Turkish Journal of Mathematics

http://journals.tubitak.gov.tr/math/

Research Article

Turk J Math
(2016) 40: 506 – 516

doi:10.3906/mat-1502-58

Combining Euclidean and adequate rings

Huanyin CHEN[1,*], **Marjan SHEIBANI**[2]
[1]Department of Mathematics, Hangzhou Normal University, Hangzhou, China
[2]Faculty of Mathematics, Statistics and Computer Science, Semnan University, Semnan, Iran

Received: 19.02.2015 • **Accepted/Published Online:** 24.07.2015 • **Final Version:** 08.04.2016

Abstract: We combine Euclidean and adequate rings, and introduce a new type of ring. A ring R is called an E-adequate ring provided that for any $a,b \in R$ such that $aR+bR=R$ and $c \neq 0$ there exists $y \in R$ such that $(a+by,c)$ is an E-adequate pair. We shall prove that an E-adequate ring is an elementary divisor ring if and only if it is a Hermite ring. Elementary matrix reduction over such rings is also studied. We thereby generalize Domsha, Vasiunyk, and Zabavsky's theorems to a much wider class of rings.

Key words: Euclidean rings, adequate rings, elementary divisor rings, elementary matrix reduction

1. Introduction

Throughout this paper, all rings are commutative with an identity. A matrix A (not necessarily square) over a ring R admits diagonal reduction if there exist invertible matrices P and Q such that PAQ is a diagonal matrix (d_{ij}), for which d_{ii} is a divisor of $d_{(i+1)(i+1)}$ for each i. A ring R is an elementary divisor ring provided that every matrix over R admits a diagonal reduction. A ring R is a Hermite ring if every 1×2 matrix over R admits a diagonal reduction. As is well known, a ring R is Hermite if and only if for all $a,b\in R$ there exist $a_1,,b_1\in R$ such that $a=a_1d, b=b_1d$ and $a_1R+b_1R=R$ ([10, Theorem 1.2.5]). After Kaplansky's work on elementary divisor rings without zero divisors, Gillman and Henriksen proved that

Theorem 1.1 [10, Theorem 1.2.11] *A ring R is an elementary divisor ring if and only if*

(1) *R is a Hermite ring;*

(2) *For all $a_1,a_2,a_3\in R$, $a_1R+a_2R+a_3R=R \Longrightarrow$ there exist $p,q\in R$ such that $pa_1R+(pa_2+qa_3)R=R$.*

One of the most attractive problems about the diagonal reduction of matrices is to investigate various conditions under which a Hermite ring is an elementary divisor ring. Many authors have studied this problem over some classes of rings, like Euclidean rings, 2-Euclidean rings, regular rings, adequate rings, and rings having almost stable range 1 (cf. [3,6,8,9,11]).

A map $\varphi : R\to \mathbb{N}$ (the nonnegative integers) is called a norm on R if it satisfies the following conditions: (1) $\varphi(0)=0$; (2) $\varphi(x)>0$ for all $x\neq 0$; (3) $\varphi(xy)\geq \varphi(x)$ wherever $xy\neq 0$. A ring R is called Euclidean provided that there exists a norm φ on R such that for any $a,b\in R, b\neq 0$, we can write $a=bq+r$ with $\varphi(r)<\varphi(b)$ (cf. [1,2]). As is well known, every Euclidean domain is an elementary divisor ring.

*Correspondence: huanyinchen@aliyun.com
2010 *AMS Mathematics Subject Classification:* 13F99, 13E15, 06F20.

An element $c \in R$ is adequate provided that for any $a \in R$ there exist some $r, s \in R$ such that (1) $c = rs$; (2) $rR + aR = R$; (3) $s'R + aR \neq R$ for each noninvertible divisor s' of s. A Bézout ring in which every nonzero element is adequate is called an adequate ring. Kaplansky proved that for the class of adequate domains being a Hermite ring is equivalent to being an elementary divisor ring. This was extended to rings with zero-divisors by Gillman and Henriksen. A ring R has stable range 1 if $aR + bR = R$ with $a, b \in R$ there exists a $y \in R$ such that $a + by \in R$ is invertible. Such a condition plays an important role in algebraic K-theory (cf. [1]). It includes many kind of rings, e.g., regular rings, semiregular rings, π-regular rings, local rings, and clean rings. Domsha and Vasiunyk combined this condition with adequate condition together. They state that a ring R has adequate range 1 if $aR + bR = R$ with $a, b \in R$ implies that there exists a $y \in R$ such that $a + by \in R$ is adequate. [3, Theorem 14] says that every Bézout domain having adequate range 1 is an elementary divisor ring.

Let R be a ring and $a, b \in R$. The pair (a, b) is said to be an E-adequate pair if there exist a $Q \in GE_2(R)$ and an adequate element $w \in R$ such that $(a, b)Q = (w, v)$ for a $v \in R$. Then R is said to be E-adequate if for any $a, b \in R$ such that $aR + bR = R$ and $0 \neq c \in R$, there exists $y \in R$ such that $(a + by, c)$ is an E-adequate pair. In Section 2 we prove that an E-adequate ring is an elementary divisor ring if and only if it is a Hermite ring. A matrix A over a ring admits an elementary reduction provided that A can be reduced to diagonal form by elementary row (column) operations. In Section 3 we explore when every matrix over an E-adequate ring admits an elementary reduction. We prove that for any $m \times n$ matrix over a type of strongly E-adequate Bézout rings R there exist $P \in GE_m(R)$ and $Q \in GE_n(R)$, such that PAQ is a diagonal matrix (d_{ij}), for which d_{ii} is a divisor of $d_{(i+1)(i+1)}$ for each i. More explicit related results are obtained as well. We thereby generalize some known results to much wider class of rings, e.g., [3, Theorem 14], [4, Theorem 5.3], and [10, Theorem 1.2.13].

