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Existence and nonexistence of sign-changing solutions to elliptic critical equations

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Abstract: We consider the nonlinear equation $-\Delta u = |u|^{p-1}u - \varepsilon u$ in $\Omega, u = 0$ on $\partial\Omega$, where Ω is a smooth bounded domain in $\mathbb{R}^n$, $n \geq 4$, ε is a small positive parameter, and $p = (n+2)/(n-2)$. We study the existence of sign-changing solutions that concentrate at some points of the domain. We prove that this problem has no solutions with one positive and one negative bubble. Furthermore, for a family of solutions with exactly two positive bubbles and one negative bubble, we prove that the limits of the blow-up points satisfy a certain condition.

Key words: Blow-up analysis, critical Sobolev exponent, sign-changing solutions

1. Introduction

In this paper, we study the sign-changing solutions for the following semilinear elliptic problem:

$$(P_\varepsilon)\begin{cases} -\Delta u = |u|^{p-1}u - \varepsilon u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where Ω is a smooth bounded domain in $\mathbb{R}^n$, $n \geq 4$, ε is a small positive parameter, and $p+1 = \frac{2n}{n-2}$ is the critical Sobolev exponent for the embedding of $H_0^1(\Omega)$ into $L^{p+1}(\Omega)$.

The Sobolev embedding $H_0^1(\Omega) \hookrightarrow L^{p+1}(\Omega)$ is known to be noncompact, and for this reason, the solvability of (P_ε) is quite delicate. Pohozaev's identity [24] shows that the problem (P_ε) has only a trivial solution if the domain Ω is assumed to be strictly star-shaped.

Moreover, during the last two decades, there has been extensive research on this problem, and much progress has been made with regard to the existence of positive solutions. It is known that there is an effect of the domain topology on the existence of positive solutions. The first attempts were made by Bahri and Coron [2], who found a positive solution for (P_0) in the case that the domain Ω satisfies some nontrivial topological conditions. Moreover, Dancer [13] and Ding [14] gave an example of contractible domains on which a solution still exists, showing that both topology and geometry of the domains play a prominent role.

The great contribution was the work of Brezis and Nirenberg [10]. Assuming that Ω is a bounded regular domain in $\mathbb{R}^n$, $n \geq 4$ and $\varepsilon \in (-\lambda_1(\Omega), 0)$, where $\lambda_1(\Omega)$ denotes the first eigenvalue of $-\Delta$ under the Dirichlet boundary condition. They proved that (P_ε) has a solution. Furthermore, for $n = 3$ there exists $\lambda_1^* > 0$ such that (P_ε) has a solution if $\varepsilon \in (-\lambda_1(\Omega), -\lambda_1^\star)$. This paper highlighted the crucial role played by the dimension

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n in the study of (P_ε). The reason for this difference relies on the presence in the equation of the lower order term εu, which makes the estimates quite different.

After the work of Brezis and Nirenberg, Han in [16] proved that the solution found by them blows up at the critical point of Robin's function defined by $\varphi(x) = H(x,x)$, where H is the regular part of Green's function as $\varepsilon < 0$ goes to zero. Conversely, in [25, 26] Rey proved that any C^1-stable critical point of Robin's function generates a family of solutions that blow up at this point as ε goes to zero. Moreover, in [19], Musso and Pistoia considered the case where $\varepsilon > 0$ is close to 0. They also proved the existence of a family of solutions that blow up and concentrate in two points if Ω is a domain with a small "hole".

The existence and qualitative behavior of sign-changing solutions for elliptic problems with critical nonlinearity have been extensively investigated during the last few decades (see [4, 5, 7, 8, 11, 12, 15, 17, 18, 20, 21]). Ben Ayed et al. in [7, 8] studied the blow-up of the low energy sign-changing solutions of $(P_{-\varepsilon})$, which converges to the value $2S^{n/2}$ as $\varepsilon \to 0$. More precisely, they proved that the solution blow-up occurs at exactly two points, which are the limits of concentration points of the positive and negative parts of the solution and whose distance from each other and from the boundary is bounded. In [11], Castro and Clapp considered a suitable symmetric domain Ω and proved the existence of one pair of solutions that change sign exactly once, provided that $n \geq 4$ and $\varepsilon < 0$ small. Micheletti and Pistoia in [18] and Bartsh et al. in [4] generalized such a result showing the existence of at least N pairs of sign-changing solutions with one positive and one negative blow-up point.

The study of asymptotic behavior would become difficult in the absence of solution positivity assumption. The major difficulty is that the limit problem of (P_ε) after a change of variable is

$$-\Delta u = |u|^{p-1}u \quad \text{in } \mathbb{R}^n, \tag{1.1}$$

having many unknown sign-changing solutions. However, interesting information about energy shows that (see [27])

$$\int_{\mathbb{R}^n} |\nabla w|^2 > 2S^{n/2}, \quad \text{for each sign-changing solution } w \text{ of (1.1)}, \tag{1.2}$$

where S denotes the best minimizer of the Sobolev inequality on the whole space; that is,

$$S = \inf\{|\nabla u|^2_{L^2(\mathbb{R}^n)}|u|^{-2}_{L^{2n/(n-2)}(\mathbb{R}^n)} : \ \nabla u \in L^2(\mathbb{R}^n),\ u \in L^{2n/(n-2)}(\mathbb{R}^n),\ u \neq 0\}.$$

When we add the positivity assumption, the solutions of (1.1) are the family

$$\delta_{(a,\lambda)}(x) = c_0 \frac{\lambda^{(n-2)/2}}{(1+\lambda^2|x-a|^2)^{(n-2)/2}},\ c_0 = (n(n-2))^{(n-2)/4},\ \lambda > 0,\ a \in \mathbb{R}^n. \tag{1.3}$$

The space $H^1_0(\Omega)$ is equipped with the norm $\|.\|$ and its corresponding inner product $< ., . >$ defined by

$$\|u\| = \Big(\int_\Omega |\nabla u|^2\Big)^{1/2} \quad \text{and} \quad < u, v >= \int_\Omega \nabla u \nabla v, \quad u, v \in H^1_0(\Omega). \tag{1.4}$$

When we study problem (1.1) in a bounded smooth domain Ω, we need to introduce the function $P\delta_{(a,\lambda)}$, which is the projection of $\delta_{(a,\lambda)}$ on $H^1_0(\Omega)$. This function satisfies the following:

$$-\triangle P\delta_{(a,\lambda)} = -\triangle \delta_{(a,\lambda)} \quad \text{in} \quad \Omega; \quad P\delta_{(a,\lambda)} = 0 \quad \text{on } \partial\Omega.$$

We are particularly interested in the existence and nonexistence of sign-changing solutions that blow up positively and negatively at different points of Ω as the parameter ε goes to zero in the sense of the following definition.

Definition 1.1 *Let (u_ε) be a family of solutions for (P_ε). We say that (u_ε) blow up positively at different k_1 points, $a_1, ..., a_{k_1}$, in Ω and blow up negatively at different k_2 points, $a_{k_1+1}, ..., a_{k_1+k_2}$, in Ω if there exist k_1+k_2 points $a_{1,\varepsilon}, ..., a_{k_1+k_2,\varepsilon} \in \Omega$ and k_1+k_2 concentration $\lambda_{1,\varepsilon}, ..., \lambda_{k_1+k_2,\varepsilon}$ with $\lim_{\varepsilon\to 0} a_{i,\varepsilon} = a_i$, $\lim_{\varepsilon\to 0} \lambda_{i,\varepsilon} = +\infty$, $\lambda_{i,\varepsilon} d(a_{i,\varepsilon}, \partial\Omega) \to +\infty$ for $i = 1, ..., k_1+k_2$ and $\varepsilon_{ij} = \left(\frac{\lambda_i}{\lambda_j} + \frac{\lambda_j}{\lambda_i} + \lambda_i\lambda_j|a_i - a_j|^2\right)^{\frac{2-n}{2}} \to 0$ as $\varepsilon \to 0$ for $i \neq j$, such that*

$$\Big\| u_\varepsilon - \Big(\sum_{i=1}^{k_1} P\delta_{(a_{i,\varepsilon},\lambda_{i,\varepsilon})} - \sum_{i=k_1+1}^{k_1+k_2} P\delta_{(a_{i,\varepsilon},\lambda_{i,\varepsilon})} \Big) \Big\| \to 0 \text{ in } H_0^1(\Omega) \text{ as } \varepsilon \to 0. \tag{1.5}$$

Our first result concerns the nonexistence of sign-changing solutions that blow up at two points.

Theorem 1.1 *Let Ω be any smooth bounded domain in $\mathbb{R}^n$, $n \geq 4$. There exists $\varepsilon_0 > 0$, such that, for each $\varepsilon \in (0, \varepsilon_0)$, problem (P_ε) has no sign-changing solutions u_ε that blow up positively at $a_{1,\varepsilon} \in \Omega$ and negatively at $a_{2,\varepsilon} \in \Omega$.*

To state the result in the case of three concentration points, we need to introduce some notations. We denote by G Green's function of the Laplace operator defined by: $\forall x \in \Omega$

$$-\Delta G(x, .) = c_n \delta_x \quad \text{in} \quad \Omega, \quad G(x, .) = 0 \quad \text{on} \quad \partial\Omega,$$

where δ_x denotes the Dirac mass at x and $c_n = (n-2)w_n$, with w_n being the area of the unit sphere of $\mathbb{R}^n$. We denote by H the regular part of G; that is,

$$H(x_1, x_2) = |x_1 - x_2|^{2-n} - G(x_1, x_2) \quad \text{for} \quad (x_1, x_2) \in \Omega^2.$$

Note that the construction of positive solutions that concentrate at different k points of Ω, with $k \geq 2$, is related to suitable critical points of the function $\Psi_k : \mathbb{R}_+^k \times \Omega^k \to \mathbb{R}$ defined by

$$\Psi_k(\Lambda, x) = \frac{1}{2}(M(x)\Lambda, \Lambda) + \frac{1}{2}\sum_{i=1}^{k} \Lambda_i^{\frac{4}{n-2}},$$

where $\Lambda =^T (\Lambda_1, ..., \Lambda_k)$, $M(x) = (m_{ij})_{1\leq i,j\leq k}$, being the matrix defined by

$$m_{ii} = H(a_i, a_i) \text{ for } i = 1, ..., k, m_{ij} = m_{ji} = -G(a_i, a_j) \text{ for } i \neq j.$$

Let $\rho(x)$ be the smallest eigenvalue of $M(x)$ and $r(x)$ the eigenvector corresponding to $\rho(x)$ whose norm is 1. We point out that we can choose $r(x)$ so that all their components are strictly positive (see [3, 6]).

Note that in the positive case, all positive solutions blow up with comparable speeds. However, for the subcritical semilinear Dirichlet problem, Pistoia and Weth in [23] constructed a family of sign-changing solutions with k bubbles, concentrated at the same point in the case where Ω is a symmetric domain with respect to the $x_1, ..., x_n$ axes. This result is generalized by Musso and Pistoia in [22], under a suitable assumption on the nondegeneracy of Robin's function. Moreover, Ben Ayed and Ould Bouh in [9] proved that the phenomenon of bubble-tower solutions cannot occur in the supercritical case. In this theorem, we prove that this phenomenon cannot occur in the case where $k = 3$, in the sense that the distances of the two positive blow-up points from each other and from the boundary are bounded.

Theorem 1.2 *Let Ω be any smooth bounded domain in $\mathbb{R}^n$, $n \geq 4$. Assume that u_ε is a solution of (P_ε) that blows up positively at $(a_{\varepsilon,1}, a_{\varepsilon,3}) \in \Omega^2$ and negatively at $a_{\varepsilon,2} \in \Omega$.*

Then, when $\varepsilon \to 0$, $a_{\varepsilon,i} \to \overline{a}_i$ for $i = 1, 3$, $|\overline{a}_1 - \overline{a}_3| \geq c_0$ and we have either $\rho(\overline{a}_1, \overline{a}_3) = 0$ and $\nabla\rho(\overline{a}_1, \overline{a}_3) = 0$ or:

If $n \geq 5$ $(\Lambda_1, \Lambda_3, \overline{a}_1, \overline{a}_3)$ is a critical point of Ψ_2, where $\Lambda_i = c\mu_i$ with $\mu_i = \lim\limits_{\varepsilon\to 0} \varepsilon^{\frac{1}{n-4}}\lambda_i > 0$ for $i \in \{1, 3\}$ and c is a positive constant.

If $n = 4$, let $\overline{\eta}_i$ denote the limit of $\lambda_{\varepsilon,i}/\lambda_{\varepsilon,j}$ ($\overline{\eta}_1 = \overline{\eta}_3^{-1}$) and $\overline{\Lambda}$ the limit of $\Lambda_i = \dfrac{c_3}{c_1}\varepsilon \log(\lambda_i)$ up to a subsequence, and then $(\overline{\eta}_i, \overline{\Lambda})$ satisfies

$$H(\overline{a}_i, \overline{a}_i) - \overline{\eta}_i G(\overline{a}_1, \overline{a}_3) + \overline{\Lambda} = 0 \quad \text{and} \quad -\frac{\partial H(\overline{a}_i, \overline{a}_i)}{\partial a_i} + 2\overline{\eta}_i \frac{\partial G(\overline{a}_1, \overline{a}_3)}{\partial a_i} = 0, \text{ for } i = 1, 3. \tag{1.6}$$

Note that in the positive case, if Ω is a domain with a small "hole", Musso and Pistoia [19] proved the existence of a family of solutions that blow up at two points. In the case of sign-changing solutions, we have the following example of the existence result.

