Turkish Journal of Mathematics

http://journals.tubitak.gov.tr/math/

Research Article

Turk J Math
(2016) 40: 540 – 552
© TÜBİTAK
doi:10.3906/mat-1504-47

Anti-invariant Riemannian submersions from Kenmotsu manifolds onto Riemannian manifolds

Ayşe BERİ, İrem KÜPELİ ERKEN*, Cengizhan MURATHAN
Department of Mathematics, Faculty of Arts and Science, Uludağ University, Bursa, Turkey

Received: 16.04.2015 • **Accepted/Published Online:** 04.09.2015 • **Final Version:** 08.04.2016

Abstract: The purpose of this paper is to study anti-invariant Riemannian submersions from Kenmotsu manifolds onto Riemannian manifolds. Several fundamental results in this respect are proved. The integrability of the distributions and the geometry of foliations are investigated. We proved the nonexistence of (anti-invariant) Riemannian submersions from Kenmotsu manifolds onto Riemannian manifolds such that the characteristic vector field ξ is a vertical vector field. We gave a method to get horizontally conformal submersion examples from warped product manifolds onto Riemannian manifolds. Furthermore, we presented an example of anti-invariant Riemannian submersions in the case where the characteristic vector field ξ is a horizontal vector field and an anti-invariant horizontally conformal submersion such that ξ is a vertical vector field.

Key words: Riemannian submersion, conformal submersion,Warped product, Kenmotsu manifold, Anti-invariant Riemannian submersion

1. Introduction

Riemannian submersions between Riemannian manifolds were studied by O'Neill [16] and Gray [9]. Riemannian submersions have several applications in mathematical physics. Indeed, Riemannian submersions have their applications in the Yang–Mills theory [4, 27], Kaluza–Klein theory [5, 10], supergravity and superstring theories [11, 28], etc. Later such submersions were considered between manifolds with differentiable structures; see [8]. Furthermore, we have the following submersions: semi-Riemannian submersion and Lorentzian submersion [8], Riemannian submersion [9], slant submersion [7, 23], almost Hermitian submersion [26], contact-complex submersion [13], quaternionic submersion [12], almost h-slant submersion and h-slant submersion [19], semi-invariant submersion [25], h-semi-invariant submersion [20], etc.

Compared with the huge literature on Riemannian submersions, it seems that there are necessary new studies in anti-invariant Riemannian submersions; an interesting paper connecting these fields is [22]. Şahin [22] introduced anti-invariant Riemannian submersions from almost Hermitian manifolds onto Riemannian manifolds. Later, he suggested to investigate anti-invariant Riemannian submersions from almost contact metric manifolds onto Riemannian manifolds [24]. The present work is another step in this direction, more precisely from the point of view of anti-invariant Riemannian submersions from Kenmotsu manifolds. Our work is structured as follows: Section 2 is focused on basic facts for Riemannian submersions and Kenmotsu manifolds. The third section is concerned with definition of anti-invariant Riemannian submersions from

*Correspondence: iremkupeli@uludag.edu.tr
2010 *AMS Mathematics Subject Classification:* Primary 53C25, 53C43, 53C55; Secondary 53D15.

Kenmotsu manifolds onto Riemannian manifolds. We investigate the integrability of the distributions and the geometry of foliations. We proved the nonexistence of (anti-invariant) Riemannian submersions from Kenmotsu manifolds onto Riemannian manifolds such that the characteristic vector field ξ is a vertical vector field. The last section is devoted to an example of anti-invariant Riemannian submersions in the case where the characteristic vector field ξ is a horizontal vector field and an anti-invariant horizontally conformal submersion such that ξ is a vertical vector field.

2. Preliminaries

In this section we recall several notions and results that will be needed throughout the paper.

Let M be a $(2m+1)$-dimensional connected differentiable manifold [3] endowed with an almost contact metric structure (ϕ, ξ, η, g) consisting of a $(1,1)$-tensor field ϕ, a vector field ξ, a 1-form η, and a compatible Riemannian metric g satisfying

$$\phi^2 = -I + \eta \otimes \xi, \quad \phi\xi = 0, \ \eta \circ \phi = 0, \quad \eta(\xi) = 1, \tag{2.1}$$

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \tag{2.2}$$

$$g(\phi X, Y) + g(X, \phi Y) = 0, \ \eta(X) = g(X, \xi), \tag{2.3}$$

for all vector fields $X, Y \in \chi(M)$.

An almost contact metric manifold M is said to be a Kenmotsu manifold [14] if it satisfies

$$(\nabla_X \phi)Y = g(\phi X, Y)\xi - \eta(Y)\phi X, \tag{2.4}$$

where ∇ is the Levi-Civita connection of the Riemannian metric g. From the above equation it follows that

$$\nabla_X \xi = X - \eta(X)\xi, \tag{2.5}$$

$$(\nabla_X \eta)Y = g(X, Y) - \eta(X)\eta(Y). \tag{2.6}$$

A Kenmotsu manifold is normal (that is, the Nijenhuis tensor of ϕ equals $-2d\eta \otimes \xi$) but not Sasakian. Moreover, it is also not compact since from equation (2.5) we get $\operatorname{div}\xi = 2m$. Finally, the fundamental 2-form Φ is defined by $\Phi(X, Y) = g(X, \phi Y)$. In [14], Kenmotsu showed:

(a) that locally a Kenmotsu manifold is a warped product $I \times_f N$ of an interval I and a Kaehler manifold N with warping function $f(t) = se^t$, where s is a nonzero constant.

(b) that a Kenmotsu manifold of constant ϕ-sectional curvature is a space of constant curvature -1 and so it is a locally hyperbolic space.

Now we will give a well-known example, which is a Kenmotsu manifold on $\mathbb{R}^5$ by using (a).