We shall use $J(R)$ and $U(R)$ to denote the Jacobson radical of R and the set of all units in R, respectively. $GL_n(R)$ stands for the n-dimensional general group of R and $GE_n(R)$ means the subgroup of $GL_n(R)$ generated by $n \times n$ elementary matrices over R.

2. Elementary divisor rings

Clearly, every ring having stable range 1 has adequate range 1. Thus, many known types of rings have adequate range 1, for instance, every semilocal ring, every π-regular ring, and every clean ring (i.e. ring in which every element is the sum of an idempotent and a unit). Moreover, we see that every adequate ring has adequate range 1. By [3, Theorem 11], every VNL ring (i.e. a or $1 - a$ is regular for any element $a \in R$) has adequate range 1. We begin with

Proposition 2.1 *If R has adequate range 1, then it is an E-adequate ring.*

Proof Suppose that R has adequate range 1. Given $aR + bR = R$ with $a, b \in R$ and $0 \neq c \in R$, we can find a $y \in R$ such that $a + by$ is an adequate element of R. Thus, we have $(a + by, c) \cdot I_2 = (a + by, c)$. This shows that the pair $(a + by, c)$ is an E-adequate pair. Therefore, R is an E-adequate ring. □

Example 2.2 *Let $R = \{a + bx \mid a \in \mathbb{Z}, b \in \mathbb{Q}, x^2 = 0\}$. Then R is an E-adequate ring.*

Proof Clearly, $J(R) = x\mathbb{Q}$. Suppose $fR + gR = R$. Case I. $f \in J(R)$. Then $g \in U(R)$; hence, $f + g \cdot 1 \in U(R)$. Case II. $f \notin J(R)$. Since $R/J(R) \cong \mathbb{Z}$ then it is adequate. Now, as $f \notin J(R)$ then from $f + J(R)$ adequate it is easy to deduce that $f \in R$ is adequate. Hence, $f + g \cdot 0 \in R$ is adequate. Thus, R has adequate range 1, and so it is an E-adequate ring, by Proposition 2.1. □

We next record a characterization of E-adequate rings in terms of divisibility chain of finite length, as done in ω-Euclidean rings (cf. [2]).

Proposition 2.3 *Let R be a ring. Then the following are equivalent:*

(1) *R is an E-adequate ring.*

(2) *For any $a, b \in R$ such that $aR + bR = R$ and $0 \neq c \in R$, there exist a $y \in R$ and two finite sequences $(q_i)_{1 \le i \le n}$ and $(r_i)_{1 \le i \le n}$ of elements of R satisfying r_n is adequate and the following equalities:*

$$a + by = cq_1 + r_1, c = r_1q_2 + r_2, \cdots, r_{n-3} = r_{n-2}q_{n-1} + r_{n-1}, r_{n-2} = r_{n-1}q_n + r_n.$$

Proof (1) $\Rightarrow$ (2) Suppose $aR + bR = R$ with $a, b \in R$ and $0 \neq c \in R$. By hypothesis, there exist a $y \in R$, a $U \in GE_2(R)$ and an adequate element $w \in R$ such that $(a + by, c)U = (w, v)$ for a $v \in R$. By adding some identity matrices, we may write U as

$$\begin{pmatrix} 1 & 0 \\ x_1 & 1 \end{pmatrix} \begin{pmatrix} 1 & x_2 \\ 0 & 1 \end{pmatrix} \cdots \begin{pmatrix} 1 & 0 \\ x_{n-1} & 1 \end{pmatrix} \begin{pmatrix} 1 & x_n \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix},$$

where $\alpha, \beta \in U(R)$ and $n \in \mathbb{N}$ is even. Thus,

$$(a + by, c) \begin{pmatrix} 1 & 0 \\ x_1 & 1 \end{pmatrix} \begin{pmatrix} 1 & x_2 \\ 0 & 1 \end{pmatrix} \cdots \begin{pmatrix} 1 & 0 \\ x_{n-1} & 1 \end{pmatrix} \begin{pmatrix} 1 & x_n \\ 0 & 1 \end{pmatrix} = (w\alpha^{-1}, v\beta^{-1}).$$

Let $(a+by, c) \begin{pmatrix} 1 & 0 \\ x_1 & 1 \end{pmatrix} = (y_1, z_1)$. Then $(a+by)+cx_1 = y_1, z_1 = c$. Let $(y_1, z_1) \begin{pmatrix} 1 & x_2 \\ 0 & 1 \end{pmatrix} = (y_2, z_2)$. Then $z_1 + y_1x_2 = z_2, y_1 = y_2$. By iteration of this process, we let

$$\begin{aligned} (y_{n-2}, z_{n-2}) \begin{pmatrix} 1 & 0 \\ x_{n-1} & 1 \end{pmatrix} &= (y_{n-1}, z_{n-1}), \\ (y_{n-1}, z_{n-1}) \begin{pmatrix} 1 & x_n \\ 0 & 1 \end{pmatrix} &= (w\alpha^{-1}, v\beta^{-1}). \end{aligned}$$