Remark 1.3 *Let D be a bounded domain in $\mathbb{R}^n$, $n \geq 5$, which is symmetric with respect to the hyperplane $T = \{x = (x_1, x_2, ..., x_n)/x_n = 0\}$ (i.e. $x = (x_1, x_2, ..., x_n) \in D$ iff $x = (x_1, x_2, ..., -x_n) \in D$). There exists $r_0 > 0$ such that, if $0 < r < r_0$ is fixed and Ω is the domain given by $\Omega = D \setminus w_1 \cup w_2$ where $w_1 \subset B(a, r)$ ($a \in D \setminus T$) and w_2 is the symmetric of w_1 with respect to the hyperplane T, then there exists $\varepsilon_0 > 0$ such that problem (P_ε) has a pair of solutions $\pm u_\varepsilon$ for any $0 < \varepsilon < \varepsilon_0$, which blow up positively at two points and negatively at two points of Ω.*

To state a more general situation in the case of four concentration points, we define the following subset of $H_0^1(\Omega)$:

$$\begin{aligned} M_\varepsilon = \{(\alpha, \lambda, a, v) \in \mathbb{R}^4 \times (\mathbb{R}_+^*)^4 \times \Omega^4 \times H_0^1(\Omega) \text{ such that } \forall\, i \in \{1, .., 4\},\ |\alpha_i - 1| < \alpha_0,\ d(a_i, \partial\Omega) \geq d_0, \\ \lambda_i \in [c_0^{-1}\varepsilon^{-1/(n-4)}, c_0\varepsilon^{-1/(n-4)}],\ |a_i - a_j| \geq d_0\ \forall i \neq j,\ v \in E,\ \|v\| \leq \eta_0\}, \end{aligned}$$

where η_0, c_0, α_0, d_0 are suitable positive constants and $E = \{P\delta_{(a_{\varepsilon,i},\lambda_{\varepsilon,i})},\ \partial P\delta_{(a_{\varepsilon,i},\lambda_{\varepsilon,i})}/\partial\lambda_{\varepsilon,i},\ \partial P\delta_{(a_{\varepsilon,i},\lambda_{\varepsilon,i})}/\partial a_i^j,$ $i \leq k;\ j \leq n\}\}^\top$.

Assume that u_ε is a family of solutions of (P_ε), with exactly two positive blow-up points and two negative blow-up points. Then, in the limit, the blow-up points have to satisfy a certain condition in terms of Green's function and its regular part and we have the following result.

Theorem 1.4 *Let Ω be any smooth bounded domain in $\mathbb{R}^n$, $n \geq 5$. Assume that u_ε is a sign-changing solution of (P_ε) of the form*

$$u_\varepsilon = \sum_{i=1}^{4} \gamma_i \alpha_{\varepsilon,i} P\delta_{(a_{\varepsilon,i},\lambda_{\varepsilon,i})} + v_\varepsilon, \tag{1.7}$$

where $\gamma_1 = \gamma_3 = 1$, $\gamma_2 = \gamma_4 = -1$, $(\alpha_\varepsilon, \lambda_\varepsilon, a_\varepsilon, v_\varepsilon) \in M_\varepsilon$. Then when $\varepsilon \to 0, \alpha_{\varepsilon,i} \to 1, a_{\varepsilon,i} \to \overline{a}_i, c\varepsilon^{-1/(n-4)}\lambda_{\varepsilon,i} \to \overline{\Lambda}_i$ for $i = 1, ..., 4$ and $(\overline{\Lambda}, \overline{a})$ is a critical point of Φ_4, where Φ_4 is defined by

$$\Phi_4(\Lambda, a) = \frac{1}{2}(M(a)\Lambda, \Lambda) + \frac{1}{2}\sum_{i=1}^{4} \Lambda_i^{\frac{4}{n-2}},$$

where $\Lambda =^T (\Lambda_1, ..., \Lambda_k)$, $M(a) = (m_{ij})_{1\leq i,j\leq 4}$ is the matrix defined by

$$m_{ii} = H(a_i, a_i) \textit{ for } i = 1, ..., 4, m_{ij} = m_{ji} = -\gamma_i\gamma_j G(a_i, a_j) \textit{ for } i \neq j.$$

This paper is organized as follows. In Section 2 we prove Theorems 1.1 and 1.2. Section 3 is devoted to the proof of Theorem 1.4. Finally, the Appendix provides some integral estimates that are needed in Section 2.

2. Proof of Theorems 1.1 and 1.2

This section is devoted to the proof of Theorems 1.1 and 1.2. It presents some ideas introduced by Bahri [1] and other technical estimates.

Let $k \geq 2$ be a fixed integer. We assume that there exists solution u_ε of (P_ε) as in Definition 1.1. Arguing as in [1, 25], we see that there is a unique way to choose $\alpha_{\varepsilon,i}$, $a_{\varepsilon,i}$, $\lambda_{\varepsilon,i}$, and v_ε such that

$$u_\varepsilon = \sum_{i=1}^{k} \gamma_i \alpha_{\varepsilon,i} P\delta_{(a_{\varepsilon,i},\lambda_{\varepsilon,i})} + v_\varepsilon, \tag{2.1}$$

with

$$\begin{cases} \gamma_i \in \{-1, 1\},\ \alpha_{\varepsilon,i} \in \mathbb{R},\ \alpha_{\varepsilon,i} \to 1, \text{ as } \varepsilon \to 0, \\ a_{i,\varepsilon} \in \Omega,\ \lambda_{i,\varepsilon} \in \mathbb{R}_+^*,\ \ \lambda_{\varepsilon,i} d(a_{\varepsilon,i}, \partial\Omega) \to +\infty, \text{ as } \varepsilon \to 0, \\ v_\varepsilon \to 0 \text{ in } H_0^1(\Omega), \text{ as } \varepsilon \to 0, \end{cases}$$

where $v_\varepsilon \in E$ such that:

$$E := \{v :< v, \varphi >= 0 \quad \forall\varphi \in Span\{P\delta_{(a_{\varepsilon,i},\lambda_{\varepsilon,i})},\ \partial P\delta_{(a_{\varepsilon,i},\lambda_{\varepsilon,i})}/\partial\lambda_{\varepsilon,i},\ \partial P\delta_{(a_{\varepsilon,i},\lambda_{\varepsilon,i})}/\partial a_{\varepsilon,i}^j,\ i \leq k;\ j \leq n\}\}, \tag{2.2}$$

where $a_{\varepsilon,i}^j$ is the jth component of $a_{\varepsilon,i}$.

To simplify the notation, we write α_i, a_i, λ_i, v, δ_i, and $P\delta_i$ instead of $\alpha_{\varepsilon,i}$, $a_{\varepsilon,i}$, $\lambda_{\varepsilon,i}$, v_ε, $\delta_{(a_{\varepsilon,i},\lambda_{\varepsilon,i})}$, and $P\delta_{(a_{\varepsilon,i},\lambda_{\varepsilon,i})}$. We denote by $f = O(g)$ as $\varepsilon \to 0$ that f/g is bounded for ε near 0 and by $f = o(g)$ as $\varepsilon \to 0$ that f/g goes to zero as $\varepsilon \to 0$.

This type of problem is usually handled by first dealing with the v-part of u_ε, so as to show that it is negligible with respect to the concentration phenomenon. Namely, we have the following estimate.

Lemma 2.1 *Let $k=2$, and the function v defined in (2.1) satisfies the following estimate:*

$$\|v\| = O\begin{cases} \displaystyle\sum_{i=1}^{2}\Big(\frac{\varepsilon}{\lambda_i^{\frac{n-2}{2}}} + \frac{1}{(\lambda_i d_i)^{n-2}}\Big) + \varepsilon_{12}(\log(\varepsilon_{12}^{-1}))^{\frac{n-2}{2}}, \ \textit{if } n=4,5, \\ \displaystyle\sum_{i=1}^{2}\Big(\frac{\varepsilon(\log\lambda_i)^{\frac{2}{3}}}{\lambda_i^2} + \frac{\log(\lambda_i d_i)}{(\lambda_i d_i)^4}\Big) + \varepsilon_{12}(\log(\varepsilon_{12}^{-1}))^{\frac{2}{3}}, \ \textit{if } n=6, \\ \displaystyle\sum_{i=1}^{2}\Big(\frac{\varepsilon}{\lambda_i^2} + \frac{1}{(\lambda_i d_i)^{\frac{n+2}{2}}}\Big) + \varepsilon_{12}^{\frac{n+2}{2(n-2)}}(\log(\varepsilon_{12}^{-1}))^{\frac{n+2}{2n}}, \ \textit{if } n>6, \end{cases}$$

where $\varepsilon_{12} = \Big(\frac{\lambda_1}{\lambda_2} + \frac{\lambda_2}{\lambda_1} + \lambda_1\lambda_2|a_1-a_2|^2\Big)^{(2-n)/2}$ *and* $d_i := d(a_i,\partial\Omega)$.

Proof Since $u_\varepsilon = \alpha_1 P\delta_1 - \alpha_2 P\delta_2 + v$ is a solution of (P_ε) and using the fact that $v\in E$ (see (2.2)), multiplying (P_ε) by v and integrating on Ω, we obtain

$$\begin{aligned}\int_\Omega -\triangle u_\varepsilon v = \|v\|^2 &= \int_\Omega |\alpha_1 P\delta_1 - \alpha_2 P\delta_2 + v|^{p-1}(\alpha_1 P\delta_1 - \alpha_2 P\delta_2 + v)v - \varepsilon\int_\Omega(\alpha_1 P\delta_1 - \alpha_2 P\delta_2 + v)v \\ &= \int_\Omega |\alpha_1 P\delta_1 - \alpha_2 P\delta_2|^{p-1}(\alpha_1 P\delta_1 - \alpha_2 P\delta_2)v + p\int_\Omega|\alpha_1 P\delta_1 - \alpha_2 P\delta_2|^{p-1}v^2 \\ &+ \ o(\|v\|^2) - \varepsilon\int_\Omega(\alpha_1 P\delta_1 - \alpha_2 P\delta_2 + v)v.\end{aligned}$$

Hence, we have

$$Q(v,v) + o(\|v\|^2) = f(v), \tag{2.3}$$

where

$$\begin{aligned} Q(v,v) &= \|v\|^2 - p\int_\Omega|\alpha_1 P\delta_1 - \alpha_2 P\delta_2|^{p-1}v^2 = \|v\|^2 - p\sum_{i=1}^{2}\int_\Omega P\delta_i^{p-1}v^2 + o(\|v\|^2), \\ f(v) &= \int_\Omega|\alpha_1 P\delta_1 - \alpha_2 P\delta_2|^{p-1}(\alpha_1 P\delta_1 - \alpha_2 P\delta_2)v - \varepsilon\int_\Omega(\alpha_1 P\delta_1 - \alpha_2 P\delta_2)v.\end{aligned}$$

According to [1], Q is a positive definite quadratic form on v, and thus there exists $c>0$ independent of ε, satisfying $Q(v,v)\ge c\|v\|^2$, for each $v\in E$. Then, from (2.3), we get

$$\|v\|^2 = O(|f(v)|).$$

It remains to estimate $f(v)$. Using the fact that $v\in E$, Holder's inequality, and the embedding theorem, we find

$$\begin{aligned}
f(v) &= \alpha_1^{\frac{n+2}{n-2}}\int_\Omega P\delta_1^{\frac{n+2}{n-2}}v - \alpha_2^{\frac{n+2}{n-2}}\int_\Omega P\delta_2^{\frac{n+2}{n-2}}v + O\Big(\sum_{i\neq j}\int_\Omega \delta_i^{\frac{4}{n-2}}\inf(\delta_i,\delta_j)|v| + \sum_{i=1}^2 \varepsilon\int_\Omega \delta_i|v|\Big)\\
&= O\Big(\sum_{i=1}^2\int_\Omega \delta_i^{\frac{4}{n-2}}(\delta_i - P\delta_i)|v| + \sum_{i\neq j}\int_\Omega \delta_i^{\frac{4}{n-2}}\inf(\delta_i,\delta_j)|v|\Big)\\
&+ O\Big(\|v\|\sum_{i=1}^2\Big(\frac{\varepsilon}{\lambda_i^{\frac{n-2}{2}}}(\text{ if } n=4,5) + \frac{\varepsilon(\log\lambda_i)^{\frac{2}{3}}}{\lambda_i^2}(\text{ if } n=6) + \frac{\varepsilon}{\lambda_i^2}(\text{ if } n>6)\Big)\Big)\\
&= \begin{cases} O\bigg(\|v\|\Big(\sum_{i=1}^2\big(\frac{\varepsilon}{\lambda_i^{\frac{n-2}{2}}} + \frac{1}{(\lambda_i d_i)^{n-2}}\big) + \varepsilon_{12}(\log(\varepsilon_{12}^{-1}))^{\frac{n-2}{n}}\Big)\bigg), & \text{if } n=4,5,\\
O\bigg(\|v\|\Big(\sum_{i=1}^2\big(\frac{\varepsilon(\log\lambda_i)^{\frac{2}{3}}}{\lambda_i^2} + \frac{\log(\lambda_i d_i)}{(\lambda_i d_i)^4}\big) + \varepsilon_{12}(\log(\varepsilon_{12}^{-1}))^{\frac{2}{3}}\Big)\bigg), & \text{if } n=6,\\
O\bigg(\|v\|\Big(\sum_{i=1}^2\big(\frac{\varepsilon}{\lambda_i^2} + \frac{1}{(\lambda_i d_i)^{\frac{n+2}{2}}}\big) + \varepsilon_{12}^{\frac{n+2}{2(n-2)}}(\log(\varepsilon_{12}^{-1}))^{\frac{n+2}{2n}}\Big)\bigg), & \text{if } n>6.\end{cases}
\end{aligned}$$

□

Now we are able to obtain the following result, which is a crucial point in the proof of Theorem 1.1.