Example 1 *We consider $M = \{(x_1, x_2, y_1, y_2, z) \in \mathbb{R}^5 : \ z \neq 0\}$. Let η be a 1-form defined by*

$$\eta = dz.$$

The characteristic vector field ξ is given by $\frac{\partial}{\partial z}$ and its Riemannian metric g and tensor field ϕ are given by

$$g = e^{2z}\sum_{i=1}^{2}((dx_i)^2 + (dy_i)^2) + (dz)^2, \quad \phi = \begin{pmatrix} 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

This gives a Kenmotsu structure on M. *The vector fields* $E_1 = e^{-z}\frac{\partial}{\partial y_1}$, $E_2 = e^{-z}\frac{\partial}{\partial y_2}$, $E_3 = e^{-z}\frac{\partial}{\partial x_1}$, $E_4 = e^{-z}\frac{\partial}{\partial x_2}$, *and* $E_5 = \xi$ *form a* ϕ*-basis for the Kenmotsu structure. On the other hand, it can be shown that* $M(\phi, \xi, \eta, g)$ *is a Kenmotsu manifold.*

Let (M, g_M) be an m-dimensional Riemannian manifold and let (N, g_N) be an n-dimensional Riemannian manifold. A Riemannian submersion is a smooth map $F : M \to N$ that is onto and satisfying the following axioms:

$S1$. F has maximal rank.

$S2$. The differential F_* preserves the lengths of horizontal vectors.

The fundamental tensors of a submersion were defined by O'Neill [16, 17]. They are $(1,2)$-tensors on M, given by the following formulas:

$$\mathcal{T}(E,F) = \mathcal{T}_E F = \mathcal{H}\nabla_{\mathcal{V}E}\mathcal{V}F + \mathcal{V}\nabla_{\mathcal{V}E}\mathcal{H}F, \tag{2.7}$$

$$\mathcal{A}(E,F) = \mathcal{A}_E F = \mathcal{V}\nabla_{\mathcal{H}E}\mathcal{H}F + \mathcal{H}\nabla_{\mathcal{H}E}\mathcal{V}F, \tag{2.8}$$

for any vector fields E and F on M. Here ∇ denotes the Levi-Civita connection of g_M. These tensors are called integrability tensors for the Riemannian submersions. Note that we denote the projection morphism on the distributions $\ker F_*$ and $(\ker F_*)^\perp$ by $\mathcal{V}$ and $\mathcal{H}$, respectively.

If the second condition $S2$. can be changed as F_* restricted to horizontal distribution of F is a conformal mapping, we get the *horizontally conformal submersion* definition [18]. In this case the second condition can be written in the following way:

$$g_M(X,Y) = e^{2\lambda(p)} g_N(F_*X, F_*Y), \forall p \in M, \forall X, Y \in \Gamma((\ker F_*)^\perp), \exists \lambda \in C^\infty(M). \tag{2.9}$$

The warped product $M = M_1 \times_f M_2$ of two Riemannian manifolds (M_1, g_1) and (M_2, g_2), is the Cartesian product manifold $M_1 \times M_2$, endowed with the warped product metric $g = g_1 + f g_2$, where f is a positive function on M_1. More precisely, the Riemannian metric g on $M_1 \times_f M_2$ is defined for pairs of vector fields X, Y on $M_1 \times M_2$ by

$$g(X,Y) = g_1(\pi_{1*}(X), \pi_{1*}(Y)) + f^2(\pi_1(.)) g_2(\pi_{2*}(X), \pi_{2*}(Y)),$$

where $\pi_1 : M_1 \times M_2 \to M_1; (p,q) \to p$ and $\pi_2 : M_1 \times M_2 \to M_2; (p,q) \to q$ are the canonical projections. We recall that these projections are submersions. If f is not a constant function of value 1, one can prove that the second projection is a conformal submersion whose vertical and horizontal spaces at any point (p,q) are respectively identified with T_pM_1, T_qM_2.

Let $\mathcal{L}(M_1)$ and $\mathcal{L}(M_2)$ be the set of lifts of vector fields on M_1 and M_2 to $M_1 \times_f M_2$, respectively. We use the same notation for a vector field and for its lift. We denote the Levi-Civita connection of the warped product metric tensor of g by ∇.

Proposition 1 *[17]* $M = M_1 \times_f M_2$ *be a warped Riemannian product manifold with the warping function* f *on* M_1. *If* $X_1, Y_1 \in \mathcal{L}(M_1)$ *and* $X_2, Y_2 \in \mathcal{L}(M_2)$, *then*

(i) $\nabla_{X_1} Y_1$ *is the lift of* $\nabla^1_{X_1} Y_1$,

(ii) $\nabla_{X_1} X_2 = \nabla_{X_2} X_1 = (X_1 f / f) X_2$,

(iii) nor $\nabla_{X_2} Y_2 = -(g(X_2, Y_2)/f) grad f$,

(iv) tan $\nabla_{X_2}Y_2 \in \mathcal{L}(M_2)$ *is the lift of* $\nabla^2_{X_2}Y_2$,

where ∇^1 *and* ∇^2 *are Riemannian connections on* M_1 *and* M_2*, respectively.*

Now we will introduce the following proposition ([6], pp. 86) for Subsection 3.2.

Proposition 2 *If* ϕ *is a submersion of* N *onto* N_1 *and if* $\psi : N_1 \to N_2$ *is a differentiable function, then the rank of* $\psi \circ \phi$ *at* p *is equal to the rank of* ψ *at* $\phi(p)$.