Then $y_{n-2} + z_{n-2}x_{n-1} = y_{n-1}, z_{n-2} = z_{n-1}, y_{n-1} = w\alpha^{-1}$, and $z_{n-1} + y_{n-1}x_n = v\beta^{-1}$. Let $q_i = -x_i$, $r_i = y_i\,(i \text{ is odd})$ and $r_i = z_i\,(i \text{ is even})$. Then we get a sequence of equations:

$$a + by = cq_1 + r_1, c = r_1q_2 + r_2, \cdots, r_{n-3} = r_{n-2}q_{n-1} + r_{n-1},$$

where $r_{n-1} = w\alpha^{-1} \in R$ is adequate, as desired.

(2) $\Rightarrow$ (1) By hypothesis, there exists a finite divisible chain:

$$a = bq_1 + r_1, b = r_1q_2 + r_2, \cdots, r_{n-2} = r_{n-1}q_n + r_n,$$

where $r_n \in R$ is adequate. Hence, we get $(a, b) \begin{pmatrix} 0 & -1 \\ 1 & q_1 \end{pmatrix} = (b, -r_1), (b, -r_1) \begin{pmatrix} 0 & -1 \\ 1 & -q_2 \end{pmatrix} = (-r_1, -r_2)$, $(-r_1, -r_2) \begin{pmatrix} 0 & -1 \\ 1 & q_3 \end{pmatrix} = (-r_2, r_3)$, $(-r_2, r_3) \begin{pmatrix} 0 & -1 \\ 1 & -q_4 \end{pmatrix} = (r_3, r_4), (r_3, r_4) \begin{pmatrix} 0 & -1 \\ 1 & q_5 \end{pmatrix} = (r_4, -r_5), \quad \cdots,$ and

$$\begin{aligned} & \left((-1)^{\frac{(n-2)(n-1)}{2}} \cdot r_{n-2}, \quad (-1)^{\frac{(n-1)n}{2}} \cdot r_{n-1}\right) \begin{pmatrix} 0 & -1 \\ 1 & (-1)^{n-1}q_n \end{pmatrix} \\ = & \left((-1)^{\frac{(n-1)n}{2}} \cdot r_{n-1}, (-1)^{\frac{n(n+1)}{2}} \cdot r_n\right). \end{aligned}$$

Clearly, $c := (-1)^{\frac{n(n+1)}{2}} \cdot r_n \in R$ is adequate. Thus,

$$\big((-1)^{\frac{(n-1)n}{2}} \cdot r_{n-1}, (-1)^{\frac{n(n+1)}{2}} \cdot r_n\big) \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \big(c, (-1)^{1+\frac{(n-1)n}{2}} \cdot r_{n-1}\big).$$

Let $F(x)$ denote the matrix $\begin{pmatrix} 0 & -1 \\ 1 & x \end{pmatrix}$. Then

$$F(x) = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$$

is the product of four elementary matrices. This completes the proof. $\square$

The following lemma is well known (see [Lam, Excrise in Classical Ring Theory, Ex. 20.10B]). We include a simple proof for completeness.

Lemma 2.4 *If R has stable range 1, then $J(R) = \{x \mid x - u \in U(R)$ for any $u \in U(R)\}$.*

Proof Let $x \in R$ and $x - u \in U(R)$ for any $u \in U(R)$. Let $r \in R$. Then $xR + (1 - xr)R = R$. Since R has stable range 1, we have a $y \in R$ such that $u := x + (1 - xr)y \in U(R)$. Hence, $x - u = -(1 - xr)y \in U(R)$, and then $1 - xr \in U(R)$. Therefore, $x \in J(R)$, and thus yielding the result. $\square$

Recall that a ring R is semiregular provided that $R/J(R)$ is regular and every idempotent lifts modulo $J(R)$. A ring R is semiregular if and only if for any $a \in R$ there exists an idempotent $e \in aR$ such that $a - ae \in J(R)$ (cf. [7]).

Lemma 2.5 *Let R be a Bézout ring, and let $a \in R$ be adequate. Then R/aR is semiregular.*

Proof Let $S = R/aR$ and $x \in S$. In view of [12, Theorem 2], $0 \in S$ is adequate. Then we have some $r, s \in R$ such that $0 = rs$, where $rS + xS = S$ and $s'S + xS \neq S$ for any noninvertible divisor s' of s. Write $rS + sS = hS$. If $h \notin U(S)$, then $s = hs'$ for some $s' \in R$. Hence, $hS + xS \neq S$. Write $r = hr'$. Then $hS + xS = S$, an absurd. This implies that $rS + sS = S$. Since $rS + xS = rS + sS = S$, we get $rS + sxS = S$. Write $rc + sxd = 1$ in S. Set $e = rc$. Then $e^2 - e = (rc)^2 - rc = (rc)(sxd) = 0$. Let u be an arbitrary invertible element of S.

Claim I. $(u - ex)S + rS = S$. If not, $(u - ex)S + rS = tS \neq S$. Then $rS \subseteq tS$, and so $u \in eS + tS \subseteq rS + tS \subseteq tS$. This implies that $t \in U(S)$, a contradiction.