Proposition 2.2 *Assume that* $u_\varepsilon = \alpha_1 P\delta_1 - \alpha_2 P\delta_2 + v$ *is a sign-changing solution of* (P_ε)*. We have the following estimate:*

$$S_n\alpha_i(1-\alpha_i^{\frac{4}{n-2}}) = O\Big(\sum_j\Big(\frac{1}{(\lambda_j d_j)^{n-2}} + \frac{\varepsilon}{\lambda_j^2} + (\text{if } n=4)\frac{\varepsilon\log\lambda_i d_j}{\lambda_j^2}\Big) + \varepsilon_{12}\Big),$$

where $i\in\{1,2\}$ *and* $S_n = \int_{\mathbb{R}^n}\delta_{(0,1)}^{\frac{2n}{n-2}}(y)dy$*.*

Proof It suffices to prove the proposition for $i=1$. Multiplying (P_ε) by $P\delta_1$ and integrating on Ω, we obtain

$$\int_\Omega -\Delta u_\varepsilon P\delta_1 = \int_\Omega |\alpha_1 P\delta_1 - \alpha_2 P\delta_2 + v|^{\frac{4}{n-2}}(\alpha_1 P\delta_1 - \alpha_2 P\delta_2 + v)P\delta_1 - \varepsilon\int_\Omega(\alpha_1 P\delta_1 - \alpha_2 P\delta_2 + v)P\delta_1. \quad (2.4)$$

By Lemma 2.1, we write

$$\begin{aligned}
\alpha_1\|P\delta_1\|^2 &- \alpha_2\Big\langle P\delta_2, P\delta_1\Big\rangle = \int_\Omega\big(\alpha_1^{\frac{n+2}{n-2}}P\delta_1^{2n/(n-2)} - \alpha_2^{\frac{n+2}{n-2}}P\delta_1 P\delta_2^{(n+2)/(n-2)}\big)\\
&- \frac{n+2}{n-2}\int_\Omega \alpha_1^{4/(n-2)}\alpha_2 P\delta_1^{\frac{n+2}{n-2}}P\delta_2 - \varepsilon\int_\Omega \alpha_1 P\delta_1^2 + O\Big(\sum_j\Big(\frac{1}{(\lambda_j d_j)^{n-2}} + \frac{\varepsilon}{\lambda_j^2}\Big) + \varepsilon_{12}\Big). \quad (2.5)
\end{aligned}$$

Using Lemmas A.1,..., A.4 and Lemma A.15, the result follows. □

Proposition 2.3 *Assume that* $u_\varepsilon = \alpha_1 P\delta_1 - \alpha_2 P\delta_2 + v$ *is a sign-changing solution of* (P_ε).

(a) For $n \geq 5$, *we have the following estimate:*

$$-\frac{n-2}{2}c_1\frac{H(a_i,a_i)}{\lambda_i^{n-2}} + c_1\Big(\lambda_i\frac{\partial\varepsilon_{12}}{\partial\lambda_i} + \frac{n-2}{2}\frac{H(a_1,a_2)}{(\lambda_1\lambda_2)^{\frac{n-2}{2}}}\Big) - \frac{c_2\varepsilon}{\lambda_i^2} = A_i \text{ with}$$

$$A_i = O\Big(\sum_{k=1}^{2}\Big(\frac{\log(\lambda_k d_k)}{(\lambda_k d_k)^n} + \varepsilon_{12}^{\frac{n}{n-2}}\log(\varepsilon_{12}^{-1}) + \varepsilon\varepsilon_{12} + \frac{\varepsilon}{(\lambda_i d_i)^{n-2}}\Big) + \varepsilon^2 R_1,$$

where $i \in \{1,2\}$, $c_1 = c_0^{\frac{2n}{n-2}}\int_{\mathbb{R}^n}\frac{dx}{(1+|x|^2)^{n+2/2}}$, $c_2 = c_0^2\int_{\mathbb{R}^n}\frac{dx}{(1+|x|^2)^{n-2}}$, *and* R_1 *satisfies*

$$R_1 = \begin{cases} O\Big(\sum_{k=1}^{2}\frac{1}{\lambda_k^4}\Big), & \text{if } n > 6,\\ O\Big(\sum_{k=1}^{2}\frac{(\log\lambda_k)^{\frac{4}{3}}}{\lambda_k^4}\Big), & \text{if } n = 6,\\ O\Big(\sum_{k=1}^{2}\frac{1}{\lambda_k^{n-2}}\Big), & \text{if } n = 4,5.\end{cases}$$

(b) For $n = 4$, *we have*

$$-\frac{n-2}{2}c_1\frac{H(a_i,a_i)}{\lambda_i^{n-2}} + c_1\Big(\lambda_i\frac{\partial\varepsilon_{12}}{\partial\lambda_i} + \frac{n-2}{2}\frac{H(a_1,a_2)}{(\lambda_1\lambda_2)^{\frac{n-2}{2}}}\Big) - c_3\varepsilon\frac{\log(\lambda_i d_i)}{\lambda_i^2} = A_i,$$

where $c_3 = \frac{1}{2}c_0^2\omega_4$, *with* ω_4 *denoting the area of the unit sphere of* $\mathbb{R}^4$.

Proof It is sufficient to prove the proposition for $i = 1$. Multiplying (P_ε) by $\alpha_1\lambda_1\partial P\delta_1/\partial\lambda_1$ and integrating on Ω, we obtain

$$\begin{aligned}-\int_\Omega \Delta u_\varepsilon \alpha_1\lambda_1\frac{\partial P\delta_1}{\partial\lambda_1} &= \int_\Omega |\alpha_1 P\delta_1 - \alpha_2 P\delta_2 + v|^{\frac{4}{n-2}}(\alpha_1 P\delta_1 - \alpha_2 P\delta_2 + v)\alpha_1\lambda_1\frac{\partial P\delta_1}{\partial\lambda_1}\\ &- \varepsilon\int_\Omega(\alpha_1 P\delta_1 - \alpha_2 P\delta_2 + v)\alpha_1\lambda_1\frac{\partial P\delta_1}{\partial\lambda_1} = I_1 - I_2. \qquad (2.6)\end{aligned}$$

Using the fact that $v \in E$ and Lemmas A.6 and A.8, we derive

$$\begin{aligned}-\int_\Omega \Delta u_\varepsilon \alpha_1\lambda_1\frac{\partial P\delta_1}{\partial\lambda_1} &= <\alpha_1 P\delta_1 - \alpha_2 P\delta_2 + v, \alpha_1\lambda_1\frac{\partial P\delta_1}{\partial\lambda_1}>\\ &= \alpha_1^2 < P\delta_1, \lambda_1\frac{\partial P\delta_1}{\partial\lambda_1}> - \alpha_1\alpha_2 < P\delta_2, \lambda_1\frac{\partial P\delta_1}{\partial\lambda_1}> + < v, \alpha_1\lambda_1\frac{\partial P\delta_1}{\partial\lambda_1}>\\ &= \alpha_1^2\frac{n-2}{2}c_1\frac{H(a_1,a_1)}{\lambda_1^{n-2}} - \alpha_1\alpha_2 c_1\Big(\lambda_1\frac{\partial\varepsilon_{12}}{\partial\lambda_1} + \frac{n-2}{2}\frac{H(a_1,a_2)}{(\lambda_1\lambda_2)^{\frac{n-2}{2}}}\Big) + R, \qquad (2.7)\end{aligned}$$

where R satisfies

$$R = O\Big(\varepsilon_{12}^{\frac{n}{n-2}}\log(\varepsilon_{12}^{-1}) + \sum_{k=1}^{2}\frac{\log(\lambda_k d_k)}{(\lambda_k d_k)^n}\Big). \qquad (2.8)$$

Using Lemma A.5 and the fact that $n \geq 4$, we derive

$$\begin{aligned}
I_1 &= \int_\Omega |\alpha_1 P\delta_1 - \alpha_2 P\delta_2|^{\frac{4}{n-2}}(\alpha_1 P\delta_1 - \alpha_2 P\delta_2)\alpha_1\lambda_1 \frac{\partial P\delta_1}{\partial\lambda_1} + \frac{n+2}{n-2}\int_\Omega |\alpha_1 P\delta_1 - \alpha_2 P\delta_2|^{\frac{4}{n-2}} v\alpha_1\lambda_1 \frac{\partial P\delta_1}{\partial\lambda_1} \\
&+ O\big(\|v\|^2\big) + R \\
&= \int_\Omega (\alpha_1 P\delta_1)^{\frac{n+2}{n-2}}\alpha_1\lambda_1\frac{\partial P\delta_1}{\partial\lambda_1} - \frac{n+2}{n-2}\int_\Omega \alpha_2 P\delta_2(\alpha_1 P\delta_1)^{\frac{4}{n-2}}\alpha_1\lambda_1\frac{\partial P\delta_1}{\partial\lambda_1} + \frac{n+2}{n-2}\int_\Omega(\alpha_1 P\delta_1)^{\frac{4}{n-2}} v\alpha_1\lambda_1\frac{\partial P\delta_1}{\partial\lambda_1} \\
&- \int_\Omega (\alpha_2 P\delta_2)^{\frac{n+2}{n-2}}\alpha_1\lambda_1\frac{\partial P\delta_1}{\partial\lambda_1} + O\Big(\int_\Omega (\delta_1\delta_2)^{\frac{n}{n-2}} + \|v\|^2\Big) + R. \qquad (2.9)
\end{aligned}$$

Using Lemmas A.5, ...,A.10, and A.19, (2.9) becomes

$$\begin{aligned}
I_1 &= 2\alpha_1^{\frac{2n}{n-2}}\Big(\frac{n-2}{2}c_1\frac{H(a_1,a_1)}{\lambda_1^{n-2}}\Big) - \alpha_1^{\frac{n+2}{n-2}}\alpha_2 c_1\Big(\lambda_1\frac{\partial\varepsilon_{12}}{\partial\lambda_1} + \frac{n-2}{2}\frac{H(a_1,a_2)}{(\lambda_1\lambda_2)^{\frac{n-2}{2}}}\Big) \\
&- \alpha_2^{\frac{n+2}{n-2}}\alpha_1 c_1\Big(\lambda_1\frac{\partial\varepsilon_{12}}{\partial\lambda_1} + \frac{n-2}{2}\frac{H(a_1,a_2)}{(\lambda_1\lambda_2)^{\frac{n-2}{2}}}\Big) + R + O\big(\|v\|^2\big). \qquad (2.10)
\end{aligned}$$

Using Lemma A.16 and Lemma A.18, we obtain for $n \geq 5$

$$\begin{aligned}
I_2 &= -\alpha_1^2\frac{c_2\varepsilon}{\lambda_1^2} + O\Big(\frac{\varepsilon}{(\lambda_1 d_1)^{n-2}}\Big) + O\Big(\varepsilon\int_\Omega \delta_1\delta_2 + \varepsilon\int_\Omega |v|\delta_1\Big), \\
&= -\alpha_1^2\frac{c_2\varepsilon}{\lambda_1^2} + O\Big(\frac{\varepsilon}{(\lambda_1 d_1)^{n-2}} + \varepsilon\varepsilon_{12} + (\text{ if } n \leq 5)\frac{\varepsilon\|v\|}{\lambda_i^{\frac{n-2}{2}}} + (\text{ if } n=6)\frac{\varepsilon\|v\|(\log\lambda_i)^{\frac{2}{3}}}{\lambda_i^2} + \frac{\varepsilon\|v\|}{\lambda_i^2}\Big). \qquad (2.11)
\end{aligned}$$

Therefore, combining (2.6), ..., (2.11), with Proposition 2.2 and using the estimate of v, the proof of Claim (a) of Proposition 2.3 follows.

To prove Claim (b), observe that we have used the fact that $n \geq 5$ only in I_2. Then we need to compute

$$\begin{aligned}
\int_\Omega P\delta_1\lambda_1\frac{\partial P\delta_1}{\partial\lambda_1} &= \int_\Omega \delta_1\lambda_1\frac{\partial\delta_1}{\partial\lambda_1} + O\Big(\frac{1}{(\lambda_1 d_1)^2}\Big) \\
&= \int_{B(a_1,d_1)}\delta_1\lambda_1\frac{\partial\delta_1}{\partial\lambda_1} + \int_{\Omega\setminus B(a_1,d_1)}\delta_1\lambda_1\frac{\partial\delta_1}{\partial\lambda_1} + O\Big(\frac{1}{(\lambda_1 d_1)^2}\Big).
\end{aligned}$$

An easy computation shows that

$$\begin{aligned}
\int_{B(a_1,d_1)}\delta_1\lambda_1\frac{\partial\delta_1}{\partial\lambda_1} &= c_0^2\int_{B(a_1,d_1)}\frac{\lambda_1^2(1-\lambda^2|x-a_1^2|)}{(1+\lambda_1^2|x-a_1|^2)^3}dx \\
&= c_0^2\frac{mes(S^3)}{\lambda_1^2}\int_0^{\lambda_1 d_1}\frac{(1-r^2)r^3}{(1+r^2)^3}dr, \\
&= -c_3\frac{\log(\lambda_1 d_1)}{\lambda_1^2} + O\Big(\frac{1}{(\lambda_1 d_1)^2\lambda_1^2}\Big),
\end{aligned}$$

where $c_3 = \dfrac{1}{2}c_0^2 mes(S^3)$.

