbing primary ideal of R for some function ϕ if and only if $I/\phi(I)$ is a weakly 2-absorbing primary ideal of $R/\phi(I)$. It is shown (Theorem 2.8 and Corollary 2.10) that if I is a ϕ-2-absorbing primary ideal of R for some function ϕ that is not a 2-absorbing primary ideal of R, then $I^3 \subseteq \phi(I)$ and $\sqrt{\phi(I)} = \sqrt{I}$. It is shown (Corollary 8) that if I is a proper ideal of a Noetherian domain R, then I is a 2-absorbing primary ideal of R if and only if I is a ϕ-2-absorbing primary ideal of R for some ϕ with $\phi \leq \phi_4$. Let R_1 and R_2 be commutative rings with $1 \neq 0$, I_1, I_2 be ideals of R_1 and R_2, respectively, and $R = R_1 \times R_2$. Let $\psi_i : S(R_i) \to S(R_i) \cup \emptyset$ $(i = 1, 2)$ be functions. Let $\phi = \psi_1 \times \psi_2$. If $I = I_1 \times I_2$ is a nonzero proper ideal of R, then it is shown (Theorem 2.30) that I is a ϕ-2-absorbing primary ideal of R that is not a 2-absorbing primary ideal of R if and only if $\phi(I) \neq \emptyset$ and one of the following conditions holds:

1. $\psi_2(R_2) = R_2$ and I_1 is a ψ_1-2-absorbing primary ideal of R_1 that is not a 2-absorbing primary ideal of R_1.

2. $\psi_1(R_1) = R_1$, and I_2 is a ψ_2-2-absorbing primary ideal of R_2 that is not a 2-absorbing primary ideal of R_2.

3. $I_2 = \psi_2(I_2)$ is a primary ideal of R_2 and $I_1 \neq R_1$ is ψ_1-primary ideal of R_1 that is not primary such that $\psi_1(I_1) \neq I_1$ (note that if $I_1 = 0$, then $I_2 \neq 0$).

4. $I_1 = \psi_1(I_1)$ is a primary ideal of R_2 and $I_2 \neq R_2$ is a ψ_2-primary ideal of R_2 that is not primary such that $\psi_2(I_2) \neq I_2$ (note that if $I_1 = 0$, then $I_2 \neq 0$).

Let $R = R_1 \times R_2 \times \cdots \times R_m$, where $3 \leq m < \infty$, and R_1, R_2, ..., R_m are commutative rings with $1 \neq 0$. Let $n \geq 2$. It is shown (Theorem 18) that every proper ideal of R is a ϕ_n-2-absorbing primary ideal of R if and only if $R_1, ..., R_m$ are von Neumann regular rings (and hence R is a von Neumann regular ring). Let I be a ϕ-2-absorbing primary ideal of R for some function ϕ. Suppose that $I_1I_2I_3 \subseteq I$, but $I_1I_2I_3 \not\subseteq \phi(I)$, for some ideals I_1, I_2 and I_3 of R such that I is a free ϕ-triple-zero with respect to $I_1I_2I_3$ (see Definition 2.5 and Definition 2.37). Then it is shown (Theorem 21) that $I_1I_2 \subseteq I$ or $I_1I_3 \subseteq \sqrt{I}$ or $I_2I_3 \subseteq \sqrt{I}$.

2. ϕ-2-absorbing primary ideals

Throughout this paper, as it is noted earlier in the introduction, if $\phi : S(R) \to S(R) \cup \emptyset$ is a function, then we always assume that $\phi(I) \subseteq I$.

Lemma 1 *Let I be a proper ideal of R and ψ_1, $\psi_2 : S(R) \to S(R) \cup \emptyset$ are functions with $\psi_1 \leq \psi_2$. If I is a ψ_1-2-absorbing primary ideal of R, then I is a ψ_2-2-absorbing primary ideal of R.*

Proof Suppose that I is a ψ_1-2-absorbing primary ideal of R and $a, b, c \in R$ such that $abc \in I - \psi_2(I)$. Since $abc \in I - \psi_2(I) \subseteq I - \psi_1(I)$, the claim is clear. □

Theorem 1 *Let I be a proper ideal of R. Then*

1. I is a 2-absorbing primary ideal of $R \Rightarrow I$ is a weakly 2-absorbing primary ideal of $R \Rightarrow I$ is a $\phi_\omega-2$-absorbing primary ideal of $R \Rightarrow I$ is a ϕ_{n+1}-2-absorbing primary ideal of R for every $n \geq 2 \Rightarrow I$ is a ϕ_n-2-absorbing primary ideal of R for every $n \geq 2 \Rightarrow I$ is an almost 2-absorbing primary ideal of R.

2. I is an idempotent ideal of $R \Rightarrow I$ is an $\phi_\omega-2$-absorbing primary ideal of R and I is a ϕ_n-2-absorbing ideal of R for every $n \geq 1$.

3. If $\sqrt{I} = I$, then I is a ϕ_n- 2-absorbing primary ideal of R if and only if I is a ϕ_n-2-absorbing ideal of R.

4. I is a ϕ_n-2-absorbing primary ideal of R for all $n \geq 2$ if and only if I is a ϕ_ω-2-absorbing primary ideal of R.

Proof (1) It is clear from Lemma 1.

(2) Suppose that I is an idempotent ideal of R. Then $I = I^n$ for all $n \geq 1$, and so $\phi_\omega(I) = \cap_{n=1}^{\infty} I^n = I$. Thus the claim is clear.

(3) Since $\sqrt{\sqrt{I}} = \sqrt{I}$, the claim is obvious.

(4) Let $a,b,c \in R$ with $abc \in I - \cap_{n=1}^{\infty} I^n$. Hence $abc \in I - I^n$ for some $n \geq 2$. Since I is a ϕ_n-2-absorbing primary ideal of R, we have $ab \in I$ or $bc \in \sqrt{I}$ or $ac \in \sqrt{I}$. The converse is clear from (1). □

Theorem 2 *Let $\phi : S(R) \to S(R) \cup \emptyset$ be a function. Set $R/\emptyset = R$, and let I be a proper ideal of R. Then*

1. *I is a ϕ-2-absorbing primary ideal of R if and only if $I/\phi(I)$ is a weakly 2-absorbing primary ideal of $R/\phi(I)$.*
2. *I is a ϕ-prime ideal of R if and only if $I/\phi(I)$ is a weakly prime ideal of $R/\phi(I)$.*
3. *I is a ϕ-primary ideal of R if and only if $I/\phi(I)$ is a weakly primary ideal of $R/\phi(I)$.*

Proof If $\phi(I) = \emptyset$, then $R/\emptyset = R$ and hence there is nothing to prove. Thus we may assume that $\phi(I) \neq \emptyset$.

(1). Suppose that I is a ϕ-2-absorbing primary ideal of R. Assume that $\phi(I) \neq (a+\phi(I))(b+\phi(I))(c+\phi(I)) \in I/\phi(I)$ for some $a,b,c \in R$. Since $abc \in I - \phi(I)$, $ab \in I$ or $ac \in \sqrt{I}$ or $bc \in \sqrt{I}$. Since $\sqrt{I/\phi(I)} = \sqrt{I}/\phi(I)$, we have $ab+\phi(I) \in I/\phi(I)$ or $ac+\phi(I) \in \sqrt{I}/\phi(I)$ or $bc+\phi(I) \in \sqrt{I}/\phi(I)$. Thus $I/\phi(I)$ is a weakly 2-absorbing primary ideal of $R/\phi(I)$.