The following lemmas are well known from [16, 17]:

Lemma 1 *For any* U, W *vertical and* X, Y *horizontal vector fields, the tensor fields* $\mathcal{T}$ *and* $\mathcal{A}$ *satisfy*

$$i)\mathcal{T}_U W = \mathcal{T}_W U, \tag{2.10}$$

$$ii)\mathcal{A}_X Y = -\mathcal{A}_Y X = \frac{1}{2}\mathcal{V}[X, Y]. \tag{2.11}$$

It is easy to see that $\mathcal{T}$ is vertical, $\mathcal{T}_E = \mathcal{T}_{\mathcal{V}E}$, and $\mathcal{A}$ is horizontal, $\mathcal{A} = \mathcal{A}_{\mathcal{H}E}$.

For each $q \in N$, $F^{-1}(q)$ is an $(m-n)$-dimensional submanifold of M. The submanifolds $F^{-1}(q)$ are called fibers. A vector field on M is called vertical if it is always tangent to fibers. A vector field on M is called horizontal if it is always orthogonal to fibers. A vector field X on M is called basic if X is horizontal and F-related to a vector field X_* on N, i.e. $F_*X_p = X_{*F(p)}$ for all $p \in M$.

Lemma 2 *Let* $F : (M, g_M) \to (N, g_N)$ *be a Riemannian submersion. If* X, Y *are basic vector fields on* M*, then*

i) $g_M(X, Y) = g_N(X_*, Y_*) \circ F$,

ii) $\mathcal{H}[X, Y]$ is basic and F-related to $[X_*, Y_*]$,

iii) $\mathcal{H}(\nabla_X Y)$ is a basic vector field corresponding to $\overset{*}{\nabla}_{X_*} Y_*$ where ∇^* is the connection on N.

iv) for any vertical vector field V, $[X, V]$ is vertical.

Moreover, if X is basic and U is vertical, then $\mathcal{H}(\nabla_U X) = \mathcal{H}(\nabla_X U) = \mathcal{A}_X U$. On the other hand, from (2.7) and (2.8) we have

$$\nabla_V W = \mathcal{T}_V W + \hat{\nabla}_V W, \tag{2.12}$$

$$\nabla_V X = \mathcal{H}\nabla_V X + \mathcal{T}_V X, \tag{2.13}$$

$$\nabla_X V = \mathcal{A}_X V + \mathcal{V}\nabla_X V, \tag{2.14}$$

$$\nabla_X Y = \mathcal{H}\nabla_X Y + \mathcal{A}_X Y, \tag{2.15}$$

for $X, Y \in \Gamma((\ker F_*)^\perp)$ and $V, W \in \Gamma(\ker F_*)$, where $\hat{\nabla}_V W = \mathcal{V}\nabla_V W$.

Note that $\mathcal{T}$ acts on the fibers as the second fundamental form of the submersion and restricted to vertical vector fields and it can be easily seen that $\mathcal{T} = 0$ is equivalent to the condition that the fibers are totally geodesic. A Riemannian submersion is called a Riemannian submersion with totally geodesic fibers if $\mathcal{T}$ vanishes identically. Let $U_1, ..., U_{m-n}$ be an orthonormal frame of $\Gamma(\ker F_*)$. Then the horizontal vector field $H = \frac{1}{m-n}\sum\limits_{j=1}^{m-n}\mathcal{T}_{U_j}U_j$ is called the mean curvature vector field of the fiber. If $H = 0$, then the Riemannian submersion is said to be minimal. A Riemannian submersion is called a Riemannian submersion with totally umbilical fibers if

$$\mathcal{T}_U W = g_M(U,W)H, \tag{2.16}$$

for $U, W \in \Gamma(\ker F_*)$. For any $E \in \Gamma(TM), \mathcal{T}_E$ and $\mathcal{A}_E$ are skew-symmetric operators on $(\Gamma(TM), g_M)$ reversing the horizontal and the vertical distributions. By Lemma 1, horizontal distribution $\mathcal{H}$ is integrable if and only if $\mathcal{A} = 0$. For any $D, E, G \in \Gamma(TM)$, one has

$$g(\mathcal{T}_D E, G) + g(\mathcal{T}_D G, E) = 0 \tag{2.17}$$

and

$$g(\mathcal{A}_D E, G) + g(\mathcal{A}_D G, E) = 0. \tag{2.18}$$

Finally, we recall the notion of harmonic maps between Riemannian manifolds. Let (M, g_M) and (N, g_N) be Riemannian manifolds and suppose that $\varphi : M \to N$ is a smooth map between them. Then the differential φ_* of φ can be viewed as a section of the bundle $Hom(TM, \varphi^{-1}TN) \to M$, where $\varphi^{-1}TN$ is the pullback bundle that has fibers $(\varphi^{-1}TN)_p = T_{\varphi(p)}N$, $p \in M$. $Hom(TM, \varphi^{-1}TN)$ has a connection ∇ induced from the Levi-Civita connection ∇^M and the pullback connection. Then the second fundamental form of φ is given by

$$(\nabla\varphi_*)(X,Y) = \nabla^{\varphi}_X \varphi_*(Y) - \varphi_*(\nabla^M_X Y), \tag{2.19}$$

for $X, Y \in \Gamma(TM)$, where ∇^{φ} is the pullback connection. It is known that the second fundamental form is symmetric. If φ is a Riemannian submersion, it can be easily proved that

$$(\nabla\varphi_*)(X,Y) = 0, \tag{2.20}$$

for $X, Y \in \Gamma((\ker F_*)^{\perp})$. A smooth map $\varphi : (M, g_M) \to (N, g_N)$ is said to be harmonic if $trace(\nabla\varphi_*) = 0$. On the other hand, the tension field of φ is the section $\tau(\varphi)$ of $\Gamma(\varphi^{-1}TN)$ defined by

$$\tau(\varphi) = div\varphi_* = \sum_{i=1}^{m} (\nabla\varphi_*)(e_i, e_i), \tag{2.21}$$

where $\{e_1, ..., e_m\}$ is the orthonormal frame on M. Then it follows that φ is harmonic if and only if $\tau(\varphi) = 0$, (for details, see [2]).