Now, we need to estimate

$$\int_{\Omega\setminus B(a_1,d_1)} \delta_1\lambda_1\frac{\partial\delta_1}{\partial\lambda_1} \leq \frac{\lambda_1}{\lambda_1^2 d_1^2}\int_{\Omega\setminus B(a_1,d_1)} \frac{c_0^2\lambda_1}{(1+\lambda_1^2|x-a_1|^2)}dx = O\Big(\frac{1}{(\lambda_1 d_1)^2}\Big).$$

Therefore,

$$\int_\Omega \delta_1\lambda_1\frac{\partial\delta_1}{\partial\lambda_1} = -c_3\frac{\log(\lambda_1 d_1)}{\lambda_1^2} + O\Big(\frac{1}{(\lambda_1 d_1)^2}\Big).$$

The proof of Claim (b) follows. □

Proof of Theorem 1.1 Arguing by contradiction, let us suppose that the problem (P_ε) has a solution u_ε as stated in Theorem 1.1. This solution has to satisfy (2.1), and from Proposition 2.3, we have

$$-\frac{n-2}{2}c_1\frac{H(a_i,a_i)}{\lambda_i^{n-2}} + c_1\Big(\lambda_i\frac{\partial\varepsilon_{12}}{\partial\lambda_i} + \frac{n-2}{2}\frac{H(a_1,a_2)}{(\lambda_1\lambda_2)^{\frac{n-2}{2}}}\Big) - \frac{c_2\varepsilon}{\lambda_i^2} = A, \text{ if } n\geq 5, \tag{2.12}$$

$$-\frac{n-2}{2}c_1\frac{H(a_i,a_i)}{\lambda_i^{n-2}} + c_1\Big(\lambda_i\frac{\partial\varepsilon_{12}}{\partial\lambda_i} + \frac{n-2}{2}\frac{H(a_1,a_2)}{(\lambda_1\lambda_2)^{\frac{n-2}{2}}}\Big) - \frac{c_3\varepsilon\log(\lambda_i d_i)}{\lambda_i^2} = A, \text{ if } n = 4, \tag{2.13}$$

where $i=1,2,$ and $A = o\Big(\varepsilon_{12} + \sum_{k=1}^2\Big(\frac{1}{(\lambda_k d_k)^{n-2}} + \frac{\varepsilon}{\lambda_k^2}\Big)\Big)$.

Furthermore, an easy computation shows that

$$\lambda_i\frac{\partial\varepsilon_{12}}{\partial\lambda_i} = -\frac{n-2}{2}\varepsilon_{12}\Big(1-2\frac{\lambda_j}{\lambda_i}\varepsilon_{12}^{\frac{2}{n-2}}\Big), \text{ for } i,j\in\{1,2\},\ i\neq j. \tag{2.14}$$

Without loss of generality, we can assume that $\lambda_2\geq\lambda_1$. We distinguish two cases and we will prove that they cannot occur. This implies our theorem.

Case 1. $\dfrac{\lambda_1\lambda_2|a_1-a_2|^2}{\lambda_2/\lambda_1}\to+\infty$. In this case, it is easy to obtain

$$\varepsilon_{12} = \frac{1}{(\lambda_1\lambda_2|a_1-a_2|^2)^{\frac{n-2}{2}}} + o(\varepsilon_{12}), \tag{2.15}$$

which implies that

$$\lambda_i\frac{\partial\varepsilon_{12}}{\partial\lambda_i} = -\frac{n-2}{2}\frac{1}{(\lambda_1\lambda_2|a_1-a_2|^2)^{\frac{n-2}{2}}} + o(\varepsilon_{12}), \text{ for } i=1,2. \tag{2.16}$$

Then from (2.16), (2.12) and (2.13) become

$$\frac{-(n-2)}{2}c_1\frac{H(a_i,a_i)}{\lambda_i^{n-2}} - \frac{(n-2)}{2}c_1\left(\frac{1}{(\lambda_1\lambda_2|a_1-a_2|^2)^{\frac{n-2}{2}}} - \frac{H(a_1,a_2)}{(\lambda_1\lambda_2)^{\frac{n-2}{2}}}\right) - \frac{c_2\varepsilon}{\lambda_i^2} = o\Big(\varepsilon_{12} + \sum_{k=1}^2\big(\frac{1}{(\lambda_k d_k)^{n-2}} + \frac{\varepsilon}{\lambda_k^2}\big)\Big), \quad \text{if } n\geq 5, \tag{2.17}$$

$$\frac{-(n-2)}{2}c_1\frac{H(a_i,a_i)}{\lambda_i^{n-2}} - \frac{(n-2)}{2}c_1\left(\frac{1}{(\lambda_1\lambda_2|a_1-a_2|^2)^{\frac{n-2}{2}}} - \frac{H(a_1,a_2)}{(\lambda_1\lambda_2)^{\frac{n-2}{2}}}\right) - \frac{c_3\varepsilon\log(\lambda_i d_i)}{\lambda_i^2} = o\Big(\varepsilon_{12} + \sum_{k=1}^2\big(\frac{1}{(\lambda_k d_k)^{n-2}} + \frac{\varepsilon}{\lambda_k^2}\big)\Big), \quad \text{if } n = 4. \tag{2.18}$$

Using the fact that

$$\varepsilon_{12} = O\Big(\frac{H(a_1,a_2)}{(\lambda_1\lambda_2)^{\frac{n-2}{2}}} + \frac{G(a_1,a_2)}{(\lambda_1\lambda_2)^{\frac{n-2}{2}}}\Big) = O\Big(\frac{1}{(\lambda_1 d_1)^{n-2}} + \frac{G(a_1,a_2)}{(\lambda_1\lambda_2)^{\frac{n-2}{2}}}\Big), \tag{2.19}$$

we derive

$$-\sum_{i=1}^{2}\frac{H(a_i,a_i)}{\lambda_i^{n-2}}(1+o(1)) - 2\frac{G(a_1,a_2)}{(\lambda_1\lambda_2)^{\frac{n-2}{2}}}(1+o(1)) - \sum_{i=1}^{2}\frac{\varepsilon}{\lambda_i^2}(c_2+o(1)) = 0, \quad \text{if } n\geq 5,$$

$$-\sum_{i=1}^{2}\frac{H(a_i,a_i)}{\lambda_i^{n-2}}(1+o(1)) - 2\frac{G(a_1,a_2)}{(\lambda_1\lambda_2)^{\frac{n-2}{2}}}(1+o(1)) - \sum_{i=1}^{2}\frac{\varepsilon\log(\lambda_i d_i)}{\lambda_i^2}(c_3+o(1)) = 0, \quad \text{if } n = 4,$$

which gives a contradiction. Hence, this case cannot occur.

Case 2: $\dfrac{\lambda_1\lambda_2|a_1-a_2|^2}{\lambda_2/\lambda_1} \to c \geq 0$. In this case, we note that $\lambda_2/\lambda_1 \to +\infty$. Multiplying (2.12) by 2 for $i=2$ and adding to (2.12) for $i=1$, we obtain

$$-\frac{n-2}{2}c_1\Big(\frac{H(a_1,a_1)}{\lambda_1^{n-2}} + 2\frac{H(a_2,a_2)}{\lambda_2^{n-2}}\Big) + c_1\Big(\lambda_1\frac{\partial\varepsilon_{12}}{\partial\lambda_1} + 2\lambda_2\frac{\partial\varepsilon_{12}}{\partial\lambda_2}\Big) + 3c_1\frac{n-2}{2}\frac{H(a_1,a_2)}{(\lambda_1\lambda_2)^{\frac{n-2}{2}}}$$
$$-\frac{c_2\varepsilon}{\lambda_1^2} - 2\frac{c_2\varepsilon}{\lambda_2^2} = o\Big(\varepsilon_{12} + \sum_{k=1}^{2}\big(\frac{1}{(\lambda_k d_k)^{n-2}} + \frac{\varepsilon}{\lambda_k^2}\big)\Big). \tag{2.20}$$

Now, using (2.14) and the fact that $\lambda_2 \geq \lambda_1$, an easy computation shows that

$$\lambda_1\frac{\partial\varepsilon_{12}}{\partial\lambda_1} + 2\lambda_2\frac{\partial\varepsilon_{12}}{\partial\lambda_2} \leq -\frac{n-2}{4}\varepsilon_{12}. \tag{2.21}$$

Furthermore, since $H(a_1,a_2) \leq c d_1^{2-n}$ and $\lambda_2/\lambda_1 \to +\infty$, we get

$$\frac{H(a_1,a_2)}{(\lambda_1\lambda_2)^{\frac{n-2}{2}}} = o\Big(\frac{1}{(\lambda_1 d_1)^{n-2}}\Big). \tag{2.22}$$

Using (2.20), (2.21), and (2.22) we have

$$-\sum_{i=1}^{2}\Big(\frac{n-2}{2}c_1\frac{H(a_i,a_i)}{\lambda_i^{n-2}}(1+o(1)) + c_2\frac{\varepsilon}{\lambda_i^2}(1+o(1))\Big) - \frac{n-2}{4}c_1\varepsilon_{12}(1+o(1)) \geq 0.$$

Then we derive a contradiction and therefore this case cannot occur for $n \geq 5$, using the same argument for $n=4$. Hence, Theorem 1.1 is proved. $\square$

Proof of Theorem 1.2 Let us assume that problem (P_ε) has a solution u_ε as stated in Theorem 1.2. This solution has to satisfy (2.1),

$$u_\varepsilon = \alpha_1 P\delta_1 - \alpha_2 P\delta_2 + \alpha_3 P\delta_3 + v, \tag{2.23}$$

with v orthogonal to each $P\delta_i$ and their derivatives with respect to λ_i and a_i^k, where a_i^k denotes the kth component of a_i.

Note that for each $i=1,2,3$, as in Proposition 2.3, we have

$$(E_i)\quad c_1\frac{n-2}{2}\frac{H(a_i,a_i)}{\lambda_i^{n-2}} + \gamma_i\sum_{j\neq i}\gamma_j c_1\Big(\lambda_i\frac{\partial\varepsilon_{ij}}{\partial\lambda_i}+\frac{n-2}{2}\frac{H(a_i,a_j)}{(\lambda_i\lambda_j)^{\frac{n-2}{2}}}\Big)+\frac{\varepsilon c_2}{\lambda_i^2}(\text{ if } n\geq 5)$$
$$+ c_3\frac{\varepsilon\log(\lambda_i d_i)}{\lambda_i^2}(\text{ if } n=4) = o\Big(\sum_{j=1}^{3}\Big(\frac{1}{(\lambda_j d_j)^{n-2}}+\frac{\varepsilon}{\lambda_j^2}\Big)+\sum_{j\neq k}\varepsilon_{jk}\Big), \qquad (2.24)$$

where $\gamma_1=\gamma_3=1$, $\gamma_2=-1$.

As in Proposition 2.3, we have the following result:

Proposition 2.4 *Assume that* $u_\varepsilon=\sum_{i=1}^{3}\alpha_i\gamma_i P\delta_i+v$ *is a sign-changing solution of* (P_ε). *We have*

$$(F_i)\quad \frac{\gamma_i}{2}\frac{c_1}{\lambda_i^{n-1}}\frac{\partial H(a_i,a_i)}{\partial a_i}-\sum_{j\neq i}\gamma_j c_1\Big(\frac{-1}{(\lambda_i\lambda_j)^{\frac{n-2}{2}}}\frac{1}{\lambda_i}\frac{\partial H(a_i,a_j)}{\partial a_i}+\frac{1}{\lambda_i}\frac{\partial\varepsilon_{ij}}{\partial a_i}\Big)=O\Big(\sum_{j\neq i}\big(\varepsilon\varepsilon_{ij}+\lambda_j|a_i-a_j|\varepsilon_{ij}^{\frac{n+1}{n-2}}\big)$$
$$+\frac{\varepsilon}{(\lambda_i d_i)^{n-1}}+\sum_{k=1}^{3}\frac{\log\lambda_k d_k}{(\lambda_k d_k)^n}+\sum_{j\neq k}\varepsilon_{kj}^{\frac{n}{n-2}}\log(\varepsilon_{kj}^{-1})+\sum_{j=1}^{3}\Big(\big(\textit{ if } n\leq 5\big)\frac{\varepsilon^2}{\lambda_j^{n-2}}+\big(\textit{ if } n=6\big)\frac{\varepsilon^2(\log\lambda_j)^{\frac{4}{3}}}{\lambda_j^4}+\frac{\varepsilon^2}{\lambda_j^4}\Big)\Big).$$

Proof As in the proof of Proposition 2.3, we get (2.6), but with $\alpha_1\lambda_1\partial P\delta_1/\partial\lambda_1$ changed by $\alpha_1(\lambda_1)^{-1}\partial P\delta_1/\partial a_1$. Thus, using Lemmas A.11,..., A.14, A.17, and A.20, the proposition follows. □

Now we distinguish many cases depending on the set

$$F:=\{(i,j): i\neq j \text{ and } \min(\lambda_i,\lambda_j)|a_i-a_j| \text{ is bounded}\}$$

and we will prove that all these cases cannot occur.

We note that if $(i,j)\in F$ we can derive $\lambda_i/\lambda_j\to 0$ or ∞ and $d_i/d_j=1+o(1)$ as $\varepsilon\to 0$.

Furthermore, the behavior of ε_{ij} depends on the set F. In fact, assuming that $\lambda_i\leq\lambda_j$, we have

$$c\Big(\frac{\lambda_i}{\lambda_j}\Big)^{\frac{n-2}{2}}\leq\varepsilon_{ij}\leq\Big(\frac{\lambda_i}{\lambda_j}\Big)^{\frac{n-2}{2}},\ \text{ if } (i,j)\in F, \qquad (2.25)$$

$$\varepsilon_{ij}=\frac{1}{\Big(\lambda_i\lambda_j|a_i-a_j|^2\Big)^{\frac{n-2}{2}}}+o(\varepsilon_{ij}),\ \text{ if } (i,j)\notin F. \qquad (2.26)$$

Lemma 2.5 *The case* $\varepsilon_{13}=o\Big(\sum_{i=1}^{3}\big(\frac{1}{(\lambda_i d_i)^{n-2}}+\frac{\varepsilon}{\lambda_i^2}\big)+\sum_{r\neq j}\varepsilon_{rj}\Big)$ *does not occur.*

Proof In the following, we focus only on proving the case $n\geq 5$. The case $n=4$ is not proved since it can be demonstrated using the same reasoning as in the first case.