Conversely, suppose that $I/\phi(I)$ is a weakly 2-absorbing primary ideal of $R/\phi(I)$. Assume that $abc \in I - \phi(I)$ for some $a,b,c \in R$. Thus $\phi(I) \neq (a+\phi(I))(b+\phi(I))(c+\phi(I)) = abc+\phi(I) \in I/\phi(I)$. Hence $ab+\phi(I) \in I/\phi(I)$ or $ac+\phi(I) \in \sqrt{I}/\phi(I)$ or $bc+\phi(I) \in \sqrt{I}/\phi(I)$. Thus $ab \in I$ or $ac \in \sqrt{I}$ or $bc \in \sqrt{I}$.

By a similar argument as in the proof of (1), one can prove (2) and (3). □

Since $\phi_n(I) = I^n$, the proof of the following result is clear by Theorem 2.

Corollary 1 *Let I be a proper ideal of R and $n \geq 2$. Then*

1. *I is a ϕ_n-2-absorbing primary ideal of R if and only if I/I^n is a weakly 2-absorbing primary ideal of R/I^n.*
2. *I is a ϕ_n-prime ideal of R if and only if I/I^n is a weakly prime ideal of R/I^n.*
3. *I is a ϕ_n-primary ideal of R if and only if I/I^n is a weakly primary ideal of R/I^n.*

Definition 1 *Let I be a ϕ-2-absorbing primary ideal of R and suppose that $abc \in \phi(I)$ for some $a,b,c \in R$ such that $ab \notin I$, $ac \notin \sqrt{I}$, and $bc \notin \sqrt{I}$, then we say (a,b,c) is a ϕ-triple-zero of I. Similarly, if I is a weakly 2-absorbing primary ideal of R and $abc = 0$ for some $a,b,c \in R$ such that $ab \notin I$, $ac \notin \sqrt{I}$, and $bc \notin \sqrt{I}$, then we say (a,b,c) is a triple-zero of I.*

Remark 1 *Note that a proper ideal I of a ring R is a ϕ-2-absorbing primary ideal of R that is not a 2-absorbing primary ideal of R if and only if I has a ϕ-triple-zero (a,b,c) for some $a,b,c \in R$.*

Lemma 2 *Let I be a proper ideal of R and suppose that I is a ϕ-2-absorbing primary ideal of R for some function ϕ. Let $a,b,c \in R$. The following statements are equivalent.*

1. (a,b,c) *is a* ϕ*-triple-zero of* I.

2. $(a+\phi(I), b+\phi(I), c+\phi(I))$ *is a triple-zero of* $I/\phi(I)$.

Proof (1) $\Rightarrow$ (2). Suppose that (a,b,c) is a ϕ-triple-zero of I. Hence $abc \in \phi(I)$, but $ab \notin I$, $ac \notin \sqrt{I}$, and $bc \notin \sqrt{I}$. Thus $ab+\phi(I) \notin I/\phi(I)$, $ac+\phi(I) \notin \sqrt{I}/\phi(I)$, and $bc+\phi(I) \notin \sqrt{I}/\phi(I)$. Since $I/\phi(I)$ is a weakly 2-absorbing primary ideal of R by Theorem 2, $(a+\phi(I), b+\phi(I), c+\phi(I))$ is a triple-zero of $I/\phi(I)$.

(2) $\Rightarrow$ (1). Suppose that $(a+\phi(I), b+\phi(I), c+\phi(I))$ is a triple-zero of $I/\phi(I)$. Then $abc \in \phi(I)$ such that $ab+\phi(I) \notin I/\phi(I)$, $ac+\phi(I) \notin \sqrt{I}/\phi(I)$, and $bc+\phi(I) \notin \sqrt{I}/\phi(I)$. Hence $ab \notin I$, $ac \notin \sqrt{I}$, and $bc \notin \sqrt{I}$. Thus (a,b,c) is a ϕ-triple-zero of I. □

Theorem 3 *Let* I *be a* ϕ*-2-absorbing primary ideal of* R *for some function* ϕ *and suppose that* (a,b,c) *is a* ϕ*-triple-zero of* I *for some* $a,b,c \in R$ *(hence* I *is not a 2-absorbing primary ideal of* R*). Then*

1. $abI,\ bcI,\ acI \subseteq \phi(I)$.

2. $aI^2,\ bI^2,\ cI^2 \subseteq \phi(I)$.

3. $I^3 \subseteq \phi(I)$.

Proof **(1)**. Since I is a ϕ-2-absorbing primary ideal of R, $I/\phi(I)$ is a weakly 2-absorbing primary ideal of $R/\phi(I)$ by Theorem 2. Since (a,b,c) is a ϕ-triple-zero of I, $(a+\phi(I), b+\phi(I), c+\phi(I))$ is a triple-zero of $I/\phi(I)$ by Lemma 2. Hence $abI+\phi(I) = bcI+\phi(I) = acI+\phi(I) = \phi(I)$ (in $R/\phi(I)$) by [8, Theorem 2.9]. Thus $abI,\ bcI,\ acI \subseteq \phi(I)$.

(2). Again, since $(a+\phi(I), b+\phi(I), c+\phi(I))$ is a triple-zero of $I/\phi(I)$ by Lemma 2 and $I/\phi(I)$ is a weakly 2-absorbing primary ideal of $R/\phi(I)$ by Theorem 2, we have $aI^2+\phi(I) = bI^2+\phi(I) = cI^2+\phi(I) = \phi(I)$ (in $R/\phi(I)$) by [8, Theorem 2.9]. Thus $aI^2,\ bI^2,\ cI^2 \subseteq \phi(I)$.

(3). Since $(a+\phi(I), b+\phi(I), c+\phi(I))$ is a triple-zero of $I/\phi(I)$ by Lemma 2 and $I/\phi(I)$ is a weakly 2-absorbing primary ideal of $R/\phi(I)$ by Theorem 2, we have $I^3+\phi(I) = \phi(I)$ (in $R/\phi(I)$) by [8, Theorem 2.10]. Thus $I^3 \subseteq \phi(I)$. □

Corollary 2 *Let* I *be a* ϕ*-2-absorbing primary ideal of* R *such that* $I^3 \nsubseteq \phi(I)$*. Then* I *is a 2-absorbing primary ideal of* R.

Proof The proof is clear by Remark 1 and Theorem 3(3). □

Corollary 3 *If* I *is a* ϕ*-2-absorbing primary ideal of* R *that is not a 2-absorbing primary ideal of* R*, then* $\sqrt{I} = \sqrt{\phi(I)}$.