Let g be a Riemannian metric tensor on the manifold $M = M_1 \times M_2$ and assume that the canonical foliations D_{M_1} and D_{M_2} intersect perpendicularly everywhere. Then g is the metric tensor of a usual product of Riemannian manifolds if and only if D_{M_1} and D_{M_2} are totally geodesic foliations [21].

3. Anti-invariant Riemannian submersions

In this section, we are going to define anti-invariant Riemannian submersions from Kenmotsu manifolds and investigate the geometry of such submersions.

Definition 1 *Let $M(\phi, \xi, \eta, g_M)$ be a Kenmotsu manifold and (N, g_N) a Riemannian manifold. A Riemannian submersion $F : M(\phi, \xi, \eta, g_M) \to (N, g_N)$ is called an anti-invariant Riemannian submersion if* $\ker F_*$ *is anti-invariant with respect to ϕ, i.e. $\phi(\ker F_*) \subseteq (\ker F_*)^{\perp}$.*

Let $F : M(\phi, \xi, \eta, g_M) \to (N, g_N)$ be an anti-invariant Riemannian submersion from a Kenmotsu manifold $M(\phi, \xi, \eta, g_M)$ to a Riemannian manifold (N, g_N). First of all, from Definition 1, we have $\phi(\ker F_*)^\perp \cap (\ker F_*) \neq \{0\}$. We denote the complementary orthogonal distribution to $\phi(\ker F_*)$ in $(\ker F_*)^\perp$ by μ. Then we have

$$(\ker F_*)^\perp = \phi \ker F_* \oplus \mu. \tag{3.1}$$

3.1. Anti-invariant Riemannian submersions admitting a horizontal structure vector field

In this subsection, we will study anti-invariant Riemannian submersions from a Kenmotsu manifold onto a Riemannian manifold such that the characteristic vector field ξ is a horizontal vector field. Using (3.1), we have $\mu = \phi\mu \oplus \{\xi\}$. For any horizontal vector field X we put

$$\phi X = BX + CX, \tag{3.2}$$

where $BX \in \Gamma(\ker F_*)$ and $CX \in \Gamma(\mu)$.

Now we suppose that V is a vertical and X is a horizontal vector field. Using the above relation and (2.2), we obtain

$$g_M(CX, \phi V) = 0. \tag{3.3}$$

By virtue of (2.2) and (3.2), we get

$$\begin{aligned} g_M(CX, \phi U) &= g_M(\phi X - BX, \phi U) \\ &= g_M(X, U) - \eta(X)\eta(U) - g_M(BX, \phi U). \end{aligned} \tag{3.4}$$

Since $\phi U \in \Gamma((\ker F_*)^\perp)$ and $\xi \in \Gamma(\ker F_*)^\perp$, (3.4) implies (3.3). From this last relation we have $g_N(F_*\phi V, F_*CX) = 0$, which implies that

$$TN = F_*(\phi(\ker F_*)) \oplus F_*(\mu). \tag{3.5}$$

The proof of the following result is the same as Theorem 10 of [15]; therefore, we omit its proof.

Theorem 1 *Let $M(\phi, \xi, \eta, g_M)$ be a Kenmotsu manifold of dimension $2m+1$ and (N, g_N) a Riemannian manifold of dimension n. Let $F : M(\phi, \xi, \eta, g_M) \to (N, g_N)$ be an anti-invariant Riemannian submersion such that $(\ker F_*)^\perp = \phi \ker F_* \oplus \{\xi\}$. Then $m+1 = n$.*

Remark 1 *We note that Example 2 satisfies Theorem 1.*

Lemma 3 *Let F be an anti-invariant Riemannian submersion from a Kenmotsu manifold $M(\phi, \xi, \eta, g_M)$ to a Riemannian manifold (N, g_N). Then we have*

$$\mathcal{A}_X \xi = 0, \tag{3.6}$$

$$\mathcal{T}_U \xi = U, \tag{3.7}$$

$$g_M(\nabla_Y CX, \phi U) = -g_M(CX, \phi \mathcal{A}_Y U), \tag{3.8}$$

for $X, Y \in \Gamma((\ker F_*)^\perp)$ and $U \in \Gamma(\ker F_*)$.

Proof Using (2.15) and (2.5), we have (3.6). Using (2.13) and (2.5), we obtain (3.7). Now using (3.3), we get

$$g_M(\nabla_Y CX, \phi U) = -g_M(CX, \nabla_Y \phi U),$$

for $X, Y \in \Gamma((\ker F_*)^\perp)$ and $U \in \Gamma(\ker F_*)$. Then (2.14) and (2.4) imply that

$$g_M(\nabla_Y CX, \phi U) = -g_M(CX, \phi \mathcal{A}_Y U) - g_M(CX, \phi(\mathcal{V}\nabla_Y U)).$$

Since $\phi(\mathcal{V}\nabla_Y U) \in \Gamma((\ker F_*)^\perp)$, we obtain (3.8). □

We now study the integrability of the distribution $(\ker F_*)^\perp$ and then we investigate the geometry of leaves of $\ker F_*$ and $(\ker F_*)^\perp$.

Theorem 2 *Let F be an anti-invariant Riemannian submersion from a Kenmotsu manifold $M(\phi, \xi, \eta, g_M)$ to a Riemannian manifold (N, g_N). Then the following assertions are equivalent to each other:*

i) $(\ker F_*)^\perp$ *is integrable,*

ii) $g_N((\nabla F_*)(Y, BX), F_*\phi V) = g_N((\nabla F_*)(X, BX), F_*\phi V)$
$\qquad + g_M(CY, \phi \mathcal{A}_X V) - g_M(CX, \phi \mathcal{A}_Y V),$

iii) $g_M(\mathcal{A}_X BY - \mathcal{A}_Y BX, \phi V) = g_M(CY, \phi \mathcal{A}_X V) - g_M(CX, \phi \mathcal{A}_Y V)$

for $X, Y \in \Gamma((\ker F_*)^\perp)$ *and* $V \in \Gamma(\ker F_*)$.