Without loss of generality, we can assume that $\lambda_1\leq\lambda_2\leq\lambda_3$. We distinguish three cases and we will prove that they cannot occur. This implies our lemma.

Case 1. $\{(1,2),(2,3)\}\in F$.

Adding $(E_1)+2(E_2)+4(E_3)$ and using (2.21), we have

$$\sum_{i=1}^{3} c_1\frac{n-2}{2}2^{i-1}\frac{H(a_i,a_i)}{\lambda_i^{n-2}}) + c_1\frac{n-2}{4}\varepsilon_{12}(1+o(1)) + c_1\frac{n-2}{2}\varepsilon_{23} + O(\varepsilon_{13}) + 5c_1\frac{n-2}{2}\frac{H(a_1,a_3)}{(\lambda_1\lambda_3)^{\frac{n-2}{2}}}$$
$$-3c_1\frac{n-2}{2}\frac{H(a_1,a_2)}{(\lambda_1\lambda_2)^{\frac{n-2}{2}}} - 6c_1\frac{n-2}{2}\frac{H(a_2,a_3)}{(\lambda_2\lambda_3)^{\frac{n-2}{2}}} + \sum_{i=1}^{3} c_2 2^{i-1}\frac{\varepsilon}{\lambda_i^2} \le o\Big(\sum_{i=1}^{3}(\frac{1}{(\lambda_i d_i)^{n-2}} + \frac{\varepsilon}{\lambda_i^2}) + \sum_{r\neq j}\varepsilon_{rj}\Big).$$

Using $\{(1,2),(2,3)\} \in F$, we obtain

$$\frac{H(a_i,a_{i+1})}{(\lambda_i\lambda_{i+1})^{\frac{n-2}{2}}} = o\Big(\frac{1}{(\lambda_i d_i)^{n-2}}\Big), \text{ for } i=1,2,$$

which gives a contradiction. Hence, this case cannot occur.

Case 2. $\{(1,2),(2,3)\} \cap F = \emptyset$.
Adding $(E_1)+2(E_2)+4(E_3)$ and using (2.19), we have

$$\sum_{i=1}^{3} c_1\frac{n-2}{2}2^{i-1}\frac{H(a_i,a_i)}{\lambda_i^{n-2}} + O(\varepsilon_{13}) + 5c_1\frac{n-2}{2}\frac{H(a_1,a_3)}{(\lambda_1\lambda_3)^{\frac{n-2}{2}}} + c_1\frac{n-2}{2}\Big(\frac{3G(a_1,a_2)}{(\lambda_1\lambda_2)^{\frac{n-2}{2}}}$$
$$+\frac{6G(a_2,a_3)}{(\lambda_2\lambda_3)^{\frac{n-2}{2}}}\Big) + \sum_{i=1}^{3} c_2 2^{i-1}\frac{\varepsilon}{\lambda_i^2} = o\Big(\sum_{i=1}^{3}(\frac{1}{(\lambda_i d_i)^{n-2}} + \frac{\varepsilon}{\lambda_i^2}) + \frac{G(a_1,a_2)}{(\lambda_1\lambda_2)^{\frac{n-2}{2}}} + \frac{G(a_2,a_3)}{(\lambda_2\lambda_3)^{\frac{n-2}{2}}}\Big).$$

Then we derive a contradiction and therefore this case cannot occur.

Case 3. $(1,2)\in F$ and $(2,3)\notin F$ or $(1,2)\notin F$ and $(2,3)\in F$. Assume that $(1,2)\in F$ and $(2,3)\notin F$.
Adding $(E_1)+2(E_2)+4(E_3)$, using (2.19) and (2.21), we have

$$\sum_{i=1}^{3} c_1\frac{(n-2)}{2}2^{i-1}\frac{H(a_i,a_i)}{\lambda_i^{n-2}} + c_1\frac{(n-2)}{4}\varepsilon_{12} - 3c_1\frac{(n-2)}{2}\frac{H(a_1,a_2)}{(\lambda_1\lambda_2)^{n-2/2}} + 6c_1\frac{(n-2)}{2}\frac{G(a_2,a_3)}{(\lambda_2\lambda_3)^{n-2/2}}$$
$$+O(\varepsilon_{13}) + \sum_{i=1}^{3} c_2 2^{i-1}\frac{\varepsilon}{\lambda_i^2} \le o\Big(\sum_{i=1}^{3}(\frac{1}{(\lambda_i d_i)^{n-2}} + \frac{\varepsilon}{\lambda_i^2}) + \varepsilon_{12} + \frac{G(a_2,a_3)}{(\lambda_2\lambda_3)^{\frac{n-2}{2}}}\Big).$$

As in case 1, $(1,2)\in F$ implies $\dfrac{H(a_1,a_2)}{(\lambda_1\lambda_2)^{n-2/2}} = o\Big(\dfrac{1}{(\lambda_1 d_1)^{n-2}}\Big)$, which gives a contradiction. Hence, Lemma 2.5 is proved. □

First, we start by providing the following crucial lemmas. We are only interested in proving the case $n\ge 5$ since the same reasoning can be used for $n=4$.

Lemma 2.6 *There exists a positive constant* $\underline{c}_0 > 0$ *such that*

1. $\underline{c}_0^{-1} \le \dfrac{d_1}{d_3} \le \underline{c}_0,$

2. $\underline{c}_0^{-1} \le \dfrac{\lambda_1}{\lambda_3} \le \underline{c}_0,$

3. $\quad \underline{c}_0^{-1} \leq \dfrac{|a_1 - a_3|}{d_i} \leq \underline{c}_0,$ *for* $i = 1, 3$.

Proof The proof will be by contradiction.

Claim 1. Assume that $d_1/d_3 \to 0$. In this case, we have

$$|a_1 - a_3| \geq cd_3 \quad \text{and} \quad \varepsilon_{13} = \frac{1}{(\lambda_1\lambda_3|a_1 - a_3|^2)^{\frac{n-2}{2}}} + o(\varepsilon_{13}), \tag{2.27}$$

which implies that $\varepsilon_{13} = o((\lambda_1 d_1)^{2-n} + (\lambda_3 d_3)^{2-n})$.

Using Lemma 2.5, we derive a contradiction. In the same way, we prove that $d_3/d_1 \not\to 0$. Hence, the proof of Claim 1 is completed.

Claim 2. Assume that $\lambda_1/\lambda_3 \to 0$. By Claim 1, we have $(\lambda_3 d_3)^{-1} = o((\lambda_1 d_1)^{-1})$. Four cases may occur.

Case 1. $\lambda_2/\lambda_3 \not\to 0$ or $\{(1,2),(2,3)\} \cap F = \emptyset$.

If $\lambda_2/\lambda_3 \not\to 0$, we have $\lambda_1/\lambda_2 \leq c\lambda_1/\lambda_3$, and then $\lambda_1/\lambda_2 \to 0$.

Therefore, by (2.14), we have $\lambda_2 \dfrac{\partial \varepsilon_{12}}{\partial \lambda_2} = -\dfrac{n-2}{2}\varepsilon_{12} + o(\varepsilon_{12})$ and $\lambda_2 \dfrac{\partial \varepsilon_{23}}{\partial \lambda_2} = -\dfrac{n-2}{2}\varepsilon_{23} + o(\varepsilon_{23})$.

Thus, (E_2) and (E_3) becomes

$$c_1\frac{n-2}{2}\Big(\frac{H(a_2,a_2)}{\lambda_2^{n-2}} - \frac{H(a_1,a_2)}{(\lambda_1\lambda_2)^{\frac{n-2}{2}}} - \frac{H(a_2,a_3)}{(\lambda_2\lambda_3)^{\frac{n-2}{2}}} + \varepsilon_{12} + \varepsilon_{23}\Big) + \frac{c_2\varepsilon}{\lambda_2^2} = o\Big(\sum_{i=1}^{3}\big(\frac{1}{(\lambda_i d_i)^{n-2}} + \frac{\varepsilon}{\lambda_i^2}\big) + \sum_{k\neq r}\varepsilon_{kr}\Big),$$

$$c_1\frac{n-2}{2}\Big(-\varepsilon_{13} - \frac{H(a_2,a_3)}{(\lambda_2\lambda_3)^{\frac{n-2}{2}}}\Big) + O(\varepsilon_{23}) = o\Big(\sum_{i=1}^{3}\big(\frac{1}{(\lambda_i d_i)^{n-2}} + \frac{\varepsilon}{\lambda_i^2}\big) + \sum_{k\neq r}\varepsilon_{kr}\Big).$$

Using the fact that $\lambda_1/\lambda_2 \to 0$, $\lambda_1/\lambda_3 \to 0$ and Claim 1, we obtain

$$\frac{H(a_1,a_i)}{(\lambda_1\lambda_i)^{\frac{n-2}{2}}} = o\Big(\frac{1}{(\lambda_1 d_1)^{n-2}}\Big) \text{ and } \frac{H(a_2,a_3)}{(\lambda_2\lambda_3)^{\frac{n-2}{2}}} = o\Big(\frac{1}{(\lambda_1 d_1)^{n-2}}\Big).$$

We can choose m a fixed large constant, so that $m(E_2)-(E_3)$ implies $\varepsilon_{13} = o\big(\sum_{i=1}^{3}(\frac{1}{(\lambda_i d_i)^{n-2}} + \frac{\varepsilon}{\lambda_i^2}) + \sum_{k\neq r}\varepsilon_{kr}\big)$.

Hence, by Lemma 2.5, this case cannot occur.

If $\{(1,2),(2,3)\} \cap F = \emptyset$. (E_2) and (E_3) imply that

$$c_1\frac{n-2}{2}\Big(\frac{H(a_2,a_2)}{\lambda_2^{\frac{n-2}{2}}} + \frac{G(a_1,a_2)}{(\lambda_1\lambda_2)^{\frac{n-2}{2}}} + \frac{G(a_2,a_3)}{(\lambda_2\lambda_3)^{\frac{n-2}{2}}}\Big) + \frac{c_2\varepsilon}{\lambda_2^2} = o\Big(\sum_{i=1}^{3}\big(\frac{1}{(\lambda_i d_i)^{n-2}} + \frac{\varepsilon}{\lambda_i^2}\big) + \sum_{k\neq r}\varepsilon_{kr}\Big),$$

$$c_1\frac{n-2}{2}\Big(-\varepsilon_{13} + \frac{G(a_2,a_3)}{(\lambda_2\lambda_3)^{\frac{n-2}{2}}}\Big) = o\Big(\sum_{i=1}^{3}\big(\frac{1}{(\lambda_i d_i)^{n-2}} + \frac{\varepsilon}{\lambda_i^2}\big) + \sum_{k\neq r}\varepsilon_{kr}\Big).$$

Using the formula $(E_2) - (E_3)$, we obtain

$$c_1\frac{n-2}{2}\Big(\varepsilon_{13} + \frac{H(a_2,a_2)}{\lambda_2^{\frac{n-2}{2}}} + \frac{G(a_1,a_2)}{(\lambda_1\lambda_2)^{\frac{n-2}{2}}}\Big) + \frac{c_2\varepsilon}{\lambda_2^2} = o\Big(\sum_{i=1}^{3}\big(\frac{1}{(\lambda_i d_i)^{n-2}} + \frac{\varepsilon}{\lambda_i^2}\big) + \sum_{k\neq r}\varepsilon_{kr}\Big),$$

which leads to $\varepsilon_{13}=o\big(\sum_{i=1}^{3}(\frac{1}{(\lambda_i d_i)^{n-2}}+\frac{\varepsilon}{\lambda_i^2})+\sum_{k\neq r}\varepsilon_{kr}\big)$. Hence, by Lemma 2.5, this case cannot occur.

Case 2. $\lambda_2/\lambda_3 \to 0$, $\{(1,2),(2,3)\}\cap F \neq \emptyset$ and $\lambda_2/\lambda_1 \to +\infty$. In this case, it is easy to obtain $\varepsilon_{13}=o(\varepsilon_{12}+\varepsilon_{23})$. Using Lemma 2.5, we derive a contradiction.

Case 3. $\lambda_2/\lambda_3 \to 0$, $(2,3)\in F$, $(1,2)\notin F$ and $\lambda_2/\lambda_1 \not\to +\infty$. In this case, we have that $\lambda_2 \mid a_2-a_3 \mid$ is bounded and $\lambda_2 \mid a_1-a_2 \mid\to +\infty$. Hence, we derive that $\lambda_2 \mid a_1-a_3 \mid\to +\infty$, which implies that $\lambda_k \mid a_1-a_3 \mid\to +\infty$ for $k=1,3$. Thus,

$$\varepsilon_{13}=\frac{1}{(\lambda_1\lambda_3\mid a_1-a_3\mid^2)^{\frac{n-2}{2}}}(1+o(1))=(\frac{\lambda_2}{\lambda_3})^{\frac{n-2}{2}}\frac{1}{(\lambda_1\lambda_2\mid a_1-a_3\mid^2)^{\frac{n-2}{2}}}(1+o(1))=o(\varepsilon_{23}).$$

Then, by Lemma 2.5, we get a contradiction.

Case 4. $\lambda_2/\lambda_3 \to 0$, $(1,2)\in F$ and $\lambda_2/\lambda_1 \not\to +\infty$. In this case, it is easy to get $\varepsilon_{23}\le(\frac{\lambda_2}{\lambda_3})^{\frac{n-2}{2}}=(\frac{\lambda_2}{\lambda_1})^{\frac{n-2}{2}}(\frac{\lambda_1}{\lambda_3})^{\frac{n-2}{2}}=o(\frac{\lambda_2}{\lambda_1})^{\frac{n-2}{2}}=o(\varepsilon_{12})$.