Proof Since I is not a 2-absorbing primary ideal of R, $I^3 \subseteq \phi(I)$ by Theorem 3. Hence $\sqrt{I} \subseteq \sqrt{\phi(I)}$. Since $\phi(I) \subseteq I$, we have $\sqrt{\phi(I)} \subseteq \sqrt{I}$. Thus $\sqrt{I} = \sqrt{\phi(I)}$. □

Corollary 4 *Let ϕ be a function and let I be a proper ideal of R such that $\sqrt{\phi(I)}$ is a prime ideal of R. Then I is a ϕ-2-absorbing primary ideal of R if and only if I is a 2-absorbing primary ideal of R.*

Proof Suppose that I is a ϕ-2-absorbing primary ideal of R. Assume that I is not a 2-absorbing primary ideal of R. Then $\sqrt{I} = \sqrt{\phi(I)}$ by Corollary 3. Thus $\sqrt{I}$ is a prime ideal of R. Since $\sqrt{I}$ is prime, we conclude that I is a 2-absorbing primary ideal of R by [7, Theorem 2.8]. □

Corollary 5 *Let I be a proper ϕ-2-absorbing primary ideal of R such that $\phi \leq \phi_4$. Then*

1. *I is a ϕ_n-2-absorbing primary ideal of R for every $n \geq 3$.*
2. *I is a ϕ_ω-2-absorbing primary ideal of R.*

Proof If I is a 2-absorbing primary ideal of R, then (1) and (2) are clear. Hence assume that I is not a 2-absorbing primary ideal of R. Thus $I^3 \subseteq \phi(I)$ by Theorem 3. Since $\phi \leq \phi_4$, we have $I^3 \subseteq \phi(I) \subseteq I^4$. Hence $I^3 = I^n = \phi(I)$ for every $n \geq 3$. Thus (1) and (2) are clear. □

Theorem 4 *Let J be a finitely generated proper ideal of R. Suppose that J is a ϕ-2-absorbing primary ideal of R for some function ϕ, where $\phi \leq \phi_4$. Then J is a ψ-2-absorbing primary ideal of R for every function ψ with $\phi_\omega < \psi$ and one of the following statements holds.*

1. *J is a weakly 2-absorbing primary ideal of R.*
2. *$J^3 = eR = eJ$ for some idempotent $e \in R$ and $I = (1-e)J$ is a weakly 2-absorbing primary ideal of $(1-e)R$.*

Proof If J is a 2-absorbing primary ideal of R, then there is nothing to prove. Hence assume that J is not a 2-absorbing primary ideal of R. Thus $J^3 \subseteq \phi(J)$ by Theorem 3. Hence $J^3 \subseteq \phi_4(J) = J^4$. Thus $\phi(J) = J^3 = J^4$. Hence $J^3 = J^6$. Thus J^3 is an idempotent ideal of R. Since J^3 is an idempotent ideal of R and $J^3 = J^4$, we have $\phi_\omega(J) = J^3 = \phi(J)$. Thus J is a ψ-2-absorbing primary ideal of R for every function ψ such that $\phi_\omega < \psi$ by Lemma 1. Since J^3 is a finitely generated idempotent ideal of R, $J^3 = eR$ for some idempotent $e \in R$ by [Ex. 2.25, [9]]. Hence $J^3 = eR\phi(J)$. We consider two cases. **Case I**. Suppose that $J^3 = 0$. Then $J^3 = \phi(J) = 0$. Thus J is a weakly 2-absorbing primary ideal of R. **Case II**. Assume that $J^3 \neq 0$. Let $I = (1-e)J$. Assume that $0 \neq abc \in I \subseteq J$ for some $a, b, c \in (1-e)R$. Since $eR \cap (1-e)J = \{0\}$ and $\phi(J) = J^3 = eR$, $abc \in J - \phi(J)$. Thus $ab \in J$ or $ac \in \sqrt{J}$ or $bc \in \sqrt{J}$. Let $\sqrt{I}_{(1-e)R}$ denotes the radical of I in $(1-e)R$. Since $a, b, c \in (1-e)R$ and $\sqrt{I}_{(1-e)R} = (1-e)\sqrt{J}$, we conclude that $ab \in I$ or $ac \in \sqrt{I}_{(1-e)R}$ or $bc \in \sqrt{I}_{(1-e)R}$. □

The proof of the following result is clear by Theorem 4.

Corollary 6 *Suppose that $\{0, 1\}$ is the set of all idempotents of R. Let I be a finitely generated proper ideal of R. Then I is a weakly 2-absorbing primary ideal of R if and only if I is a ϕ-2-absorbing primary ideal of R for some ϕ with $\phi \leq \phi_4$. In particular, I is a weakly 2-absorbing primary ideal of R if and only if I is a ϕ_n-2-absorbing primary ideal of R for some $n \geq 4$.*

Recall that a commutative ring with $1 \neq 0$ is said to be a *quasi-local* ring if R has exactly one maximal ideal.

Corollary 7 *Suppose that (R, M) is a quasi-local commutative ring. Then a finitely generated ideal I of R is a ϕ-2-absorbing primary ideal of R for some ϕ with $\phi \leq \phi_4$ if and only if I is a weakly 2-absorbing primary ideal of R. In particular, I is a weakly 2-absorbing primary ideal of R if and only if I is a ϕ_n-2-absorbing primary ideal of R for some $n \geq 4$.*

Proof Since (R, M) is a quasi-local commutative ring, $\{0, 1\}$ is the set of all idempotents of R. Hence the claim is clear by Corollary 6. □

Since a proper ideal of an integral domain is weakly 2-absorbing primary if and only if it is 2-absorbing primary, in view of Corollary 6 we have the following result.

Corollary 8 *Let I be a proper ideal of a Noetherian domain R. Then I is a 2-absorbing primary ideal of R if and only if I is a ϕ-2-absorbing primary ideal of R for some ϕ with $\phi \leq \phi_4$. In particular, I is a 2-absorbing primary ideal of R if and only if I is a ϕ_n-2-absorbing primary ideal of R for some $n \geq 4$.*

Theorem 5 *Let $a \in R$ be nonunit. Let $(0 : a) \subseteq \sqrt{(a)}$. Then (a) is ϕ-2-absorbing primary, for some ϕ with $\phi \leq \phi_3$ if and only if (a) is a 2-absorbing primary ideal of R.*

Proof Suppose that (a) is ϕ_3-2-absorbing primary. Let $x_1x_2x_3 \in (a)$. If $x_1x_2x_3 \notin (a^3)$, then $x_1x_2 \in (a)$ or $x_2x_3 \in \sqrt{(a)}$ or $x_1x_3 \in \sqrt{(a)}$. Now suppose that $x_1x_2x_3 \in (a^3)$. Thus $(x_1+a)x_2x_3 \in (a)$. If $(x_1+a)x_2x_3 \notin (a^3)$, then $(x_1 + a)x_2 \in (a)$ or $x_2x_3 \in \sqrt{(a)}$ or $(x_1 + a)x_3 \in \sqrt{(a)}$. So we have $x_1x_2 \in (a)$ or $x_2x_3 \in \sqrt{(a)}$ or $x_1x_3 \in \sqrt{(a)}$. If $(x_1 + a)x_2x_3 \in (a^3)$, then $x_1x_2x_3 \in (a^3)$ gives $ax_2x_3 \in (a^3)$. Therefore $ax_2x_3 = ra^3$, for some $r \in R$. Thus $a(x_2x_3 - ra^2) = 0$, and so $x_2x_3 - ra^2 \in (0 : a)$. Hence $x_2x_3 \in (0 : a) + (a)) \subseteq \sqrt{(a)}$, and thus $x_2x_3 \in \sqrt{(a)}$. The converse part is obvious by Theorem 1. □