Proof From (2.2) and (2.4), one easily obtains

$$\begin{aligned} g_M([X, Y], V) &= g_M(\nabla_X Y, V) - g_M(\nabla_Y X, V) \\ &= g_M(\nabla_X \phi Y, \phi V) - g_M(\nabla_Y \phi X, \phi V). \end{aligned}$$

for $X, Y \in \Gamma((\ker F_*)^\perp)$ and $V \in \Gamma(\ker F_*)$. Then from (3.2), we have

$$\begin{aligned} g_M([X, Y], V) &= g_M(\nabla_X BY, \phi V) + g_M(\nabla_X CY, \phi V) - g_M(\nabla_Y BX, \phi V) \\ &\quad - g_M(\nabla_Y CX, \phi V). \end{aligned}$$

Taking into account that F is a Riemannian submersion and using (2.8), (2.14), and (3.8), we obtain

$$\begin{aligned} g_M([X, Y], V) &= g_N(F_*\nabla_X BY, F_*\phi V) - g_M(CY, \phi \mathcal{A}_X V) \\ &\quad - g_N(F_*\nabla_Y BX, F_*\phi V) + g_M(CX, \phi \mathcal{A}_Y V). \end{aligned}$$

Thus, from (2.19) we have

$$\begin{aligned} g_M([X, Y], V) &= g_N(-(\nabla F_*)(X, BY) + (\nabla F_*)(Y, BX), F_*\phi V) \\ &\quad + g_M(CX, \phi \mathcal{A}_Y V) - g_M(CY, \phi \mathcal{A}_X V) \end{aligned}$$

which proves $(i) \Leftrightarrow (ii)$. On the other hand, using (2.19), we get

$$(\nabla F_*)(Y, BX) - (\nabla F_*)(X, BY) = -F_*(\nabla_Y BX - \nabla_X BY).$$

Then (2.14) implies that

$$(\nabla F_*)(Y, BX) - (\nabla F_*)(X, BY) = -F_*(\mathcal{A}_Y BX - \mathcal{A}_X BY).$$

From (2.8) it follows that $\mathcal{A}_Y BX - \mathcal{A}_X BY \in \Gamma((\ker F_*)^\perp)$; this shows that $(ii) \Leftrightarrow (iii)$. □

Remark 2 *We assume that* $(\ker F_*)^\perp = \phi \ker F_* \oplus \{\xi\}$. *Using (3.2) one can prove that* $CX = 0$ *for* $X \in \Gamma((\ker F_*)^\perp)$.

Hence we can give the following corollary.

Corollary 1 *Let* $M(\phi, \xi, \eta, g_M)$ *be a Kenmotsu manifold of dimension* $2m+1$ *and* (N, g_N) *a Riemannian manifold of dimension* n. *Let* $F : M(\phi, \xi, \eta, g_M) \to (N, g_N)$ *be an anti-invariant Riemannian submersion such that* $(\ker F_*)^\perp = \phi \ker F_* \oplus \{\xi\}$. *Then the following assertions are equivalent to each other:*

i) $(\ker F_*)^\perp$ *is integrable,*

ii) $(\nabla F_*)(X, \phi Y) = (\nabla F_*)(\phi X, Y), \;\; X, Y \in \Gamma((\ker F_*)^\perp)$,

iii) $\mathcal{A}_X \phi Y = \mathcal{A}_Y \phi X$.

Theorem 3 *Let* $M(\phi, \xi, \eta, g_M)$ *be a Kenmotsu manifold of dimension* $2m+1$ *and* (N, g_N) *a Riemannian manifold of dimension* n. *Let* $F : M(\phi, \xi, \eta, g_M) \to (N, g_N)$ *be an anti-invariant Riemannian submersion. Then the following assertions are equivalent to each other:*

i) $(\ker F_*)^\perp$ *defines a totally geodesic foliation on* M,

ii) $g_M(\mathcal{A}_X BY, \phi V) = g_M(CY, \phi \mathcal{A}_X V)$,

iii) $g_N((\nabla F_*)(X, \phi Y), F_* \phi V) = -g_M(CY, \phi \mathcal{A}_X V)$,

for $X, Y \in \Gamma((\ker F_*)^\perp)$ *and* $V \in \Gamma(\ker F_*)$.

Proof From (2.2) and (2.4), we obtain

$$g_M(\nabla_X Y, V) = g_M(\nabla_X \phi Y, \phi V),$$

for $X, Y \in \Gamma((\ker F_*)^\perp)$ and $V \in \Gamma(\ker F_*)$. By virtue of (3.2), we get

$$g_M(\nabla_X Y, V) = g_M(\nabla_X BY + \nabla_X CY, \phi V).$$

Using (2.14) and (3.8), we have

$$g_M(\nabla_X Y, V) = g_M(\mathcal{A}_X BY + \mathcal{V}\nabla_X BY, \phi V) - g_M(CY, \phi \mathcal{A}_X V).$$

The last equation shows $(i) \Leftrightarrow (ii)$.