Claim II. $(u - ex)S + sS = S$. If not, $(u - ex)S + sS = tS \neq S$. Then t is a noninvertible divisor of s, and so $tS + xS \neq S$. It follows from $eS + sS = S$ that $eS + tS = S$. Write $u - ex = tw$ with $w \in R$. Then $e = ex + tw$, and so $eS + tS \subseteq tS + xS \neq S$, an absurd.

Finally, $(u - ex)S + rsS = S$. we get $u - ex \in U(S)$. This implies that $\big(x - (1 - e)x\big) - u = ex - u \in U(S)$. In light of [12, Theorem 3], S has stable range 1. It follows from Lemma 2.4 that $x - (1 - e)x \in J(S)$. Clearly, $1 - e = sxd \in xS$. Therefore, the lemma is true. $\square$

Lemma 2.6 (cf. [10, Theorem 1.2.9]) *Let R be a Hermite ring. Then the following are equivalent:*

(1) *R is an elementary divisor ring.*

(2) *Every matrix $\begin{pmatrix} a & 0 \\ b & c \end{pmatrix} \in M_2(R)$ with $aR + bR + cR = R$ admits an elementary reduction.*

We have accumulated all the information necessary to prove the following.

Theorem 2.7 *Let R be an E-adequate ring. Then R is an elementary divisor ring if and only if R is a Hermite ring.*

Proof $\Longrightarrow$ This is obvious by Theorem 1.1.

$\Longleftarrow$ Suppose that $A = \begin{pmatrix} a & 0 \\ b & c \end{pmatrix} \in M_2(R)$ with $aR + bR + cR = R$. It will suffice to prove that A admits an elementary reduction, in terms of Lemma 2.6. If $c = 0$, then $aR + bR = R$. Write $ax + by = 1$. Then

$$\begin{pmatrix} x & y \\ -b & a \end{pmatrix} A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

We check that $det \begin{pmatrix} x & y \\ -b & a \end{pmatrix} = 1$, as desired.

If $c \neq 0$, it will suffice to prove that A admits an elementary diagonal reduction. Write $ax + by + cz = 1$ for some $x, y, z \in R$. Then $bR + (ax + cz)R = R$. By hypothesis, there exists some $t \in R$ such that $b + (ax + cz)t = w \in R$ and (w, c) is an E-adequate pair. Thus, we see that

$$\begin{pmatrix} 1 & 0 \\ xt & 1 \end{pmatrix} A \begin{pmatrix} 1 & 0 \\ zt & 1 \end{pmatrix} = \begin{pmatrix} a & 0 \\ w & c \end{pmatrix}.$$

As (w, c) is an E-adequate pair, there exist a $Q \in GE_2(R)$ and a $d \in R$ such that $(w, c)Q = (e, d)$ where $e \in R$ is adequate. Hence,

$$\begin{pmatrix} a & 0 \\ w & c \end{pmatrix} Q \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} * & * \\ d & e \end{pmatrix}.$$

As R is a Hermite ring, we can find a $P \in GL_2(R)$ such that $(d, e)P = (q, 0)$ for a $q \in R$. Thus, $(d, e)P \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = (0, q)$. Set $\big(P \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\big)^{-1} = (p_{ij})$. Then $(d, e) = (0, q)(p_{ij})$, and so $e = qp_{22}$. Further, we see that

$$\begin{pmatrix} 1 & 0 \\ xt & 1 \end{pmatrix} A \begin{pmatrix} 1 & 0 \\ zt & 1 \end{pmatrix} Q \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} P \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} p & r \\ 0 & q \end{pmatrix}.$$

As $aR + bR + cR = R$, we see that $rR + pR + qR = R$. Since $e \in R$ is adequate, R/eR is semiregular by Lemma 2.5, and so $(R/eR)/J(R/eR)$ is regular. For any $\overline{\alpha} \in R/eR$, we can find some $\beta \in R$ such that $\big(\alpha - \alpha\beta\alpha\big) + eR \in J(R/eR)$. It follows from $eR \subseteq qR$ that $\big(\alpha - \alpha\beta\alpha\big) + qR \in J(R/qR)$. Thus, $(R/qR)/J(R/qR)$ is regular; hence, $(R/qR)/J(R/qR)$ has stable range 1. This implies that R/qR has stable range 1.

Clearly, $\overline{r}(R/qR) + \overline{p}(R/qR) = R/qR$. Then we can find some $t \in R$ such that $\overline{r + pt} \in R/qR$ is invertible. Hence, $(r + pt)k + ql = 1$ for some $k, l \in R$. One easily checks that

$$\begin{pmatrix} k & l \\ -q & r + pt \end{pmatrix} \begin{pmatrix} p & r \\ 0 & q \end{pmatrix} \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & -kp \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -qp \end{pmatrix}.$$

Since $det \begin{pmatrix} k & l \\ -q & r + pt \end{pmatrix} = 1$, we see that $\begin{pmatrix} k & l \\ -q & r + pt \end{pmatrix} \in GL_2(R)$, and therefore A admits diagonal reduction, as asserted. $\square$

We now extend [3, Theorem 14] to Bézout rings (not necessarily domains).