Theorem 6 *Suppose that a proper ideal I of R is a ϕ-prime ideal of R for some ϕ and suppose that $\phi(I) \subseteq \phi(J)$ for some radical ideal J of R such that $J \subset I$ $(J \neq I)$. Then I is a prime ideal of R.*

Proof Suppose that I is not a prime ideal of R. Then $I^2 \subseteq \phi(I)$ by [1, Theorem 5]. Hence $\sqrt{I} = \sqrt{\phi(I)}$. Since $\phi(I) \subseteq \phi(J) \subseteq J$ and J is a radical ideal of R, we have $\sqrt{I} = \sqrt{\phi(J)} \subseteq J$. Hence $I \subseteq J$, a contradiction. Thus I is a prime ideal of R. □

Theorem 7 *Let J, K be proper ideals of R such that $J \subseteq K$, and let $n \geq 2$. If K is a ϕ_n-2-absorbing primary ideal of R, then K/J is a ϕ_n-2-absorbing primary ideal of R/J.*

Proof Suppose that K is a ϕ_n-2-absorbing primary ideal of R. Assume that $(a + J)(b + J)(c + J) \in K/J - (K/J)^n$ for some $a, b, c \in R$. Since $J \subseteq K$, we have $abc \in K - K^n$. Thus $ab \in K$ or $ac \in \sqrt{K}$ or $bc \in \sqrt{K}$. Since $J \subseteq K$, $\sqrt{K/J} = \sqrt{K}/J$. Hence $(a + J)(b + J) \in K/J$ or $(a + J)(c + J) \in \sqrt{K}/J$ or $(b + J)(c + J) \in \sqrt{K}/J$. Thus K/J is a ϕ_n-2-absorbing primary ideal of R/J. □

The proof of the following result is similar to the proof of Theorem 7. Hence we leave the proof to the reader.

Theorem 8 *Let J, K be proper ideals of R such that $J \subseteq K$. If K is a ϕ_ω-2-absorbing primary ideal of R, then K/J is a ϕ_ω-2-absorbing primary ideal of R/J.*

Definition 2 *Let $\phi : S(R) \to S(R) \cup \emptyset$ be a function. We remind the reader that we always assume that $\phi(I) \subseteq I$. Let I be a proper ideal of R and S be a multiplicatively closed subset of R. Then*

1. *A proper ideal L/I of R/I, where L is a proper ideal of R such that $I \subseteq L$, is called a ϕ_I-2-absorbing primary ideal of R/I if whenever $a, b, c \in R/I$ with $abc \in L/I - (\phi(L) + I)/I$ implies $ab \in L/I$ or $ac \in \sqrt{L/I}$ or $bc \in \sqrt{L/I}$.*

2. *A proper ideal L_S of R_S, where L is a proper ideal of R such that $L \cap S = \emptyset$, is called a ϕ_S-2-absorbing primary ideal of R_S if whenever $a, b, c \in R_S$ with $abc \in L_S - \phi(L)_S$ implies $ab \in L_S$ or $ac \in \sqrt{L_S}$ or $bc \in \sqrt{L_S}$.*

Theorem 9 *Let $\phi : S(R) \to S(R) \cup \emptyset$ be a function, P be a proper ideal of R and let I be an ideal of R such that $I \subseteq P$. If P is a ϕ-2-absorbing primary ideal of R, then P/I is a ϕ_I-2-absorbing primary ideal of R/I.*

Proof Let $a, b, c \in R$ such that $(a+I)(b+I)(c+I) = abc + I \in P/I - (\phi(P) + I)/I$. Hence $abc \in P - \phi(P)$. Thus $ab \in P$ or $ac \in \sqrt{P}$ or $bc \in \sqrt{P}$. Since $I \subseteq P$, $\sqrt{P/I} = \sqrt{P}/I$. Thus $(a+I)(b+I) \in P/I$ or $(a+I)(c+I) \in \sqrt{P}/I$ or $(b+I)(c+I) \in \sqrt{P}/I$. □

Theorem 10 *Let $\phi : S(R) \to S(R) \cup \emptyset$ be a function and let P be a proper ideal of R. Suppose that I is a proper ideal of R such that $I \subseteq \phi(P)$. The following statements are equivalent.*

1. *P is a ϕ-2-absorbing primary ideal of R.*

2. *P/I is a ϕ_I-2-absorbing primary ideal of R/I.*

3. *P/I^n is a ϕ_{I^n}-2-absorbing primary ideal of R/I^n for every $n \geq 1$.*

Proof (1) ⇒ (2). It is clear by Theorem 9. (2) ⇒ (3). Let $n \geq 1$. Since $I \subseteq \phi(P)$, we have $I^n \subseteq I \subseteq \phi(P)$. Suppose that $(a+I^n)(b+I^n)(c+I^n) \in P/I^n - \phi(P)/I^n$ for some $a, b, c \in R$. Hence $abc \notin \phi(P)$. Since $I \subseteq \phi(P)$ and $abc \notin \phi(P)$, $abc \notin I$. Thus $(a+I)(b+I)(c+I) \in P/I - \phi(P)/I$. Since $\sqrt{P/I} = \sqrt{P/I^n} = \sqrt{P}/I^n$ and P/I is a ϕ_I-2-absorbing primary ideal of R, one can conclude that $ab \in P$ or $ac \in \sqrt{P}$ or $bc \in \sqrt{P}$. Thus $ab + I^n \in P/I^n$ or $ac + I^n \in \sqrt{P}/I^n$ or $bc + I^n \in \sqrt{P}/I^n$. (3) ⇒ (1). Let $n = 1$. Suppose that $abc \in P - \phi(P)$ for some $a, b, c \in R$. Since $I \subseteq \phi(P)$, $abc \notin I$. Since $I \subseteq \phi(P) \subset P$, we have $(a+I)(b+I)(c+I) = abc + I \in P/I - \phi(P)/I$. Since $\sqrt{P/I} = \sqrt{P}/I$ and P/I is a ϕ_I-2-absorbing primary ideal of R, one can conclude that $ab \in P$ or $ac \in \sqrt{P}$ or $bc \in \sqrt{P}$. □

Corollary 9 *Let $\phi : S(R) \to S(R) \cup \emptyset$ be a function and let P be a proper ideal of R that is not a weakly 2-absorbing primary ideal of R. The following statements are equivalent.*

1. *P is a ϕ-2-absorbing primary ideal of R.*

2. *P/P^3 is a ϕ_{P^3}-2-absorbing primary ideal of R/P^3.*

3. *P/P^n is a ϕ_{P^n}-2-absorbing primary ideal of R/P^n for every $n \geq 3$.*

Proof Since P is not a weakly 2-absorbing primary ideal of R (and hence P is not a 2-absorbing primary ideal of R), we have $P^3 \subseteq \phi(P)$ by Theorem 3. Hence we are done by Theorem 10. □

For a commutative ring R with $1 \neq 0$. Let $Z(R)$ be the set of all zero-divisors of R.