For $X, Y \in \Gamma((\ker F_*)^\perp)$ and $V \in \Gamma(\ker F_*)$,

$$g_M(\mathcal{A}_X BY, \phi V) = g_M(CY, \phi \mathcal{A}_X V) \tag{3.9}$$

Since differential F_* preserves the lengths of horizontal vectors the relation (3.9) forms

$$g_M(CY, \phi \mathcal{A}_X V) = g_N(F_* \mathcal{A}_X BY, F_* \phi V) \tag{3.10}$$

By using (2.14) and (2.19) in (3.10), we obtain

$$g_M(CY, \phi \mathcal{A}_X V) = g_N(-(\nabla F_*)(X, \phi Y), F_* \phi V)$$

which tells that $(ii) \Leftrightarrow (iii)$. □

Corollary 2 *Let $F : M(\phi, \xi, \eta, g_M) \to (N, g_N)$ be an anti-invariant Riemannian submersion such that $(\ker F_*)^{\perp} = \phi \ker F_* \oplus \{\xi\}$, where $M(\phi, \xi, \eta, g_M)$ is a Kenmotsu manifold and (N, g_N) is a Riemannian manifold. Then the following assertions are equivalent to each other:*

i) $(\ker F_*)^{\perp}$ *defines a totally geodesic foliation on* M,

ii) $\mathcal{A}_X \phi Y = 0$,

iii) $(\nabla F_*)(X, \phi Y) = 0$ *for* $X, Y \in \Gamma((\ker F_*)^{\perp})$ *and* $V \in \Gamma(\ker F_*)$.

The following result is a consequence from (2.12) and (3.7).

Theorem 4 *Let F be an anti-invariant Riemannian submersion from a Kenmotsu manifold $M(\phi, \xi, \eta, g_M)$ to a Riemannian manifold (N, g_N). Then $(\ker F_*)$ does not define a totally geodesic foliation on M.*

Using Theorem 4, one can give the following result.

Theorem 5 *Let $F : M(\phi, \xi, \eta, g_M) \to (N, g_N)$ be an anti-invariant Riemannian submersion where $M(\phi, \xi, \eta, g_M)$ is a Kenmotsu manifold and (N, g_N) is a Riemannian manifold. Then F is not a totally geodesic map.*

Remark 3 *Now we suppose that $\{e_1, ..., e_m\}$ is a local orthonormal frame of $\Gamma(\ker F_*)$. From the well-known equation $H = \frac{1}{m} \sum_{i=1}^{m} \mathcal{T}_{e_i} e_i$, (2.12), and (2.17) we have*

$$\begin{aligned} mg(H, \xi) &= g(T_{e_1} e_1, \xi) + g(T_{e_2} e_2, \xi) + \cdots + g(T_{e_m} e_m, \xi) \\ &= -g(T_{e_1} \xi, e_1) - g(T_{e_2} \xi, e_2) - \cdots - g(T_{e_m} \xi, e_m) \\ &= -g(e_1, e_1) - g(e_2, e_2) - \cdots - g(e_m, e_m) \\ &= -m \end{aligned}$$

We get $g(H, \xi) = -1$. Therefore, $\ker F_*$ does not have minimal fibers.

By virtue of Remark 3, we have the following theorem.

Theorem 6 *Let $F : M(\phi, \xi, \eta, g_M) \to (N, g_N)$ be an anti-invariant Riemannian submersion where $M(\phi, \xi, \eta, g_M)$ is a Kenmotsu manifold and (N, g_N) is a Riemannian manifold. Then F is not harmonic.*

3.2. Anti-invariant Riemannian submersions admitting a vertical structure vector field

In this subsection, we will prove that there do not exist (anti-invariant) Riemannian submersions from Kenmotsu manifolds onto Riemannian manifolds such that characteristic vector field ξ is a vertical vector field. Moreover, we will give a method to get horizontally conformal submersion examples from warped product manifolds onto Riemannian manifolds.

It is easy to see that μ is an invariant distribution of $(\ker F_*)^{\perp}$, under the endomorphism ϕ. Thus, for $X \in \Gamma((\ker F_*)^{\perp})$, we have

$$\phi X = BX + CX, \tag{3.11}$$

where $BX \in \Gamma(\ker F_*)$ and $CX \in \Gamma(\mu)$. On the other hand, since $F_*((\ker F_*)^{\perp}) = TN$ and F is a Riemannian submersion, using (3.11) we derive $g_N(F_* \phi V, F_* CX) = 0$, for every $X \in \Gamma((\ker F_*))^{\perp}$ and $V \in \Gamma(\ker F_*)$, which implies that

$$TN = F_*(\phi(\ker F_*)) \oplus F_*(\mu). \tag{3.12}$$

Theorem 7 *Let $(M^{m+1} = I \times_f L^m, g_M = dt^2 + f^2 g_L)$ be a warped product manifold of an interval I and a Riemannian manifold L. If $F : (M^{m+1}, g_M) \to (N^n, g_N)$ is a Riemannian submersion with vertical vector field $\frac{\partial}{\partial t} = \partial_t$ then the warped product manifold is a Riemannian product manifold.*

Proof Let $\sigma = (t, x_1, x_2, ..., x_m)$ be a coordinate system for M at $p \in M$ and $y_1, y_2, ..., y_n$ be a coordinate system for N at $F(p)$. Since ∂_t is a vertical vector field, we have

$$0 = F_*(\partial_t)_p = \sum_{i=1}^{n} \frac{\partial(y_i \circ F)}{\partial_t}(p) \frac{\partial}{\partial y_i} \mid_{F(p)} .$$

Therefore, the component functions $y_i \circ F = f_i$ of F do not contain t parameter. Namely,

$$F : I \times_f L \to N, (t, x) \to F(t, x) = (f_1(x), ..., f_n(x)),$$

where $x = (x_1, x_2, ..., x_m)$ and also $(\ker F_*)^{\perp} \mid_{(t,x)} \subseteq T_{(t,x)}(\{t\} \times L) \cong T_x L$ at point $p = (t, x) \in M$. That is, if $\tilde{X} \in (\ker F_*)^{\perp}$, there is a vector field $X \in \Gamma(TN)$ such that the lift of X to $I \times L$ is the vector field $\tilde{X}$, $\pi_{2*}(\tilde{X}_p) = X_{\pi_2(p)}$ for all $p \in M$. For the sake of simplification we use the same notation for a vector field and for its lift.