Corollary 2.8 *If R has adequate range 1, then R is an elementary divisor ring if and only if it is a Bézout ring.*

Proof $\Longrightarrow$ In light of Theorem 1.1, R is a Hermite ring. Thus, it is a Bézout ring by [10, Corollary 2.1.3].

$\Longleftarrow$ Suppose that $aR+bR+cR=R$ with $a,b,c\in R$. Write $ax+by+cz=1$ for some $x,y,z\in R$. Then there exists $s\in R$ such that $w:=a+bys+czs\in R$ is adequate. In view of Lemma 2.5, R/wR is semiregular; hence, it has stable range 1. Clearly, $wR+bR+cR=R$, and so $\overline{b}\big(R/wR\big)+\overline{c}\big(R/wR\big)=R/wR$. Thus, we can find $p\in R$ such that $\overline{b+cp}\in U\big(R/wR\big)$. This implies that $wR+(b+cp)R=R$, i.e. $(a+bys+czt)k+(b+cp)l=1$ for some $k,l\in R$. It follows that

$$(a+(b+cp)ys+c(zt-pys))k+(b+cp)l=1.$$

Thus, $\big(a+c(zt-pys)\big)R+(b+cp)R=R$. This proves that R has stable range 2. In light of [10, Theorem 2.1.2], R is a Hermite ring. Hence, R is an elementary divisor ring, by Theorem 2.7. □

Recall that a Bézout ring is an adequate ring if every nonzero element in R is adequate. Every adequate ring has adequate range 1. It is convenient to prove that every adequate ring is an elementary divisor ring (cf. [4, 11]).

Corollary 2.9 *Every E-adequate Bézout domain is an elementary divisor ring.*

Proof Let R be an E-adequate Bézout domain. In view of [10, Theorem 1.2.8], R is a Hermite ring, and therefore we complete the proof, by Theorem 2.7. □

3. Elementary matrix reduction

Among all rings in the class of elementary divisor rings there is a subclass of rings such that every matrix over these rings can be reduced to canonical diagonal form, using only elementary operations. There are elementary divisor rings over which there exist matrices that have no elementary matrix reduction, e.g., the principal ring $\mathbb{R}[x,y]/(x^2+y^2+1)$ ([10, p.38]). In fact, using elementary operations is crucial in matrix reduction. The aim of this section is to investigate when every matrix over a ring admits an elementary reduction. Let R be a ring, and let $a,b\in R$. We say that (a,b) is a strongly E-adequate pair provided that there exist a $Q\in GE_2(R)$ such that $(a,b)Q=(w,0)$ for an adequate $w\in R$. Clearly, every strongly E-adequate pair in a ring is an E-adequate pair. A ring R is strongly E-adequate provided that for any $a,b\in R$ such that $aR+bR=R$ and $0\neq c\in R$, there exists a $y\in R$ such that $(a+by,c)$ is a strongly E-adequate pair.

Proposition 3.1 *Every ring having stable range 1 is strongly E-adequate.*

Proof Suppose that R has stable range 1. Given $aR+bR=R$ with $a,b\in R$ and $0\neq c\in R$, there exists a $y\in R$ such that $u:=a+by\in U(R)$. Hence,

$$(a+by,c)\begin{pmatrix}1 & -u^{-1}c\\ 0 & 1\end{pmatrix}=(u,0),$$

thus yielding the result. □

Thus, we see that regular rings, π-regular rings, local rings, semilocal rings, and clean rings are all strongly E-adequate.

Lemma 3.2 *Let R be a ring. Then the following are equivalent:*

(1) *R is a strongly E-adequate ring.*

(2) *For any $a, b \in R$ such that $aR+bR$ and $0 \neq c \in R$, there exist a $y \in R$ and two finite sequences $(q_i)_{1\leq i\leq n}$ and $(r_i)_{1\leq i\leq n-1}$ of elements of R satisfying r_{n-1} is adequate and the following equalities:*

$$a+by = cq_1 + r_1, c = r_1q_2 + r_2, \cdots, r_{n-3} = r_{n-2}q_{n-1} + r_{n-1}, r_{n-2} = r_{n-1}q_n.$$

Proof (1) $\Rightarrow$ (2) Suppose $aR + bR = R$ with $a, b \in R$ and $0 \neq c \in R$. By hypothesis, there exist a $y \in R$ and a $U \in GE_2(R)$ such that $(a+by, c)U = (w, 0)$ for some adequate element $w \in R$. As in the proof of Proposition 2.3, we can find $x_1, \cdots, x_n$ and $\alpha \in U(R)$ such that

$$(a,b)\begin{pmatrix} 1 & 0 \\ x_1 & 1 \end{pmatrix}\begin{pmatrix} 1 & x_2 \\ 0 & 1 \end{pmatrix}\cdots\begin{pmatrix} 1 & 0 \\ x_{n-1} & 1 \end{pmatrix}\begin{pmatrix} 1 & x_n \\ 0 & 1 \end{pmatrix} = (w\alpha^{-1}, 0).$$

We may assume that $n \in \mathbb{N}$ is even. Let

$$\begin{aligned}
(a,b)\begin{pmatrix} 1 & 0 \\ x_1 & 1 \end{pmatrix} &= (y_1, z_1), \\
(y_1,z_1)\begin{pmatrix} 1 & x_2 \\ 0 & 1 \end{pmatrix} &= (y_2, z_2), \\
&\vdots \\
(y_{n-2},z_{n-2})\begin{pmatrix} 1 & 0 \\ x_{n-1} & 1 \end{pmatrix} &= (y_{n-1}, z_{n-1}), \\
(y_{n-1},z_{n-1})\begin{pmatrix} 1 & x_n \\ 0 & 1 \end{pmatrix} &= (w\alpha^{-1}, 0).
\end{aligned}$$