Theorem 11 *Let $\phi : S(R) \to S(R) \cup \emptyset$ be a function. Let P be a proper ideal of R and S be a multiplicatively closed subset of R such that $S \cap Z(R) = S \cap P = \emptyset$. The following statements are equivalent.*

1. *P is a ϕ-2-absorbing primary ideal of R.*

2. *P_S is a ϕ_S-2-absorbing primary ideal of R_S.*

Proof (1) $\Rightarrow$ (2). Suppose that $x_1x_2x_3 = y \in P_S - \phi(P)_S$ for some $x_1, x_2, x_3 \in R_S$. Hence there is an $s \in S$ and $a, b, c, d \in R$ such that $x_1 = a/s, x_2 = b/s, x_3 = c/s$, and $y = s^2d/s^3$. Thus $\frac{abc}{s^3} = \frac{s^2d}{s^3} \in P_S - \phi(P)_S$. Since $Z(R) \cap S = \emptyset$, we have $abc = s^2d \in P - \phi(P)$. Thus $ab \in P$ or $ac \in \sqrt{P}$ or $bc \in \sqrt{P}$. Since $Z(R) \cap S = \emptyset$, $\sqrt{P_S} = \sqrt{P}_S$. Thus $x_1x_2 \in P_S$ or $x_1x_3 \in \sqrt{P}_S$ or $x_2x_3 \in \sqrt{P}_S$.

(2) $\Rightarrow$ (1). Suppose that $abc \in P - \phi(P)$ for some $a, b, c \in R$. Thus $abc \in P_S - \phi(P)_S$. Hence $ab \in P_S$ or $ac \in \sqrt{P}_S$ or $bc \in \sqrt{P}_S$. Hence $ab \in P$ or $ac \in \sqrt{P}$ or $bc \in \sqrt{P}$. □

The proof of the following result is easily verified, and hence we omit the proof.

Lemma 3 *Let $\phi : S(R) \to S(R) \cup \emptyset$ be a function. Set $R/\emptyset = R$, and let I be a proper ideal of R. Then*

1. *I is a 2-absorbing primary ideal of R if and only if $I/\phi(I)$ is a 2-absorbing primary ideal of $R/\phi(I)$.*

2. *I is a prime ideal of R if and only if $I/\phi(I)$ is a prime ideal of $R/\phi(I)$.*

3. *I is a primary ideal of R if and only if $I/\phi(I)$ is a primary ideal of $R/\phi(I)$.*

Remark 2 *Let $R_1, R_2, ..., R_n$ be commutative rings with $1 \neq 0$ ($n \geq 1$) and $R = R_1 \times \cdots \times R_n$. For each i, $1 \leq i \leq n$, let $\psi_i : S(R_i) \to S(R_i) \cup \emptyset$ be a function, and let $\phi = \psi_1 \times \cdots \times \psi_n$. Let $I = I_1 \times \cdots \times I_n$ be an ideal of R, where $I_1, ..., I_n$ are ideals of R_1, ..., R_n, respectively. Suppose that $\psi_i(I_i) = \emptyset$ for some i, $1 \leq i \leq n$. Then $I - \phi(I) = I$. Hence $\phi(I) = \emptyset$ if and only if $\psi_i(I_i) = \emptyset$ for some i, $1 \leq i \leq n$. If $\phi(I) = \emptyset$, then we set $R/\phi(I) = R$.*

Theorem 12 *Let R_1 and R_2 be commutative rings with $1 \neq 0$, I_1 be a proper ideal of R_1, and $R = R_1 \times R_2$. Let $\psi_i : S(R_i) \to S(R_i) \cup \emptyset$ ($i = 1, 2$) be functions such that $\psi_2(R_2) \neq R_2$, and let $\phi = \psi_1 \times \psi_2$. Then the following statements are equivalent.*

1. *$I_1 \times R_2$ is a ϕ-2-absorbing primary ideal of R.*

2. *$I_1 \times R_2$ is a 2-absorbing primary ideal of R.*

3. *I_1 is a 2-absorbing primary ideal of R_1.*

Proof Suppose that $\psi_1(I_1) = \emptyset$ or $\psi_2(R_2) = \emptyset$. Then $\phi(I_1 \times R_2) = \emptyset$ by Remark 2. Hence (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3). Thus assume that $\phi(I_1 \times R_2) \neq \emptyset$, and hence neither $\psi_1(I_1) = \emptyset$ nor $\psi_2(R_2) = \emptyset$.

(1) $\Rightarrow$ (2). It is clear that I_1 is a ψ_1-2-absorbing primary ideal of R_1. If I_1 is a 2-absorbing primary ideal of R_1, then we are done. Hence assume that I_1 is not a 2-absorbing ideal of R_1. Thus I_1 has a ψ_1-triple-zero (a, b, c) for some $a, b, c \in R_1$. Since $\psi_2(R_2) \neq R_2$, we have $(a, 1)(b, 1)(c, 1) \in I_1 \times R_2 - \psi_1(I_1) \times \psi_2(R_2)$. Thus $ab \in I_1$ or $ac \in \sqrt{I_1}$ or $bc \in \sqrt{I_1}$, a contradiction. Thus I_1 is a 2-absorbing primary ideal of R_1. Hence $I_1 \times R_2$ is a 2-absorbing primary ideal of R.

(2) $\Rightarrow$ (3). It is clear.

(3) $\Rightarrow$ (1). It is clear. □

Theorem 13 *Let R_1 and R_2 be commutative rings with $1 \neq 0$, I_1 be a proper ideal of R_1, and $R = R_1 \times R_2$. Let $\psi_i : S(R_i) \to S(R_i) \cup \emptyset$ $(i = 1, 2)$ be functions and let $\phi = \psi_1 \times \psi_2$. Then the following statements are equivalent.*

1. *$I_1 \times R_2$ is a ϕ-2-absorbing primary ideal of R that is not a 2-absorbing primary ideal of R.*
2. *$\phi(I_1 \times R_2) \neq \emptyset$, $\psi_2(R_2) = R_2$, and I_1 is a ψ_1-2-absorbing primary ideal of R_1 that is not a 2-absorbing primary ideal of R_1.*

Proof (1) $\Rightarrow$ (2). Since $I_1 \times R_2$ is not a 2-absorbing primary ideal of R, it is clear that $\phi(I_1 \times R_2) \neq \emptyset$ and $\psi_2(R_2) = R_2$ by Theorem 12. Since $I_1 \times R_2$ is a ϕ-2-absorbing primary ideal of R, it is clear that I_1 is a ψ_1-2-absorbing primary ideal of R_1. Since $I_1 \times R_2$ is not a 2-absorbing primary ideal of R, I_1 is not a 2-absorbing primary ideal of R_1 by [7, Theorem 2.23].