Using Proposition 1 (ii), we obtain

$$\nabla_X \partial_t = \frac{f'}{f} X \tag{3.13}$$

for $X \in \Gamma((\ker F_*)^{\perp})$. From (2.14) and (3.13) we have

$$\mathcal{A}_X \partial_t = \frac{f'}{f} X \tag{3.14}$$

for $X \in \Gamma((\ker F_*)^{\perp})$.

By applying (2.11), (2.18), and (3.14), we find

$$g_M(\mathcal{A}_X Y, \partial_t) = -\frac{f'}{f} g_M(X, Y) = -\frac{f'}{f} g_M(Y, X) = g_M(\mathcal{A}_Y X, \partial_t) = -g_M(\mathcal{A}_X Y, \partial_t)$$

for $X, Y \in \Gamma((\ker F_*)^{\perp})$. Thus, we obtain

$$g_M(\mathcal{A}_X Y, \partial_t) = -\frac{f'}{f} g_M(X, Y) = 0. \tag{3.15}$$

It follows from (3.15) that $f' = 0$. Hence warping function f must be constant. Therefore, up to a change of scale, M is a Riemannian product manifold. □

Theorem 8 *Let $M(\phi, \xi, \eta, g_M)$ be a Kenmotsu manifold of dimension $2m+1$ and (N, g_N) is a Riemannian manifold of dimension n. There is no Riemannian submersion $F : M(\phi, \xi, \eta, g_M) \to (N, g_N)$ such that the characteristic vector field ξ is a vertical vector field.*

Proof From [14] we know that locally a Kenmotsu manifold is a warped product $I \times_f L$ of an interval I and a Kaehler manifold L with metric $g_M = dt^2 + f^2 g_L$ and warping function $f(t) = se^t$, where s is a positive constant. Let $\xi = \frac{\partial}{\partial t} = \partial_t$ be a vertical vector field. It follows from Theorem 7 that M is a Riemannian product manifold. Since $f(t) = se^t$ is not constant, M cannot be a Riemannian product manifold. This is a contradiction that completes the proof of theorem. □

Theorem 9 *Let $M = M_1 \times_f M_2$ be a warped product manifold with metric $g = g_1 + f^2 g_2$, $\pi_2 : M_1 \times M_2 \to M_2$ second canonical projection, and (M_3, g_3) Riemannian manifold. If f_1 is a Riemannian submerison from M_2 onto M_3 then $f_2 = f_1 \circ \pi_2 : M \to M_3$ is a horizontally conformal submersion.*

Proof Since f_1 is a Riemannian submersion, rank $f_1 = \dim M_3$. Using Proposition 2, we have rank $f_{2|_{(p,q)}} =$ rank $f_1 |_{f_{1(q)}} = \dim M_3$ for any point $(p, q) \in M$. Consequently f_2 is a submersion. Since π_2 is a natural horizontally conformal submersion for a warped product manifold, we get $\ker \pi_{2*|_{(p,q)}} = T_{(p,q)} M_1 \equiv T_{(p,q)}(M_1 \times \{q\}) \cong T_p M_1$. Therefore, $\ker f_{2*|_{(p,q)}} \cong T_p M_1 \times \ker f_{1*q}$ and $(\ker f_{2*})^{\perp}_{|_{(p,q)}} = \{p\} \times (\ker f_{1*})^{\perp}_{|q} \cong (\ker f_{1*})^{\perp}_{|q}$. Hence,

$$\begin{aligned} g(X, Y) &= f^2(p) g_2(\pi_{2*}(X), \pi_{2*}(Y)) \\ &= f^2(p) g_3(f_{1*}(\pi_{2*}(X)), f_{1*}(\pi_{2*}(Y)) \\ &= f^2(p) g_3(f_{2*}(X), f_{2*}(Y)) \end{aligned}$$

for $X, Y \in \Gamma((\ker f_{2*})^{\perp})$. Thus we get the requested result. □

Remark 4 *Theorem 9 gives a chance to produce horizontally conformal submersion examples.*

4. Examples

We now give some examples for anti-invariant submersion and anti-invariant horizontally conformal submersions from Kenmotsu manifolds.

Example 2 *Let M be a Kenmotsu manifold as in Example 1. Let N be $\mathbb{R} \times_{e^z} \mathbb{R}^2$. The Riemannian metric tensor field g_N is defined by $g_N = e^{2z}(du \otimes du + dv \otimes dv) + dz \otimes dz$ on N.*

Let $F : M \to N$ be a map defined by $F(x_1, x_2, y_1, y_2, t) = (\frac{x_1+y_2}{\sqrt{2}}, \frac{x_2+y_1}{\sqrt{2}}, z)$. Then a simple calculation gives

$$\ker F_* = span\{V_1 = \frac{1}{\sqrt{2}}(E_2 - E_3),\ V_2 = \frac{1}{\sqrt{2}}(E_1 - E_4)\}$$

and

$$(\ker F_*)^{\perp} = span\{H_1 = \frac{1}{\sqrt{2}}(E_1 + E_4),\ H_2 = \frac{1}{\sqrt{2}}(E_2 + E_3),\ H_3 = E_5 = \xi\}.$$

Then it is easy to see that F is a Riemannian submersion. Moreover, $\phi V_1 = -H_1$, $\phi V_2 = -H_2$ imply that $\phi(\ker F_*) \subset (\ker F_*)^{\perp} = \phi(\ker F_*) \oplus \{\xi\}$. Thus F is an anti-invariant Riemannian submersion such that ξ is a horizontal vector field.