Then $y_{n-2} + z_{n-2}x_{n-1} = y_{n-1}, z_{n-2} = z_{n-1}, y_{n-1} = w\alpha^{-1}$, and $z_{n-1} + y_{n-1}x_n = 0$. Let $q_i = -x_i$, $r_i = y_i\,(i \text{ is odd})$ and $r_i = z_i\,(i \text{ is even})$. Then we get a sequence of equations:

$$a = bq_1 + r_1, b = r_1q_2 + r_2, \cdots, r_{n-3} = r_{n-2}q_{n-1} + r_{n-1}, r_{n-2} = r_{n-1}q_n,$$

where $r_{n-1} = w\alpha^{-1} \in R$ is adequate, as required.

(2) $\Rightarrow$ (1) For any $a, b \in R$ such that $aR + bR = R$ and $0 \neq c \in R$, there exist a $y \in R$ and two finite sequences $(q_i)_{1\leq i\leq n}$ and $(r_i)_{1\leq i\leq n-1}$ of elements of R satisfying r_{n-1} is adequate and the following equalities:

$$a+by = cq_1 + r_1, c = r_1q_2 + r_2, \cdots, r_{n-3} = r_{n-2}q_{n-1} + r_{n-1}, r_{n-3} = r_{n-1}q_n.$$

As in the proof of Lemma 2.3, we can find a $Q \in GE_2(R)$ such that $(a+by, c)Q = \big(0, (-1)^{1+\frac{(n-1)n}{2}} \cdot r_{n-1}\big)$. Therefore, $(a+by, c)Q\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \big((-1)^{1+\frac{(n-1)n}{2}} \cdot r_{n-1}, 0\big)$, as desired. $\square$

Theorem 3.3 *Every Euclidean ring is a strongly E-adequate ring.*

Proof Let R be a Euclidean ring. Suppose that $aR+bR=R$ and $0\neq c\in R$. Since R is a Euclidean ring, it is principal, and so R is neat by [5, Proposition 2.4], and so R/cR has stable range 1. Since $\overline{a}(R/cR)+\overline{b}(R/cR)=R/cR$, there exists a $y\in R$ such that $\overline{a+by}\in U(R/cR)$. It follows that $(a+by)R+cR=R$. By hypothesis, we have a norm φ on R, and that $a+by=cq_1+r_1$ with $\varphi(r_1)<\varphi(c)$ or $r_1=0$. If $r_1\neq 0$, then there exists a sequence of equations: $c=r_1q_2+r_2$ with $\varphi(r_2)<\varphi(r_1)$ or $r_2=0$. If $r_2\neq 0$, then we have $r_1=r_2q_3+r_3$ with $\varphi(r_3)<\varphi(r_2)$ or $r_3=0$. By iteration of this process, we shall find an infinite inequality: $\cdots<\varphi(r_m)<\cdots<\varphi(r_1)$, an absurd. Otherwise, we get a sequence of equations:

$$a+by=cq_1+r_1, c=r_1q_2+r_2, r_1=r_2q_3+r_3,\cdots,r_{n-2}=r_{n-1}q_n.$$

Clearly, $r_{n-2}\in r_{n-1}R$ and $r_{n-3}\in r_{n-2}R+r_{n-1}R\subseteq r_{n-1}R$. Repeating this process, we see that $r_{n-4},\cdots,c,a+by\subseteq r_{n-1}R$. Hence, $R=(a+by)R+cR\subseteq r_{n-1}R\subseteq R$. This implies that $r_{n-1}R=R$, and thus $r_{n-1}\in U(R)$. Therefore, R is a strongly E-adequate ring, by Lemma 3.2. □

The converse of Theorem 3.3 is not true as the following shows.

Example 3.4 *Let $R=F[[X,Y]]$ where F is a field. Then R is a strongly E-adequate ring while it is not a Euclidean ring.*

Proof Since R is a local ring, it has stable range 1. Thus it is a strongly E-adequate ring by Proposition 3.1. However, R is not a Bézout domain, since the ideal $XR+YR$ cannot be generated be an element. Hence, R is not a principal ideal ring. Therefore, R is not a Euclidean ring, as required. □

A ring R is elementary principal if, for any $a,b\in R$, there exist $c\in R$ and $Q\in GE_2(R)$ such that $(a,b)Q=(c,0)$. Examples of elementary principal rings are Euclidean rings, valuation rings, and regular rings ([13]).