(2) $\Rightarrow$ (1). Since $\phi(I_1 \times R_2) \neq \emptyset$ and $\psi_2(R_2) = R_2$, $R/\phi(I_1 \times R_2)$ is ring-isomorphic to $R_1/\psi_1(I_1)$. Since I_1 is a ψ_1-2-absorbing primary ideal of R_1 that is not a 2-absorbing primary ideal of R_1, $I_1/\psi_1(I_1)$ is a weakly 2-absorbing primary ideal of $R_1/\psi_1(I_1)$ that is not a 2-absorbing primary ideal of $R_1/\psi_1(I_1)$ by Theorem 2 and Lemma 3. Hence $(I_1 \times R_2)/\phi(I_1 \times R_2)$ is a weakly 2-absorbing primary ideal of $R/\phi(I_1 \times R_2)$ that is not a 2-absorbing primary ideal of $R/\phi(I_1 \times R_2)$. Thus $I_1 \times R_2$ is a ϕ-2-absorbing primary ideal of R that is not a 2-absorbing primary ideal of R by Theorem 2 and Lemma 3. □

Theorem 14 *Let R_1 and R_2 be commutative rings with $1 \neq 0$, I_1, I_2 be ideals of R_1 and R_2, respectively, and $R = R_1 \times R_2$. Let $\psi_i : S(R_i) \to S(R_i) \cup \emptyset$ $(i = 1, 2)$ be functions. Let $\phi = \psi_1 \times \psi_2$. If $I = I_1 \times I_2$ is a nonzero proper ideal of R and $\phi(I) \neq I_1 \times I_2$, then I is a ϕ-2-absorbing primary ideal of R that is not a 2-absorbing primary ideal of R if and only if $\phi(I) \neq \emptyset$ and one of the following conditions holds.*

1. *$\psi_2(R_2) = R_2$ and I_1 is a ψ_1-2-absorbing primary ideal of R_1 that is not a 2-absorbing primary ideal of R_1.*
2. *$\psi_1(R_1) = R_1$ and I_2 is a ψ_2-2-absorbing primary ideal of R_2 that is not a 2-absorbing primary ideal of R_2.*
3. *$I_2 = \psi_2(I_2)$ is a primary ideal of R_2 and $I_1 \neq R_1$ is ψ_1-primary ideal of R_1 that is not primary such that $\psi_1(I_1) \neq I_1$ (note that if $I_1 = 0$, then $I_2 \neq 0$).*

4. *$I_1 = \psi_1(I_1)$ is a primary ideal of R_2 and $I_2 \neq R_2$ is a ψ_2-primary ideal of R_2 that is not primary such that $\psi_2(I_2) \neq I_2$ (note that if $I_1 = 0$, then $I_2 \neq 0$).*

Proof Suppose that I is a ϕ-2-absorbing primary ideal of R that is not a 2-absorbing primary ideal of R. Hence $\phi(I) \neq \emptyset$. Assume that $I_1 = R_1$. Then $\psi_1(R_1) = R_1$ and I_2 is a ψ_2-2-absorbing primary ideal of R_2 that is not a 2-absorbing primary ideal of R_2 by Theorem 13. Assume that $I_2 = R_2$. Then $\psi_2(R_2) = R_2$ and I_1 is a ψ_1-2-absorbing primary ideal of R_1 that is not a 2-absorbing primary ideal of R_1 by Theorem 13. Thus assume that $I_1 \neq R_1$ and $I_2 \neq R_2$. Since $\phi(I) \neq I_1 \times I_2$, we conclude that $I/\phi(I)$ is a nonzero weakly 2-absorbing primary ideal of $R/\phi(I)$ that is not a 2-absorbing primary ideal of $R/\phi(I)$ by Theorem 2. Hence $I_1/\psi_1(I_1) \times I_2/\psi_2(I_2)$ is a nonzero weakly 2-absorbing primary ideal of $R_1/\psi_1(I_1) \times R_2/\psi_2(I_2)$ that is not a 2-absorbing primary ideal of $R_1/\psi_1(I_1) \times R_2/\psi_2(I_2)$. Thus by [8, Theorem 2.23], we have either $I_1/\psi_1(I_1) = \psi_1(I_1)/\psi_1(I_1)$ is a primary ideal ideal of $R_1/\psi_1(I_1)$ and $I_2/\psi_2(I_2)$ is a nonzero weakly primary ideal of $R_2/\psi_2(I_2)$ that is not primary or $I_2/\psi_2(I_2) = \psi_2(I_2)/\psi_2(I_2)$ is a primary ideal of $R_2/\psi_2(I_2)$ and $I_1/\psi_1(I_1)$ a nonzero weakly primary ideal of $R_1/\psi_1(I_1)$ that is not primary. Thus (3) or (4) must hold by Theorem 2.

Conversely, suppose that $\phi(I) \neq \emptyset$. If (1) or (2) holds, then I is a ϕ-2-absorbing primary ideal of R that is not a 2-absorbing primary ideal of R by Theorem 13. Suppose that (3) or (4) holds, then $I/\phi(I)$ is a nonzero weakly 2-absorbing primary ideal of $R/\phi(I)$ that is not 2-absorbing primary by [8, Theorem 2.23]. Thus I is a ϕ-2-absorbing primary ideal of R that is not a 2-absorbing primary ideal of R by Theorem 2. □

Theorem 15 *Let R_1 and R_2 be commutative rings with $1 \neq 0$, I_1, I_2 be nonzero ideals of R_1 and R_2, respectively, and $R = R_1 \times R_2$. Let $\psi_i : S(R_i) \to S(R_i) \cup \emptyset$ $(i = 1, 2)$ be functions such that $\psi_1(I_1) \neq I_1$ and $\psi_2(I_2) \neq I_2$. Let $\phi = \psi_1 \times \psi_2$. If $I_1 \times I_2$ is a proper ideal of R, then the following statements are equivalent.*

1. $I_1 \times I_2$ is a ϕ-2-absorbing primary ideal of R.
2. $I_1 = R_1$ and I_2 is a 2-absorbing primary ideal of R_2 or $I_2 = R_2$ and I_1 is a 2-absorbing primary ideal of R_1 or I_1, I_2 are primary ideals of R_1, R_2, respectively.
3. $I_1 \times I_2$ is a 2-absorbing primary ideal of R.

Proof Suppose that $\psi_1(I_1) = \emptyset$ or $\psi_2(I_2) = \emptyset$. Then $\phi(I_1 \times I_2) = \emptyset$ by Remark 2. Hence (1) ⇔ (2) ⇔ (3) by [7, Theorem 2.23]. Thus assume that $\phi(I_1 \times I_2) \neq \emptyset$, and hence neither $\psi_1(I_1) = \emptyset$ nor $\psi_2(I_2) = \emptyset$.

(1) ⇒ (2). Suppose that $I_1 \times I_2$ is a ϕ-2-absorbing primary ideal of R. Hence $I_1/\psi_1(I_1) \times I_2/\psi_2(I_2)$ is a weakly 2-absorbing primary ideal of $R_1/\psi_1(I_1) \times R_2/\psi_2(I_2)$ by Theorem 2. Hence by [8, Theorem 2.22], we conclude that $I_1/\psi_1(I_1) = R_1/\psi_1(I_1)$ and $I_2/\psi_2(I_2)$ is a 2-absorbing primary ideal of $R_2/\psi_2(I_2)$ or $I_2 = R_2/\psi_2(I_2)$ and $I_1/\psi_(I_1)$ is a 2-absorbing primary ideal of $R_1/\psi_1(I_1)$ or $I_1/\psi_1(I_1)$, $I_2/\psi_2(I_2)$ are primary ideals of $R_1/\psi_1(I_1)$, $R_2/\psi_2(I_2)$, respectively. Thus $I_1 = R_1$ and I_2 is a 2-absorbing primary ideal of R_2 or $I_2 = R_2$ and I_1 is a 2-absorbing primary ideal of R_1 or I_1, I_2 are primary ideals of R_1, R_2, respectively, by Lemma 3.