Example 3 *Let M be a Kenmotsu manifold as in Example 1 and N be $\mathbb{R}^2$. The Riemannian metric tensor field g_N is defined by $g_N = e^{2z}(du \otimes du + dv \otimes dv)$ on N.*

Let $F : M \to N$ be a map defined by $F(x_1, x_2, y_1, y_2, z) = (\frac{x_1+y_2}{\sqrt{2}}, \frac{x_2+y_1}{\sqrt{2}})$. Then by direct calculations we have

$$\ker F_* = span\{V_1 = \frac{1}{\sqrt{2}}(E_3 - E_2),\ V_2 = \frac{1}{\sqrt{2}}(E_4 - E_1),\ V_3 = E_5 = \xi = \frac{\partial}{\partial z}\}$$

and

$$(\ker F_*)^{\perp} = span\{H_1 = \frac{1}{\sqrt{2}}(E_3 + E_2),\ H_2 = \frac{1}{\sqrt{2}}(E_4 + E_1)\}.$$

Then it is easy to see that F is a horizontally conformal submersion. Moreover, $\phi V_1 = H_2$, $\phi V_2 = H_1$, $\phi V_3 = 0$ imply that $\phi(\ker F_*) = (\ker F_*)^{\perp}$. As a result, F is an anti-invariant horizontally conformal submersion such that ξ is a vertical vector field.

Remark 5 *Recently Akyol and Şahin [1] studied conformal anti-invariant submersions from almost Hermitian manifolds onto Riemannian manifolds. Therefore, it will be worth examining this study area, which is anti-invariant (horizontally) conformal submersion from almost contact metric manifolds onto Riemannian manifolds.*

Acknowledgment

The authors are grateful to the referee for his/her valuable comments and suggestions.

References

[1] Akyol MA, Şahin B. Conformal anti-invariant submersions from almost Hermitian manifolds. Turk J Math2016; 40: 43-47.

[2] Baird P, Wood JC. Harmonic Morphisms Between Riemannian Manifolds. London Mathematical Society Monographs. Oxford, UK: Oxford University Press, The Clarendon Press, 2003.

[3] Blair DE. Riemannian geometry of contact and symplectic manifolds. Progress in Mathematics.Boston, MA, USA: Birkhaüser , 2002.

[4] Bourguignon JP, Lawson HB. Stability and isolation phenomena for Yang-mills fields. Commun Math Phys 1981; 79: 189-230.

[5] Bourguignon JP, Lawson HB. A Mathematician's visit to Kaluza-Klein theory. Rend Semin Mat Torino Fasc Spec 1989; 143-163.

[6] Brickell F, Clark RS. Differentiable Manifolds. London, UK: VN Reinhold Co, 1970.

[7] Chen BY. Geometry of slant submanifolds. Leuven, Belgium: Katholieke Universiteit Leuven, 1990.

[8] Falcitelli M, Ianus S, Pastore AM. Riemannian Submersions and Related Topics. Singapore: World Scientific Publishing Co, 2004.

[9] Gray A. Pseudo-Riemannian almost product manifolds and submersions. J Math Mech 1967; 16: 715-737.

[10] Ianus S, Visinescu M. Kaluza-Klein theory with scalar fields and generalized Hopf manifolds. Class Quantum Gravity 1987; 4: 1317-1325.

[11] Ianus S, Visinescu M. Space-time compactification and Riemannian submersions. The Mathematical Heritage of CF Gauss, 358-371. River Edge, NJ, USA: World Sci Publ, 1991.

[12] Ianus S, Mazzocco R, Vilcu GE. Riemannian submersions from quaternionic manifolds. Acta Appl Math 2008; 104: 83-89.

[13] Ianus S, Ionescu AM, Mazzocco R, Vilcu GE. Riemannian submersions from almost contact metric manifolds. Abh Math Semin Univ Hambg 2011; 81: 101-114.

[14] Kenmotsu K. A class of almost contact Riemannian manifolds. Tohoku Math J 1972; 24: 93-103.

[15] Murathan C, Kupeli Erken I. Anti-invariant Riemannian submersions from cosymplectic manifolds. Filomat 2015; 29: 1429-1444.

[16] O'Neill B. The fundamental equations of submersion. Michigan Math J 1966; 13: 459-469.

[17] O'Neill B. Semi-Riemannian geometry with applications to relativity. New York, NY, USA: Academic Press, 1983.

[18] Ornea L, Romani G. The fundamental equations of conformal submersions. Beiträge Algebra Geom 1993; 34: 233-243.

[19] Park KS. H-slant submersions. Bull Korean Math Soc 2012; 49: 329-338.

[20] Park KS. H-semi-invariant submersions. Taiwanese J Math 2012; 16: 1865-1878.

[21] Ponge R, Reckziegel H. Twisted products in pseudo–Riemannian geometry. Geom Dedicate 1993; 48: 15-25.

[22] Şahin B. Anti-invariant Riemannian submersions from almost Hermitian manifolds. Cent Eur J Math 2010; 8: 437-447.

[23] Şahin B. Slant submersions from almost Hermitian manifolds. Bull Math Soc Sci Math Roumanie Tome 2011; 54: 93-105.

[24] Şahin B. Riemannian submersions from almost Hermitian manifolds. Taiwanese J Math 2012; 17: 629-659.

[25] Şahin B. Semi-invariant submersions from almost Hermitian manifolds. Canad Math Bull 2013; 56: 173-183.

[26] Watson B. Almost Hermitian submersions. J Differential Geom 1976; 11: 147-165.

[27] Watson B. G, $G^{'}$-Riemannian submersions and nonlinear gauge field equations of general relativity. Global analysis -analysis on manifolds, Teubner-Texte Math, Teubner, Leipzig, 1983; 57: 324-349.

[28] Mustafa MT. Applications of harmonic morphisms to gravity. J Math Phys 2000; 41: 6918-6929.