Lemma 3.5 *Every strongly E-adequate Bézout ring is elementary principal.*

Proof Let R be a strongly E-adequate Bézout ring. For any $a,b\in R$, there exist $d,s,t,x,y\in R$ such that $as+bt=d, a=dx$, and $b=dy$. If $y=0$, then $(a,b)=(a,0)$. We now assume that $y\neq 0$. Let $w=xs+yt-1$. Then $dw=0$ and $xR+(yt-w)R=R$. By hypothesis, there exists a $z\in R$ and a $Q\in GE_2(R)$ such that $\big(x+(yt-w)z,y\big)Q=(v,0)$ for an adequate element $v\in R$. Thus, $(a+btz,b)Q=d\big(x+(yt-w)z,y\big)Q=d(v,0)=(dv,0)$. That is,

$$(a,b)\begin{pmatrix} 1 & 0\\ tz & 1\end{pmatrix}Q=(dv,0).$$

Thus, R is elementary principal, as asserted. □

Theorem 3.6 *Let R be a strongly E-adequate ring. Then the following are equivalent:*

(1) *Every matrix over R admits an elementary reduction.*

(2) *R is a Bézout ring.*

Proof (1) $\Rightarrow$ (2) This is obvious.

(2) $\Rightarrow$ (1) By virtue of Lemma 3.5, R is elementary principal, and so R is a Hermite ring. It follows by [13, Theorem 1] that R is a quasi-Euclidean ring. In light of [13, Proposition 2], it suffices to consider

matrices $A=\begin{pmatrix} a & 0 \\ b & c \end{pmatrix}$, where $aR+bR+cR=R$. Since R is strongly E-adequate, it is E-adequate. In light of Theorem 2.7, we can find $U,V\in GL_2(R)$ such that UAV admits a diagonal reduction. Clearly, every elementary principal ring is a GE_2-ring, and so $GL_2(R)=GE_2(R)$. Hence, $U,V\in GE_2(R)$. This completes the proof. $\square$

As an immediate consequence of Theorem 3.6 and Theorem 3.3, we now derive a known result of Kaplansky's (cf. [4]).

Corollary 3.7 Every matrix over Euclidean rings admits an elementary reduction.

A ring R is adequately stable provided that for any adequate $a\in R, aR+bR=R\Longrightarrow \exists y\in R$ such that $a+by\in U(R)$. We turn to E-adequate rings under adequately stable condition.

Lemma 3.8 *Let R be an E-adequate ring. If R is adequately stable, then $aR+bR=R$ with $a,b\in R$ implies that there exists some $U\in GE_2(R)$ such that $(a,b)U=(1,0)$.*

Proof Suppose $aR+bR=R$ with $a,b\in R$. Case I. $b=0$. Then $a\in U(R)$. Hence, $(a,b)Q=(1,0)$ where $Q=\begin{pmatrix} a^{-1} & 0 \\ 0 & 1 \end{pmatrix}\in GE_2(R)$. Case II. $b\neq 0$. By hypothesis, we can find a $y\in R$ such that $(a+by,b)$ is an E-adequate pair. Thus, there exist a $Q\in GE_2(R)$ and an adequate $w\in R$ such that $(a+by,b)Q=(w,v)$ for a $v\in R$. It follows that

$$(a,b)\begin{pmatrix} 1 & 0 \\ y & 1 \end{pmatrix}Q=(w,v).$$

Since $wR+vR=R$, we have a $z\in R$ such that $u:=w+vz\in U(R)$. Therefore,

$$(a,b)\begin{pmatrix} 1 & 0 \\ y & 1 \end{pmatrix}Q\begin{pmatrix} 1 & 0 \\ z & 1 \end{pmatrix}\begin{pmatrix} 1 & -u^{-1}v \\ 0 & 1 \end{pmatrix}\begin{pmatrix} u^{-1} & 0 \\ 0 & 1 \end{pmatrix}=(1,0),$$

the result follows. $\square$

Recall that a ring R has stable range 2 if $a_1R+a_2R+a_3R=R$ implies that there exist some $b_1,b_2\in R$ such that $(a_1+a_3b_1)R+(a_2+a_3b_2)R=R$ (cf. [1]). For instance, every elementary divisor ring has stable range 2.

Lemma 3.9 *Let R be an E-adequate ring having stable range 2. If R is adequately stable, then $a_1R+a_2R+\cdots+a_nR=R\,(n\geq 2)$ implies that there exists some $U\in GE_n(R)$ such that $(a_1,a_2,\cdots,a_n)U=(1,0,\cdots,0)$.*

Proof By virtue of Lemma 3.8, the result holds for $n=2$. Assume that the result holds for $n=k(k\geq 2)$. Let $n=k+1$. Then $(a_1+a_{k+1}c_1)R+\cdots+(a_k+a_{k+1}c_k)R=R$ for some $c_1,\cdots,c_k\in R$. Hence,

$$(a_1,a_2,\cdots,a_n)\begin{pmatrix} 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ c_1 & c_2 & \cdots & c_k & 1 \end{pmatrix}=(a_1+a_{k+1}c_1,\cdots,a_k+a_{k+1}c_k,a_n).$$

By hypothesis, there exists some $Q\in GE_k(R)$ such that $(a_1+a_{k+1}c_1,\cdots,a_k+a_{k+1}c_k)Q=(1,0,\cdots,0)$.