Using the formula $\Big(2(E_1)+(E_2)-4(E_3)\Big)$, we obtain a contradiction, and Claim 2 follows.

Claim 3. Without loss of generality, we can assume that $d_1\le d_3$. First, as in the proof of Claim 1, we get $\mid a_1-a_3\mid\le c_0 d_1$. Now assume that $\mid a_1-a_3\mid/d_1\to 0$, which implies

$$\frac{H(a_i,a_i)}{\lambda_i^{n-2}}=o(\varepsilon_{13}),\ \text{ for } i=1,3.$$

We distinguish two cases and we will prove that they cannot occur.

Case 1. $\lambda_1\le\lambda_2$ or $\{(1,2),(2,3)\}\cap F=\emptyset$. Two cases may occur. If $\lambda_1\le\lambda_2$, using Claim 2, we have $c_0^{-1}\lambda_3\le\lambda_2$, and hence

$$\lambda_2\frac{\partial\varepsilon_{2i}}{\partial\lambda_2} = -\frac{n-2}{2}\varepsilon_{2i}+o(\varepsilon_{2i}) \quad \text{for } i=1,3, \tag{2.28}$$

$$\lambda_1\frac{\partial\varepsilon_{13}}{\partial\lambda_1} = -\frac{n-2}{2}\varepsilon_{13}+o(\varepsilon_{13}) \quad \text{and } \lambda_1\frac{\partial\varepsilon_{12}}{\partial\lambda_1}=O(\varepsilon_{12}), \tag{2.29}$$

$$\frac{H(a_2,a_i)}{(\lambda_2\lambda_i)^{\frac{n-2}{2}}} = o(\varepsilon_{13}) \quad \text{for } i=1,3. \tag{2.30}$$

Using (2.28), (2.29), (2.30), (E_1), and (E_2), we obtain

$$\frac{H(a_2,a_2)}{\lambda_2^{n-2}}=o(\varepsilon_{13}),\quad \varepsilon_{i2}=o(\varepsilon_{13}) \quad \text{for } i=1,3 \text{ and } \quad \frac{\varepsilon}{\lambda_2^2}=o(\varepsilon_{13}). \tag{2.31}$$

On the other hand, we have

$$\frac{1}{(\lambda_1\lambda_i)^{\frac{n-2}{2}}}\frac{1}{\lambda_1}\frac{\partial H(a_1,a_i)}{\partial a_1^k} = o(\varepsilon_{13}^{\frac{n-1}{n-2}}) \quad \text{for } i=2,3, \tag{2.32}$$

$$\frac{1}{\lambda_1}\frac{\partial \varepsilon_{13}}{\partial a_1^k} = -\frac{(n-2)\lambda_3(a_1-a_3)_k}{(\lambda_1\lambda_3|a_1-a_3|^2)^{\frac{n}{2}}}(1+o(1)), \tag{2.33}$$

$$\frac{1}{\lambda_1}\frac{\partial \varepsilon_{12}}{\partial a_1^k} = -\frac{(n-2)\lambda_2(a_1-a_2)_k}{(\frac{\lambda_1}{\lambda_2}+\frac{\lambda_2}{\lambda_1}+\lambda_1\lambda_2|a_1-a_2|^2)^{\frac{n}{2}}} = -(n-2)\lambda_2(a_1-a_2)_k\varepsilon_{12}\varepsilon_{12}^{\frac{2}{n-2}}. \tag{2.34}$$

By (2.31), ..., (2.34) and (F_1), we obtain

$$\frac{(n-2)\lambda_3(a_1-a_3)_k}{(\lambda_1\lambda_3|a_1-a_3|^2)^{\frac{n}{2}}} - (n-2)\lambda_2(a_1-a_2)_k\varepsilon_{12}^{\frac{2}{n-2}}\varepsilon_{12} = o(\varepsilon_{13}^{\frac{n-1}{n-2}}) + O\Big(\varepsilon_{12}^{\frac{n+1}{n-2}}\lambda_2|a_2-a_1|\Big). \tag{2.35}$$

If $|a_1-a_2|\geq \frac{1}{2}|a_1-a_3|$, we have $\lambda_2(a_1-a_2)_k\varepsilon_{12}^{2/n-2} \leq \dfrac{\lambda_2|a_1-a_2|}{\lambda_1\lambda_2|a_1-a_2|^2} \leq \dfrac{2}{(\lambda_1^2|a_1-a_3|^2)^{1/2}} = O\big(\varepsilon_{13}^{1/n-2}\big)$. Then $\lambda_2(a_1-a_2)_k\varepsilon_{12}^{2/n-2}\varepsilon_{12} = O(\varepsilon_{12}\varepsilon_{13}^{1/n-2}) = o(\varepsilon_{13}^{\frac{n-1}{n-2}})$. (2.35) becomes

$$\varepsilon_{13}^{\frac{n-1}{n-2}} = o(\varepsilon_{13}^{\frac{n-1}{n-2}}),$$

which gives a contradiction. Hence, this case cannot occur.

If $|a_1-a_2|\leq \frac{1}{2}|a_1-a_3|$, we have $|a_3-a_2|\geq|a_3-a_1|-|a_1-a_2|\geq 1/2|a_1-a_3|$. Using (F_3), the same argument as in (2.35), we obtain a contradiction.

If $\{(1,2),(2,3)\}\cap F=\emptyset$.

Using the same reasoning, we derive a contradiction and therefore this case cannot occur.

Case 2. $\lambda_2\leq\lambda_1$ and $\{(1,2),(2,3)\}\cap F\neq\emptyset$. Let $k\in\{1,3\}$ such that $(2,k)\in F$. Using Claim 2 and the fact that $\lambda_2\leq\lambda_1$, we derive that $\varepsilon_{2k}\geq c(\lambda_2/\lambda_k)^{(n-2)/2}$, which implies that $d_2\sim d_k$, $\lambda_2/\lambda_k\to 0$, and that $\lambda_2\mid a_2-a_k\mid$ is bounded.

Using (F_i) for $i=k$, we get

$$-(n-2)\Big(\lambda_2(a_2-a_k)_j\varepsilon_{2k}^{\frac{n}{n-2}} - \frac{\lambda_1\lambda_3}{\lambda_k}(a_1-a_3)_j\varepsilon_{13}^{\frac{n}{n-2}}\Big) = o\Big(\frac{1}{(\lambda_2 d_2)^{n-1}} + \sum_{r\neq i}\varepsilon_{ri}^{\frac{n-1}{n-2}} + \frac{\varepsilon}{\lambda_2^2}\Big)$$
$$+ O\Big(\varepsilon_{k2}^{\frac{n+1}{n-2}}\lambda_2|a_2-a_k| + \varepsilon_{13}^{\frac{n+1}{n-2}}\lambda_1|a_1-a_3|\Big).$$

Since $\lambda_2\mid a_2-a_k\mid$ is bounded and $\varepsilon_{13}=(\lambda_1\lambda_3\mid a_1-a_3\mid^2)^{(2-n)/2}(1+o(1))$, we derive that

$$\varepsilon_{13}^{\frac{n-1}{n-2}} = o\Big(\frac{1}{(\lambda_2 d_2)^{n-1}} + \sum_{k\neq r}\varepsilon_{kr}^{\frac{n-1}{n-2}} + \frac{\varepsilon}{\lambda_2^2}\Big),$$

which implies that

$$\varepsilon_{13} = o\Big(\frac{1}{(\lambda_2 d_2)^{n-2}} + \sum_{k\neq r}\varepsilon_{kr} + \frac{\varepsilon}{\lambda_2^2}\Big). \tag{2.36}$$

By Lemma 2.5, we get a contradiction. $\square$

Lemma 2.7 *There exists a positive constant* $\underline{c}_0'$ *such that*

$$(i) \quad \underline{c}_0'\lambda_i \leq \lambda_2\,; \quad (ii) \quad d_i \geq \underline{c}_0' \quad \textit{for } i = 1, 3.$$

Proof The proof of this lemma is similar to that of Lemma 4.2 of [9] and therefore is omitted. □

We turn now to the proof of Theorem 1.2. By the previous lemmas, we know that λ_1 and λ_3 are of the same order, $\mid a_1 - a_3 \mid \geq c$, $\lambda_2 \geq \underline{c}_0'\lambda_i$ and $d_i \geq \underline{c}_0'$, for $i = 1, 3$ where $c, \underline{c}_0'$ are positive constants. From (E_2), we obtain

$$c_1\frac{n-2}{2}\Big(\frac{H(a_2,a_2)}{\lambda_2^{n-2}}(1+o(1)) + \varepsilon_{12}(1+o(1)) + \varepsilon_{23}(1+o(1))\Big) + \frac{c_2\varepsilon}{\lambda_2^2}(1+o(1)) = o\big(\frac{1}{\lambda_1^{n-2}} + \frac{\varepsilon}{\lambda_1^2}\big). \tag{2.37}$$

Then

$$\frac{H(a_2,a_2)}{\lambda_2^{n-2}} = o\big(\frac{1}{\lambda_1^{n-2}} + \frac{\varepsilon}{\lambda_1^2}\big), \quad \varepsilon_{i2} = o\big(\frac{1}{\lambda_1^{n-2}} + \frac{\varepsilon}{\lambda_1^2}\big) \quad \text{for } i = 1, 3. \tag{2.38}$$

Using (2.38), (E_i) and (F_i) for $i = 1, 3$ imply that

$$\frac{H(a_i,a_i)}{\lambda_i^{n-2}} - \frac{G(a_1,a_3)}{(\lambda_1\lambda_3)^{\frac{n-2}{2}}} + \frac{c_2}{c_1}\frac{2}{n-2}\frac{\varepsilon}{\lambda_i^2} = o\Big(\frac{1}{\lambda_i^{n-2}} + \frac{\varepsilon}{\lambda_i^2}\Big), \quad \text{if } n \geq 5, \tag{2.39}$$

$$\frac{H(a_i,a_i)}{\lambda_i^2} - \frac{G(a_1,a_3)}{\lambda_1\lambda_3} + \frac{c_3}{c_1}\varepsilon\frac{\log(\lambda_i)}{\lambda_i^2} = o\Big(\frac{1}{\lambda_i^2} + \varepsilon\frac{\log(\lambda_i)}{\lambda_i^2}\Big), \quad \text{if } n = 4, \tag{2.40}$$

$$-\frac{1}{\lambda_i^{n-2}}\frac{\partial H(a_i,a_i)}{\partial a_i} + \frac{2}{(\lambda_1\lambda_3)^{\frac{n-2}{2}}}\frac{\partial G(a_1,a_3)}{\partial a_i} = o\Big(\frac{1}{\lambda_i^{n-2}}\Big). \tag{2.41}$$

Three cases may occur.

Case 1. $\dfrac{1}{(\lambda_i d_i)^{n-2}} = o\Big(\dfrac{\varepsilon}{\lambda_i^2}\Big)$, if $n \geq 5$, $\quad \dfrac{1}{(\lambda_i d_i)^2} = o\Big(\dfrac{\varepsilon\log(\lambda_i)}{\lambda_i^2}\Big)$, if $n = 4$, for $i = 1, 3$.

We obtain $\varepsilon_{13} = o\Big(\dfrac{\varepsilon}{\lambda_i^2}\Big)$. Hence, this case cannot occur.

Case 2. $\dfrac{\varepsilon}{\lambda_i^2} = o\Big(\dfrac{1}{(\lambda_i d_i)^{n-2}}\Big)$, if $n \geq 5$, $\quad \dfrac{\varepsilon\log(\lambda_i)}{\lambda_i^2} = o\Big(\dfrac{1}{(\lambda_i d_i)^2}\Big)$, if $n = 4$, for $i = 1, 3$.

Let $\Lambda_i = \lambda_i^{(2-n)/2}$, $\Lambda = (\Lambda_1, \Lambda_3)$, and $x = (a_1, a_3)$. From (2.39) and (2.40), we have

$$M(x).\frac{{}^t\Lambda}{\parallel \Lambda \parallel} = o(1). \tag{2.42}$$

The scalar product of (2.42) by $r(x)$ gives

$$\rho(x)r(x).\frac{{}^t\Lambda}{\parallel \Lambda \parallel} = o(1). \tag{2.43}$$

Since the components of $r(x)$ are positive and λ_1, λ_3 are of the same order, there exists a positive constant C, such that $r(x).\frac{{}^t\Lambda}{\|\Lambda\|} \geq C > 0$. Hence, we get

$$\rho(x) = o(1). \tag{2.44}$$

Denoting by $\overline{a} = (\overline{a}_1, \overline{a}_3) \in \Omega^2$ the limit of (a_1, a_3) (up to a subsequence) and using (2.44), we get $\rho(\overline{a}) = 0$. It remains to prove that $\rho'(\overline{a}) = 0$.