(2) ⇒ (3). Suppose that $I_1 = R_1$ and I_2 is a 2-absorbing primary ideal of R_2 or $I_2 = R_2$ and I_1 is a 2-absorbing primary ideal of R_1 or I_1, I_2 are primary ideals of R_1, R_2, respectively. Then $I_1 \times I_2$ is a 2-absorbing primary ideal of R by [8, Theorem 2.22].

(3) $\Rightarrow$ (1). Suppose that $I_1 \times I_2$ is a 2-absorbing primary ideal of R. Then it is clear that $I_1 \times I_2$ is a ϕ-2-absorbing primary ideal of R by Lemma 3. □

Theorem 16 *Let $R = R_1 \times R_2 \times \cdots \times R_n$, where $2 < n < \infty$, and $R_1, R_2, ..., R_n$ are commutative rings with $1 \neq 0$. For each i, $1 \leq i \leq n$, let $\psi_i : S(R_i) \to S(R_i) \cup \emptyset$ be a function, and let $\phi = \psi_1 \times \cdots \times \psi_n$. Let $I = I_1 \times \cdots \times I_n$ be a nonzero proper ideal of R, where $I_1, ..., I_n$ are ideals of $R_1, ..., R_n$, respectively. Let $M = \{i | I_i$ is a proper ideal of $R_i, 1 \leq i \leq n\}$. If $|M| = 1$ or n, then assume that $\psi_x(I_x) \neq I_x$ for some $x \in M$ (note that $|M| \geq 1$ since I is a proper ideal of R). If $|M| \neq n$, then assume that $\psi_y(R_y) \neq R_y$ for some $y \in \{1, ..., n\} \setminus M$. Then the following statements are equivalent.*

1. I is a ϕ-2-absorbing primary ideal of R.
2. I is a 2-absorbing primary ideal of R.
3. Either $I = \times_{j=1}^{n} I_j$ such that for some $k \in \{1, ..., n\}$, I_k is a 2-absorbing primary ideal of R_k, and $I_j = R_j$ for every $j \in \{1, ..., n\} - \{k\}$, or $I = \times_{j=1}^{n} I_j$ such that for some k, $m \in \{1, ..., n\}$, I_k is a primary ideal of R_k, I_m is a primary ideal of R_m, and $I_j = R_j$ for every $j \in \{1, ..., n\} - \{k, m\}$.

Proof If $\phi(I) = \emptyset$, then (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3) by [7, Theorem 2.24]. Hence assume that $\phi(I) \neq \emptyset$.

(1) $\Rightarrow$ (2). Since I is a ϕ-2-absorbing primary ideal of R, $I/\phi(I)$ is weakly 2-absorbing primary ideal of $R/\phi(I)$ by Theorem 2. Let $N = \{i | \psi_i(I_i) \neq R_i, 1 \leq i \leq n\}$. Then by hypothesis $|N| \geq 2$. Since $R/\phi(I)$ is ring-isomorphic to $L = \times_{i \in N} R_i/\psi_i(I_i)$, $J = \times_{i \in N} I_i/\psi_i(I_i)$ is a weakly 2-absorbing ideal of L. Suppose that $|N| \geq 3$. Since J is a nonzero weakly 2-absorbing ideal of L, we conclude that J is a 2-absorbing primary ideal of L by [8, Theorem 2.24]. Thus I is a 2-absorbing primary ideal of R by Lemma 3. Hence assume that $|N| = 2$. Then by hypothesis there are $x, y \in \{1, ..., n\}$ such that I_x is a proper ideal of R_x with $\psi_x(I_x) \neq I_x$ and $I_y = R_y$ with $\psi_y(R_y) \neq R_y$. Thus $R/\phi(I)$ is ring-isomorphic to $F = R_x/\psi_x(I_x) \times R_y/\psi_y(I_y)$. Since $I_x/\psi_x(I_x)$, $R_y/\psi_y(R_y)$ are nonzero ideals of $R/\psi_x(I_x)$ and $R_y/\psi_y(I_y)$, respectively, and $H = I_x/\psi_x(I_x) \times R_y/\psi_y(R_y)$ is a weakly 2-absorbing primary ideal of F, we conclude that H is a 2-absorbing primary ideal of F by [8, Theorem 2.22]. Thus I is a 2-absorbing primary ideal of R by Lemma 3.

(2) $\Rightarrow$ (3). It is clear by [7, Theorem 2.24].

(3) $\Rightarrow$ (1). If (3) holds, then I is a 2-absorbing primary ideal of R by [7, Theorem 2.24]. Thus I is a ϕ-2-absorbing primary ideal of R. □

Theorem 17 *Let $R = R_1 \times R_2 \times \cdots \times R_m$, where $3 \leq m < \infty$, and $R_1, R_2, ..., R_m$ are commutative rings with $1 \neq 0$. For each i, $1 \leq i \leq m$, let $\psi_i : S(R_i) \to S(R_i) \cup \emptyset$ such that $\psi_i(J) \neq \emptyset$ for every $J \in S(R_i)$. Let $\phi = \psi_1 \times \cdots \times \psi_m$. The following statements are equivalent.*

1. *Every proper ideal of R is a ϕ-2-absorbing primary ideal of R.*
2. *$\psi_i(I) = I$ for every proper ideal $I \in S(R_i)$, where $1 \leq i \leq m$. If $m \geq 4$, then $\phi = \phi_1$. If $m = 3$ and $\psi_d(R_d) \neq R_d$ for some d, $1 \leq d \leq 3$, then every proper ideal of R_i is primary for every $i \neq d$, $1 \leq i \leq 3$.*

Proof (1) $\Rightarrow$ (2). Let I_k be a proper ideal of R_k, where $1 \leq k \leq m$. We show that $\psi_k(I_k) = I_k$. Suppose that $\psi_k(I_k) \neq I_k$. Let $I = I_1 \times \cdots \times I_k \times \cdots \times I_m$ such that $I_i = 0$ for each $i \neq k$, $1 \leq i \leq m$. Since $\psi_k(I_k) \neq I_k$, I is a nonzero proper ideal of R. Since $\psi_k(I_k) \neq I_k$ and $3 \leq m < \infty$, we conclude that I is a 2-absorbing ideal of R by Theorem 16, which is impossible since I is not in the form given by Theorem 16(3). Thus $\psi_i(I_i) = I_i$ for every proper ideal I_i of R_i. Assume that $m \geq 4$ and suppose that $\psi_d(R_d) \neq R_d$ for some d, $1 \leq d \leq m$. Let $I = I_1 \times \cdots \times R_d \times \cdots \times I_m$ such that $I_i = 0$ for each $i \neq d$, $1 \leq i \leq m$. Since $\psi_i(I_i) = I_i$ for every proper ideal I_i of R_i, we conclude that $J = I_1 \times \cdots \times R_d/\psi_d(R_d) \times \cdots \times I_m = 0_1 \times \cdots \times R_d/\psi_d(R_d) \times \cdots 0_m$ is a nonzero weakly 2-absorbing primary ideal of $R_1 \times \cdots \times R_d/\psi_d(R_d) \times \cdots \times R_m \cong R/\phi(I)$ by Theorem 2, which is impossible since $m \geq 4$ and J is not in the form given by [8, Theorem 2.24]. Thus $\phi = \phi_1$. Assume that $m = 3$ and suppose that $\psi_d(R_d) \neq R_d$ for some d, $1 \leq d \leq 3$. Without loss of generality, we may assume that $d = 1$. For every $i \neq 1$, $2 \leq i \leq 3$, let I_i be a proper ideal of R_i, and let $I = R_1 \times I_2 \times I_3$. Since $\psi_i(I_i) = I_i$ for every proper ideal I_i of R_i, we conclude that $J = R_1/\psi_1(R_1) \times I_2/I_2 \times I_3/I_3$ is a nonzero weakly 2-absorbing primary ideal of $R_1/\psi_1(R_1) \times R_2/I_2 \times R_3/I_3 \cong R/\phi(I)$ by Theorem 2. Thus I_2/I_2 is a primary ideal of R/I_2 and I_3/I_3 is a primary ideal of R_3/I_3 by [8, Theorem 2.24]. Thus I_2 is a primary ideal of R_2 and I_3 is a primary ideal of R_3.