Therefore,

$$\begin{aligned}
&(a_1, a_2, \cdots, a_n) \begin{pmatrix} 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ c_1 & c_2 & \cdots & c_k & 1 \end{pmatrix} \begin{pmatrix} Q & \\ & 1 \end{pmatrix} \\
&= (a_1 + a_{k+1}c_1, \cdots, a_k + a_{k+1}c_k, a_n) \begin{pmatrix} Q & \\ & 1 \end{pmatrix} \\
&= (1, 0, \cdots, 0, a_n).
\end{aligned}$$

Let

$$U = \begin{pmatrix} 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ c_1 & c_2 & \cdots & c_k & 1 \end{pmatrix} \begin{pmatrix} Q & \\ & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & \cdots & 0 & -a_n \\ 0 & 1 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 1 \end{pmatrix}.$$

Then $(a_1, a_2, \cdots, a_n)U = (1, 0, \cdots, 0)$. By induction, we obtain the result. $\square$

Lemma 3.10 *Let R be an E-adequate ring having stable range 2. If R is adequately stable, then R is a GE-ring, i.e. $GL_n(R) = GE_n(R)$ for all $n \in \mathbb{N}$.*

Proof Let $A = (a_{ij}) \in GL_n(R)$. Then $a_{11}R + \cdots + a_{1n}R = R$. In view of Lemma 3.9, there exists some $U \in GE_n(R)$ such that $(a_{11}, \cdots, a_{1n})U = (1, 0, \cdots, 0)$. Thus we deduce that

$$AU = \begin{pmatrix} 1 & 0 \\ C & B \end{pmatrix}.$$

This implies that

$$\begin{pmatrix} 1 & 0 \\ -C & I_{n-1} \end{pmatrix} AU = \begin{pmatrix} 1 & \\ & B \end{pmatrix}$$

for some $B \in GL_{n-1}(R)$. By iteration of this process, we can find $V, W \in GE_n(R)$ such that $VAW = diag(1, 1, \cdots, 1, u)$ for an invertible $u \in R$, and therefore R is a GE-ring. $\square$

Theorem 3.11 *Let R be an E-adequate ring. If R is adequately stable, then the following are equivalent:*

(1) *Every matrix over R admits an elementary reduction.*

(2) *R is a Hermite ring.*

Proof (1) $\Rightarrow$ (2) Clearly, R is an elementary divisor ring, and then R is a Hermite ring, by Theorem 1.1.

(2) $\Rightarrow$ (1) Let A be an $m \times n$ matrix over R. In light of Theorem 2.7, there exist $U \in GL_m(R)$ and $V \in GL_n(R)$ such that

$$UAV = \begin{pmatrix} \varepsilon_1 & & & & & \\ & \varepsilon_2 & & & & \\ & & \ddots & & & \\ & & & \varepsilon_r & \cdots & 0 \\ & & & \vdots & & \vdots \\ & & & 0 & \cdots & 0 \end{pmatrix} \text{ where } \varepsilon_i \text{ is a divisor } \varepsilon_{i+1}.$$

As R is a Hermite ring, R has stable range 2, by [10, Corollary 2.1.1]. Thus, it follows by Lemma 3.10 that R is a GE-ring. Hence, we prove that $U \in GE_m(R)$ and $V \in GE_n(R)$, and therefore the result follows. □

Corollary 3.12 *If R has stable range 1, then the following are equivalent:*

(1) *Every matrix over R admits an elementary reduction.*

(2) *R is a Bézout ring.*

Proof One direction is obvious. Conversely, assume that R is a Bézout ring. Then R is a Hermite ring, by [10, Corollary 2.1.8]. Obviously, R is an E-adequate ring and adequately stable. Therefore, we complete the proof by Theorem 3.11. □

As an immediate consequence of Corollary 3.12, we prove that every matrix over regular rings admits an elementary reduction.

Acknowledgments

The authors are grateful to the referee for his/her useful suggestions, which make the new version more clearer. The research of the first author was supported by the Natural Science Foundation of Zhejiang Province, China (No. LY13A010019).

References

[1] Chen H. Rings Related Stable Range Conditions. Hackensack, NJ, USA: World Scientific, 2011.

[2] Cooke GE. A weakening of the Euclidean property for integral domains and applications to algebraic number theory I. J Reine Angew Math 1976; 282: 133-156.

[3] Domsha OV, Vasiunyk IS. Combining local and adequate rings. Book of abstracts of the International Algebraic Conference. Ukraine, Kyiv: Taras Shevchenko National University of Kyiv, 2014.

[4] Kaplansky I. Elementary divisors and modules. Trans Amer Math Soc 1949; 66: 464-491.

[5] McGovern WW. Neat rings. J Pure Appl Algebra 2006; 205: 243-265.

[6] McGovern WW. Bézout rings with almost stable range 1. J Pure Appl Algebra 2008; 212: 340-348.

[7] Nicholson WK. Semiregular modules and rings. Canad J Math 1975; 28: 1105-1120.

[8] Roitman M. The Kaplansky condition and rings of almost stable range 1. Proc Amer Math Soc 2013; 141: 3013-3018.

[9] Zabavsky BV. Rings over which every matrix admits the diagonal reduction by elementary transformations. Math Stud 1997; 8: 136–139.

[10] Zabavsky BV. Diagonal Reduction of Matrices over Rings. Ukraine: VNTL Publisher, 2012.

[11] Zabavsky BV, Bilavska SI. Stable rank of adequate rings. Math Stud 2010; 33: 212-214.

[12] Zabavsky BV, Bilavska SI. Every zero adequate ring is an exchange ring. J Math Sci 2012; 187: 153-156.

[13] Zabavsky BV, Romaniv O.M. Rings with elementary reduction of matrices. Ukr Math J 2000; 52: 1872-11881.