We deduce from (2.41) that

$$\frac{\partial M}{\partial a_i}(x).{}^t\Lambda = o(\| \Lambda \|). \tag{2.45}$$

Observe that Λ may be written under the form

$$\Lambda = \beta r(x) + \overline{r}(x), \quad \text{with} \quad r(x).\overline{r}(x) = 0, \ \| \overline{r} \| = o(\beta) \quad \text{and} \quad \beta \sim \| \Lambda \| \, . \tag{2.46}$$

Using (2.45), we get

$$\beta \frac{\partial M}{\partial a_i}(x).{}^t r(x) + \frac{\partial M}{\partial a_i}(x).\overline{r}(x) = o(\| \Lambda \|). \tag{2.47}$$

Since $d_i \geq \underline{c}'_0$ for $i = 1, 3$ and $| \ a_1 - a_3 \ | \geq c$, the matrix $\dfrac{\partial M}{\partial a_i}(x)$ is bounded. Furthermore, we have $\| \overline{r} \| = o(\| \Lambda \|)$, which implies that

$$\frac{\partial M}{\partial a_i}(x).{}^t\overline{r}(x) = o(\| \Lambda \|).$$

The scalar product of (2.47) with $r(x)$ gives

$$\beta r(x) \frac{\partial M}{\partial a_i}(x).{}^t r(x) = o(\| \Lambda \|). \tag{2.48}$$

Let us consider the equality

$$M(x).{}^t r(x) = \rho(x).{}^t r(x),$$

and its derivative with respect to a_i implies

$$\frac{\partial M}{\partial a_i}(x).{}^t r(x) + M(x) \frac{\partial^t r}{\partial a_i}(x) = \frac{\partial \rho}{\partial a_i}(x).{}^t r(x) + \rho(x) \frac{\partial^t r}{\partial a_i}(x).$$

The scalar product with $r(x)$ gives

$$r(x). \frac{\partial M}{\partial a_i}(x).{}^t r(x) = \frac{\partial \rho}{\partial a_i}(x). \tag{2.49}$$

Passing to the limit in (2.48) and (2.49), we obtain

$$\frac{\partial \rho}{\partial a_i}(\overline{a}) = 0. \tag{2.50}$$

Hence the results.

Case 3. $\dfrac{1}{(\lambda_i d_i)^{n-2}} \sim \dfrac{\varepsilon}{\lambda_i^2}$, if $n \geq 5$ and $\dfrac{1}{(\lambda_i d_i)^2} \sim \dfrac{\log(\lambda_i)\varepsilon}{\lambda_i^2}$, if $n = 4$, for $i = 1, 3$.

Let us perform the change of variables

$$\lambda_i = \Lambda_i^{-\frac{2}{n-2}} \varepsilon^{-\frac{1}{n-4}} \left(\frac{c_2}{c_1}\right)^{-\frac{1}{n-4}}, \quad \text{if } n \geq 5.$$

Note that

$$\Lambda_i \varepsilon^{\frac{n-2}{2(n-4)}} \to 0 \text{ as } \varepsilon \to 0, \quad \frac{1}{c_0} \leq \Lambda_i \leq c_0,$$

and that (2.39) and (2.41) imply, for $i, j = 1, 3$ and $j \neq i$,

$$H(a_i, a_i)\Lambda_i - G(a_i, a_j)\Lambda_j + \frac{2}{n-2}\Lambda_i^{\frac{6-n}{n-2}} = o(1), \tag{2.51}$$

$$-\frac{\partial H}{\partial a_i}(a_i, a_i)\Lambda_i + 2\frac{\partial G}{\partial a_i}(a_i, a_j)\Lambda_j = o(1). \tag{2.52}$$

Denoting by $(\overline{a}_1, \overline{a}_3) \in \Omega^2$ the limit of a_1, a_3 and by $(\overline{\Lambda}_1, \overline{\Lambda}_3) \in (\mathbb{R}_+^*)^2$ the limit of Λ_1, Λ_3 (up to a subsequence), from passing to the limit in (2.51) and (2.52), we obtain

$$H(\overline{a}_i, \overline{a}_i)\overline{\Lambda}_i - G(\overline{a}_i, \overline{a}_j)\overline{\Lambda}_j + \frac{2}{n-2}\overline{\Lambda}_i^{\frac{6-n}{n-2}} = 0,$$

$$\frac{\partial H}{\partial a_i}(\overline{a}_i, \overline{a}_i)\overline{\Lambda}_i - 2\frac{\partial G}{\partial a_i}(\overline{a}_i, \overline{a}_j)\overline{\Lambda}_j = 0.$$

This means that $\frac{\partial \Psi_2}{\partial \Lambda_i}(\overline{\Lambda}_1, \overline{\Lambda}_3, \overline{a}_1, \overline{a}_3) = 0$ and $\frac{\partial \Psi_2}{\partial a_i}(\overline{\Lambda}_1, \overline{\Lambda}_3, \overline{a}_1, \overline{a}_3) = 0$, for $i \in \{1, 3\}$. The proof of Theorem 1.2 is thereby completed for $n \geq 5$.

If $n = 4$, denoting by $\eta_i = \lambda_i/\lambda_j$ with $j \neq i$ and $\Lambda_i = \frac{c_3}{c_1}\varepsilon \log(\lambda_i)$, then (2.40) and (2.41) imply

$$H(a_i, a_i) - \eta_i G(a_i, a_j) + \Lambda_i = o(1), \tag{2.53}$$

$$-\frac{\partial H}{\partial a_i}(a_i, a_i) + 2\eta_i \frac{\partial G(a_i, a_j)}{\partial a_i} = o(1). \tag{2.54}$$

From Lemma 2.6, we derive that η_i converges to a constant $\overline{\eta}_i$, with $\overline{\eta}_1 = \overline{\eta}_3^{-1} := \overline{\eta}$ (up to a subsequence). Furthermore, since $\overline{a}_i \in \Omega$ and $\overline{a}_1 \neq \overline{a}_3$, using (2.53), we get that Λ_i is bounded above and below, for $i = 1, 3$. Thus, up to a subsequence, Λ_i converges to a constant $\overline{\Lambda}_i$, and it is easy to prove that $\overline{\Lambda}_1 = \overline{\Lambda}_3 := \overline{\Lambda}$ ($\lim_{\varepsilon \to 0}(\Lambda_1 - \Lambda_3) = 0$). Passing to the limit in (2.53) and (2.54), we get

$$H(\overline{a}_i, \overline{a}_i) - \overline{\eta}_i G(\overline{a}_1, \overline{a}_3) + \overline{\Lambda} = 0$$

$$-\frac{\partial H(\overline{a}_i, \overline{a}_i)}{\partial \overline{a}_i} + 2\overline{\eta}_i \frac{\partial G(\overline{a}_1, \overline{a}_3)}{\partial \overline{a}_i} = 0.$$

This ends the proof of Theorem 1.2. $\square$

3. Proof of Theorem 1.4

First of all, let us introduce the general setting. We define on $H_0^1(\Omega)$ the functional:

$$J_\varepsilon(u) = \frac{1}{2}\|u\|^2 - \frac{n-2}{2n}\int_\Omega |u|^{\frac{2n}{n-2}} + \frac{\varepsilon}{2}\int_\Omega u^2.$$

If u is a critical point of J_ε, u satisfies on Ω the equation (P_ε). Conversely, we see that any solution of (P_ε) is a critical point of J_ε.

We introduce the following subset of $H_0^1(\Omega)$:

$$M_\varepsilon = \{(\alpha,\lambda,a,v) \in \mathbb{R}^4 \times (\mathbb{R}_+^*)^4 \times \Omega^4 \times H_0^1(\Omega) \text{ such that } \forall\, i \in \{1,..,4\}\ |\alpha_i - 1| < \eta_0,$$
$$d(a_i,\partial\Omega) \geq d_0,\ \lambda_i \in [c_0^{-1}\varepsilon^{-1/(n-4)}, c_0\varepsilon^{-1/(n-4)}],\ |a_i - a_j| \geq d_0\ \forall i \neq j,\ v \in E,\ \|v\| \leq \eta_0\},$$

where η_0, c_0, d_0 are suitable positive constants.

Let us define the functional K_ε by the map

$$K_\varepsilon : M_\varepsilon \to \mathbb{R}, \quad K_\varepsilon(\alpha,\lambda,a,v) = J_\varepsilon\Big(\sum_{i=1}^4 \alpha_i\gamma_i P\delta_{(a_i,\lambda_i)} + v\Big), \tag{3.1}$$

where $\gamma_1 = \gamma_3 = 1$, $\gamma_2 = \gamma_4 = -1$.

Note that (α,λ,a,v) is a critical point of K_ε if and only if $u = \sum_{i=1}^4 \alpha_i\gamma_i P\delta_{(a_i,\lambda_i)} + v$ is a critical point of J_ε.

Assume that u_ε is a sign-changing solution of (P_ε), which has the form (1.7) where $(\alpha_\varepsilon,\lambda_\varepsilon,a_\varepsilon,v_\varepsilon) \in M_\varepsilon$. We first deal with the v-part of u, and we prove the following:

Lemma 3.1 *There exists $\varepsilon_0 > 0$ such that, for $0 < \varepsilon < \varepsilon_0$, there exists a C^1-map for which to any (α,λ,a) with $(\alpha,\lambda,a,0) \in M_\varepsilon$ associates $\overline{v}_\varepsilon = v_{(\alpha,\lambda,a)} \in E$. Such a $\overline{v}_\varepsilon$ minimizes $K_\varepsilon(\alpha,\lambda,a,v)$ with respect to v in E, $\|v\| \leq \eta_0$, with η_0 small enough, and we have the estimate*

$$\|\overline{v}_\varepsilon\| = O\begin{cases} \displaystyle\sum_{i=1}^4\Big(\frac{\varepsilon}{\lambda_i^{\frac{3}{2}}} + \frac{1}{\lambda_i^3}\Big) + \sum_{i\neq j}\varepsilon_{ij}(\log(\varepsilon_{ij}^{-1}))^{\frac{3}{5}}, & \text{if } n = 5,\\ \displaystyle\sum_{i=1}^4\Big(\frac{\varepsilon(\log\lambda_i)^{\frac{2}{3}}}{\lambda_i^2} + \frac{\log(\lambda_i)}{\lambda_i^4}\Big) + \sum_{i\neq j}\varepsilon_{ij}(\log(\varepsilon_{ij}^{-1}))^{\frac{2}{3}}, & \text{if } n = 6,\\ \displaystyle\sum_{i=1}^4\Big(\frac{\varepsilon}{\lambda_i^2} + \frac{1}{\lambda_i^{\frac{n+2}{2}}}\Big) + \sum_{i\neq j}\varepsilon_{ij}^{\frac{n+2}{2(n-2)}}(\log(\varepsilon_{ij}^{-1}))^{\frac{n+2}{2n}}, & \text{if } n > 6. \end{cases}$$

Moreover, there exists $(B_i, C_i, D_i) \in \mathbb{R}^4 \times \mathbb{R}^4 \times (\mathbb{R}^4)^n$ such that

$$\frac{\partial K_\varepsilon}{\partial v}(\alpha,\lambda,a,\overline{v}_\varepsilon) = \sum_{i=1}^4\Big(B_i P\delta_i + C_i\lambda_i\frac{\partial P\delta_i}{\partial\lambda_i} + \sum_{k=1}^n D_{i_k}\frac{1}{\lambda_i}\frac{\partial P\delta_i}{\partial a_i^k}\Big), \tag{3.2}$$

where the a_i^k is the kth component of a_i.

Next, we prove a useful expansion of the derivative of the function K_ε with respect to α_i, λ_i, a_i.

Proposition 3.2 *Assume that $(\alpha, \lambda, a, v) \in M_\varepsilon$ and let $v := \overline{v}_\varepsilon$ be the function obtained in Lemma 3.1. Then the following expansions hold:*

1. $$\frac{\partial K_\varepsilon}{\partial \alpha_i}(\alpha, \lambda, a, v) = \alpha_i S_n (1 - \alpha_i^{\frac{4}{n-2}}) + O\Big(\sum_{j=1}^{4} (\frac{1}{\lambda_j^{n-2}} + \frac{\varepsilon}{\lambda_j^2})\Big),$$

2. $$\lambda_i \frac{\partial K_\varepsilon}{\partial \lambda_i}(\alpha, \lambda, a, v) = -2\alpha_i^2 c_2 \frac{\varepsilon}{\lambda_i^2} + \alpha_i^2 (1 - 2\alpha_i^{4/(n-2)}) \frac{c_1 (n-2) H(a_i, a_i)}{2\lambda_i^{n-2}} - c_1 \gamma_i \sum_{j \neq i} \gamma_j \alpha_j \alpha_i (1 - \alpha_j^{4/(n-2)} - \alpha_i^{4/(n-2)}) \frac{(n-2) G(a_i, a_j)}{2(\lambda_i \lambda_j)^{\frac{n-2}{2}}} + o\Big(\sum_{k=1}^{4} \Big(\frac{1}{\lambda_k^{n-2}} + \frac{\varepsilon}{\lambda_k^2}\Big) + \sum_{r \neq k} \varepsilon_{kr}\Big),$$

3. $$\frac{1}{\lambda_i} \frac{\partial K_\varepsilon}{\partial a_i}(\alpha, \lambda, a, v) = \alpha_i^2 (2\alpha_i^{4/(n-2)} - 1) \frac{c_1 \partial H(a_i, a_i)}{2\lambda_i^{n-1} \partial a_i} + c_1 \lambda_i \sum_{j \neq i} \gamma_j \alpha_j \alpha_i (1 - \alpha_j^{4/(n-2)} - \alpha_i^{4/(n-2)}) \Big(\frac{1}{(\lambda_i \lambda_j)^{\frac{n-2}{2}}} \frac{1}{\lambda_i} \frac{\partial G(a_i, a_j)}{\partial a_i}\Big) + o\Big(\sum_{r \neq k} \varepsilon_{kr}^{\frac{n-1}{n-2}} + \sum_{k=1}^{4} (\frac{1}{\lambda_k^{n-1}} + \frac{\varepsilon}{\lambda_k^3})\Big).$$

We now estimate the numbers B_i, C_i, D_{ik} defined in Lemma 3.1. Taking the scalar product of (3.2) with respect to $P\delta_i$, $\dfrac{\partial P\delta_i}{\partial \lambda_i}$, and $\dfrac{\partial P\delta_i}{\partial a_i^k}$ for $i = 1, ..., 4$ and $k = 1, ..., n$ and using Proposition 3.2, the solution of the system in B_i, C_i, D_{i_k} shows the following result.