(2) $\Rightarrow$ (1). If $m \geq 4$ and $\phi = \phi_1$, then the claim is clear. If $m = 3$, then the given conditions in this case imply (1) by Theorem 2 and [8, Theorem 2.24]. □

Let $n \geq 2$. We remind the reader that a commutative ring R with $1 \neq 0$ is a von Neumann regular ring if and only if $I^n = I$ for every proper ideal I of R.

Theorem 18 *Let $R = R_1 \times R_2 \times \cdots \times R_m$, where $3 \leq m < \infty$, and R_1, R_2, ..., R_m are commutative rings with $1 \neq 0$. Let $n \geq 2$. The following statements are equivalent.*

1. *Every proper ideal of R is a ϕ_n-2-absorbing primary ideal of R.*

2. *$R_1, ..., R_m$ are von Neumann regular rings (and hence R is a von Neumann regular ring).*

Proof (1) $\Rightarrow$ (2). For each i, $1 \leq i \leq m$, let $\psi_i : S(R_i) \to S(R_i) \cup \emptyset$ such that $\psi_i(J) = J^n$ for every $J \in S(R_i)$. Then $\psi_i(J) \neq \emptyset$ for every $J \in S(R_i)$. Let $\phi = \psi_1 \times \cdots \times \psi_m$. Then $\phi = \phi_n$. Thus $\phi = \phi_n = \phi_1$ by Theorem 17. Hence $\phi_n(I) = I^n = I$ for every ideal I of R. Thus $\psi_k(J) = J^n = J$ for every ideal J of R_k. Hence each R_i is a von Neumann regular ring, and thus R is a von Neumann regular ring.

(2) $\Rightarrow$ (1). Since R is a von Neumann regular ring, $I^n = I$ for every proper ideal I of R. Thus every proper ideal of R is a ϕ_n-2-absorbing primary ideal of R. □

The hypothesis that $m \geq 3$ is crucial in Theorem 17 and Theorem 18. In the following result, we show that when $m = 2$, then it is possible that every proper ideal of R is a ϕ_n-2-absorbing primary ideal of R, but R need not be a von Neuemann regular ring.

Theorem 19 *Let A, B be quasilocal commutative rings with $1 \neq 0$ that are not fields with maximal ideals $\sqrt{0_A}$, $\sqrt{0_B}$, respectively. Let $R = A \times B$ (hence neither A nor B nor R is a von Neumann regular ring). Then every proper ideal of R is a 2-absorbing primary ideal of R. In particular, if $\phi : S(R) \to S(R) \cup \emptyset$ is a function, then every proper ideal of R is a ϕ-2-absorbing primary ideal of R.*

Proof It is clear that every proper ideal of A is a primary ideal of A and every proper ideal of B is a primary ideal of B. It is also clear that every primary ideal is a 2-absorbing primary ideal. Hence every proper ideal of

R is a 2-absorbing primary ideal of R by [7, Theorem 2.23]. □

In view of the proof of Theorem 19 and [7, Theorem 2.23], we have the following result.

Theorem 20 *Let $R = R_1 \times R_2$, where R_1 and R_2 are commutative rings with $1 \neq 0$. The following statements are equivalent.*

1. *Every proper ideal of R is a 2-absorbing primary ideal of R.*
2. *Every proper ideal of R_1 is a primary ideal of R_1 and every proper ideal of R_2 is a primary ideal of R_2.*

Definition 3 *Let I be a ϕ-2-absorbing primary ideal of R for some function ϕ. Suppose that $I_1I_2I_3 \subseteq I$ but $I_1I_2I_3 \nsubseteq \phi(I)$, for some ideals I_1, I_2, and I_3 of R. We say I is a free-ϕ-triple-zero with respect to $I_1I_2I_3$ if (a, b, c) is not a ϕ-triple-zero of I for every $a \in I_1, b \in I_2$, and $c \in I_3$.*

Recall from [7] that if I is a weakly 2-absorbing primary ideal of R such that $0 \neq I_1I_2I_3 \subseteq I$ for some ideals I_1, I_2, and I_3 of R and (a, b, c) is not a triple-zero of I for every $a \in I_1, b \in I_2$, and $c \in I_3$, then we say that I is a *free-triple-zero with respect to* $I_1I_2I_3$.

Conjecture 1 *Let I be a ϕ-2-absorbing primary ideal of R for some function ϕ. Suppose that $I_1I_2I_3 \subseteq I$ but $I_1I_2I_3 \nsubseteq \phi(I)$, for some ideals I_1, I_2, and I_3 of R. Then I is a free-ϕ-triple-zero with respect to $I_1I_2I_3$.*

Theorem 21 *Let I be a ϕ-2-absorbing primary ideal of R for some function ϕ. Suppose that $I_1I_2I_3 \subseteq I$, but $I_1I_2I_3 \nsubseteq \phi(I)$ for some ideals I_1, I_2 and I_3 of R such that I is a free ϕ-triple-zero with respect to $I_1I_2I_3$. Then $I_1I_2 \subseteq I$ or $I_1I_3 \subseteq \sqrt{I}$ or $I_2I_3 \subseteq \sqrt{I}$.*

Proof Let $J_1 = (I_1+\phi(I))/\phi(I)$, $J_2 = (I_2+\phi(I))/\phi(I)$, and $J_3 = (I_3+\phi(I))/\phi(I)$. Then J_1, J_2, J_3 are ideals of $R/\phi(I)$. Since I is a ϕ-2-absorbing primary ideal of R, $I/\phi(I)$ is a weakly 2-absorbing primary ideal of $R/\phi(I)$ by Theorem 2. Since I is a free ϕ-triple-zero with respect to $I_1I_2I_3$, it is clear that $0 \neq J_1J_2J_3 \subseteq I/\phi(I)$ and $I/\phi(I)$ is a free-triple-zero with respect to $J_1J_2J_3$. Thus by [8, Theorem 3.11], we have $J_1J_2 \subseteq I/\phi(I)$ or $J_1J_3 \subseteq \sqrt{I}/\phi(I)$ or $J_1J_2 \subseteq \sqrt{I}/\phi(I)$. Thus $I_1I_2 \subseteq I$ or $I_1I_3 \subseteq \sqrt{I}$ or $I_2I_3 \subseteq \sqrt{I}$. □

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