Proposition 3.3 *The coefficients B_i, C_i, and D_{i_k} that occur in Lemma 3.1 satisfy the estimates*

$$\begin{cases} B_i = O\Big(\sum_{j=1}^{4} (\frac{1}{\lambda_j^{n-2}} + \frac{\varepsilon}{\lambda_j^2})\Big), \\ C_i = O\Big(\sum_{j=1}^{4} (\frac{1}{\lambda_j^{n-2}} + \frac{\varepsilon}{\lambda_j^2})\Big), \\ D_{i_k} = O\Big(\sum_{j=1}^{4} (\frac{1}{\lambda_j^{n-2}} + \frac{\varepsilon}{\lambda_j^2})\Big). \end{cases} \tag{3.3}$$

For (λ, a), our aim is to study the α-part of u. Namely, we prove the following result.

Proposition 3.4 *There exists $\varepsilon_0 > 0$ such that, for $0 < \varepsilon < \varepsilon_0$, there exists a C^1-map for which to any (λ, a) associates $\alpha = \alpha_{(\lambda,a)}$ that satisfies $\dfrac{\partial K_\varepsilon}{\partial \alpha_i}(\alpha, \lambda, a, \overline{v}_\varepsilon) = 0$ for each i, and we have the following estimate:*

$$|\alpha_i - 1| = O\Big(\sum_{j=1}^{4} (\frac{\varepsilon}{\lambda_j^2} + \frac{1}{\lambda_j^{n-2}})\Big).$$

Now we have to find (λ, a) such that

$$\frac{\partial K_\varepsilon}{\partial \lambda_i} = C_i\big(\frac{\partial^2 P\delta_i}{\partial^2 \lambda_i}, \overline{v}_\varepsilon\big) + \sum_{k=1}^{n} D_{i_k}\big(\frac{\partial^2 P\delta_i}{\partial \lambda_i \partial a_i^k}, \overline{v}_\varepsilon\Big), \quad \forall i, \tag{3.4}$$

$$\frac{\partial K_\varepsilon}{\partial a_i^r} = C_i\big(\frac{\partial^2 P\delta_i}{\partial \lambda_i \partial a_i^r}, \overline{v}_\varepsilon\big) + \sum_{k=1}^{n} D_{i_k}\big(\frac{\partial^2 P\delta_i}{\partial a_i^r \partial a_i^k}, \overline{v}_\varepsilon\Big), \quad \forall i, \ \forall r. \tag{3.5}$$

Using Lemma 3.1, Proposition 3.2, Proposition 3.3, and Proposition 3.4, we deduce that (3.4) and (3.5) are equivalent to

$$-\frac{c_1(n-2)H(a_i,a_i)}{2\lambda_i^{n-2}} + c_1\gamma_i \sum_{j\neq i} \gamma_j \frac{(n-2)G(a_i,a_j)}{2(\lambda_i\lambda_j)^{\frac{n-2}{2}}} - 2c_2\frac{\varepsilon}{\lambda^2} = o\Big(\sum_{k=1}^{4}\big(\frac{1}{\lambda_k^{n-2}} + \frac{\varepsilon}{\lambda_k^2}\big)\Big), \tag{3.6}$$

$$\frac{c_1}{2\lambda_i^{n-1}}\frac{\partial H(a_i,a_i)}{\partial a_i} - c_1\gamma_i \sum_{j\neq i}\gamma_j\Big(\frac{1}{(\lambda_i\lambda_j)^{\frac{n-2}{2}}}\frac{1}{\lambda_i}\frac{\partial G(a_i,a_j)}{\partial a_i}\Big) = o\Big(\sum_{k=1}^{4}\big(\frac{1}{\lambda_k^{n-1}} + \frac{\varepsilon}{\lambda_k^3}\big)\Big). \tag{3.7}$$

Let us perform the change of variables

$$\lambda_i = \Lambda_i^{-2/(n-2)}\varepsilon^{-1/(n-4)}\big(\frac{c_2}{c_1}\big)^{-1/(n-4)}. \tag{3.8}$$

Note that

$$\Lambda_i \to \overline{\Lambda}_i \in \mathbb{R}_+^* \text{ and } a_i \to \overline{a}_i, \text{ as } \varepsilon \to 0, \text{ for all } i.$$

Passing to the limit in (3.6) and (3.7) and using (3.8), we obtain

$$H(\overline{a}_i,\overline{a}_i)\overline{\Lambda}_i - \sum_{j\neq i}\gamma_i\gamma_j G(\overline{a}_i,\overline{a}_j)\overline{\Lambda}_j + \frac{4}{n-2}\overline{\Lambda}_i^{\frac{6-n}{n-2}} = 0,$$

$$\frac{\partial H}{\partial a_i}(\overline{a}_i,\overline{a}_i)\overline{\Lambda}_i - \sum_{j\neq i}\gamma_i\gamma_j\frac{\partial G}{\partial a_i}(\overline{a}_i,\overline{a}_j)\overline{\Lambda}_j = 0.$$

This means that $\frac{\partial \Phi_4}{\partial \Lambda_i}(\overline{\Lambda},\overline{a}) = 0$ and $\frac{\partial \Phi_4}{\partial a_i}(\overline{\Lambda},\overline{a}) = 0$, for $i \in \{1,...,4\}$. This concludes the proof of Theorem 1.4.

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Appendix

In this appendix, we collect the estimates of the different integral quantities presented in the paper. These estimates were originally introduced by Bahri [1] and Bahri and Coron [2]. For the proof, we refer the interested reader to the literature [1, 2, 25]. We suppose that $\lambda_i d_i$ is large enough and ε_{ij} is small enough. We have the following estimates

Lemma A. 1

$$< P\delta, P\delta >= S_n - c_1 \frac{H(a,a)}{\lambda^{n-2}} + O\Big(\frac{\log(\lambda d)}{(\lambda d)^n}\Big),$$

where S_n is defined in Proposition 2.2 and c_1 is defined in Proposition 2.3.

Lemma A. 2

$$\int_\Omega P\delta^{\frac{2n}{n-2}} = S_n - \frac{2n}{n-2} c_1 \frac{H(a,a)}{\lambda^{n-2}} + O\Big(\frac{\log(\lambda d)}{(\lambda d)^n}\Big).$$

Lemma A. 3 *For $i \neq j$*

$$< P\delta_i, P\delta_j >= c_1 \Big(\varepsilon_{ij} - \frac{H(a_i,a_j)}{(\lambda_i\lambda_j)^{(n-2)/2}}\Big) + O\Big(\varepsilon_{ij}^{\frac{n}{n-2}} \log(\varepsilon_{ij}^{-1}) + \sum_{k=i,j} \frac{\log(\lambda_k d_k)}{(\lambda_k d_k)^n}\Big).$$

Lemma A. 4 *For $i \neq j$*

$$\int_\Omega P\delta_i^{\frac{n+2}{n-2}} P\delta_j =< P\delta_i, P\delta_j > + O\Big(\varepsilon_{ij}^{\frac{n}{n-2}} \log(\varepsilon_{ij}^{-1}) + \sum_{k=i,j} \frac{\log(\lambda_k d_k)}{(\lambda_k d_k)^n}\Big).$$

Lemma A. 5 *For $i \neq j$*

$$\int_\Omega \big(\delta_i\delta_j\big)^{\frac{n}{n-2}} = O\Big(\varepsilon_{ij}^{\frac{n}{n-2}} \log(\varepsilon_{ij}^{-1}\Big).$$

Lemma A. 6

$$\Big\langle P\delta, \lambda\frac{\partial P\delta}{\partial\lambda}\Big\rangle = \frac{n-2}{2} c_1 \frac{H(a,a)}{\lambda^{n-2}} + O\Big(\frac{\log(\lambda d)}{(\lambda d)^n}\Big).$$

Lemma A. 7

$$\int_\Omega P\delta^{\frac{n+2}{n-2}} \lambda\frac{\partial P\delta}{\partial\lambda} = 2\Big\langle P\delta, \lambda\frac{\partial P\delta}{\partial\lambda}\Big\rangle + O\Big(\frac{\log(\lambda d)}{(\lambda d)^n}\Big).$$

Lemma A. 8 *For $i \neq j$*

$$\Big\langle P\delta_j, \lambda\frac{\partial P\delta_i}{\partial\lambda_i}\Big\rangle = c_1 \Big(\lambda_i \frac{\partial\varepsilon_{ij}}{\partial\lambda_i} + \frac{n-2}{2}\frac{H(a_i,a_j)}{(\lambda_i\lambda_j)^{(n-2)/2}}\Big) + O\Big(\varepsilon_{ij}^{\frac{n}{n-2}} \log(\varepsilon_{ij}^{-1}) + \sum_{k=i,j} \frac{\log(\lambda_k d_k)}{(\lambda_k d_k)^n}\Big).$$

Lemma A. 9 *For $i \neq j$*

$$\int_\Omega P\delta_j^{\frac{n+2}{n-2}} \lambda_i \frac{\partial P\delta_i}{\partial\lambda_i} = \Big\langle P\delta_j, \lambda\frac{\partial P\delta_i}{\partial\lambda_i}\Big\rangle + O\Big(\varepsilon_{ij}^{\frac{n}{n-2}} \log(\varepsilon_{ij}^{-1}) + \sum_{k=i,j} \frac{\log(\lambda_k d_k)}{(\lambda_k d_k)^n}\Big).$$

Lemma A. 10 *For $i \neq j$*

$$\frac{n+2}{n-2}\int_\Omega P\delta_j\Big(P\delta_i^{\frac{4}{n-2}}\lambda_i\frac{\partial P\delta_i}{\partial\lambda_i}\Big) = \Big\langle P\delta_j, \lambda\frac{\partial P\delta_i}{\partial\lambda_i}\Big\rangle + O\Big(\varepsilon_{ij}^{\frac{n}{n-2}}\log(\varepsilon_{ij}^{-1}) + \sum_{k=i,j}\frac{\log(\lambda_k d_k)}{(\lambda_k d_k)^n}\Big).$$

Lemma A. 11

$$\Big\langle P\delta, \frac{1}{\lambda}\frac{\partial P\delta}{\partial a}\Big\rangle = \frac{-1}{2}\frac{c_1}{\lambda^{n-1}}\frac{\partial H}{\partial a}(a,a) + O\Big(\frac{1}{(\lambda d)^n}\Big).$$

Lemma A. 12

$$\int_\Omega P\delta^{\frac{n+2}{n-2}}\frac{1}{\lambda}\frac{\partial P\delta}{\partial a} = 2\Big\langle P\delta, \frac{1}{\lambda}\frac{\partial P\delta}{\partial a}\Big\rangle + O\Big(\frac{\log(\lambda d)}{(\lambda d)^n}\Big).$$

Lemma A. 13 *For $i \neq j$*

$$\Big\langle P\delta_j, \frac{1}{\lambda}\frac{\partial P\delta_i}{\partial a_i}\Big\rangle = -\frac{c_1}{(\lambda_i\lambda_j)^{(n-2)/2}}\frac{1}{\lambda_i}\frac{\partial H}{\partial a_i}(a_i,a_j) + c_1\frac{1}{\lambda_i}\frac{\partial\varepsilon_{ij}}{\partial a_i} + O\Big(\sum_{k=i,j}\frac{1}{(\lambda_k d_k)^n} + \varepsilon_{ij}^{\frac{n+1}{n-2}}\lambda_j|a_i-a_j|\Big).$$

Lemma A. 14

$$\int_\Omega P\delta_j^{\frac{n+2}{n-2}}\frac{1}{\lambda_i}\frac{\partial P\delta_i}{\partial a_i} = \Big\langle P\delta_j, \frac{1}{\lambda_i}\frac{\partial P\delta_i}{\partial a_i}\Big\rangle + O\Big(\sum_{k=i,j}\frac{1}{(\lambda_k d_k)^n} + \varepsilon_{ij}^{\frac{n}{n-2}}\log(\varepsilon_{ij}^{-1})\Big).$$

Lemma A. 15 *For $n \geq 5$, we have*

$$\int_\Omega P\delta_i^2 = \frac{c_2}{\lambda_i^2} + O\Big(\frac{1}{(\lambda_i d_i)^{n-2}}\Big),$$

where c_2 is defined in Proposition 2.3.

Lemma A. 16 *For $n \geq 5$, we have*

$$\int_\Omega P\delta_i\lambda_i\frac{\partial P\delta_i}{\partial\lambda_i} = -\frac{c_2}{\lambda_i^2} + O\Big(\frac{1}{(\lambda_i d_i)^{n-2}}\Big).$$

Lemma A. 17

$$\int_\Omega P\delta_i\frac{1}{\lambda_i}\frac{\partial P\delta_i}{\partial a_i} = O\Big(\frac{1}{(\lambda_i d_i)^{n-1}}\Big).$$

Lemma A. 18 *For $i \neq j$*

$$\int_\Omega \delta_i\delta_j = O(\varepsilon_{ij}).$$

Lemma A. 19 *For $v \in E$, we have*

$$\int_\Omega P\delta_i^{\frac{4}{n-2}}\lambda_i\frac{\partial P\delta_i}{\partial\lambda_i}v = \|v\|O\Big((\text{ if } n\leq 5)\frac{1}{(\lambda_i d_i)^{n-2}} + (\text{ if } n=6)\frac{\log(\lambda_i d_i)}{(\lambda_i d_i)^4} + \frac{1}{(\lambda_i d_i)^{\frac{n+2}{2}}}\Big).$$

Lemma A. 20 *For $v \in E$, we have*

$$\int_\Omega P\delta_i^{\frac{4}{n-2}}\frac{1}{\lambda_i}\frac{\partial P\delta_i}{\partial a_i}v = \|v\|O\Big((\text{ if } n\leq 5)\frac{1}{(\lambda_i d_i)^{n-2}} + (\text{ if } n=6)\frac{\log(\lambda_i d_i)}{(\lambda_i d_i)^4} + \frac{1}{(\lambda_i d_i)^{\frac{n+2}{2}}}\Big).$$