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Uniqueness of $p(f)$ and $P[f]$

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Abstract: Let f be a nonconstant meromorphic function, $a(\not\equiv 0,\infty)$ be a meromorphic function satisfying $T(r,a)=o(T(r,f))$ as $r\to\infty$, and $p(z)$ be a polynomial of degree $n\geq 1$ with $p(0)=0$. Let $P[f]$ be a nonconstant differential polynomial of f. Under certain essential conditions, we prove that $p(f)\equiv P[f]$, when $p(f)$ and $P[f]$ share a with weight $l\geq 0$. Our result generalizes the results due to Zhang and Lü, Banerjee and Majumdar, and Bhoosnurmath and Kabbur and answers a question asked by Zhang and Lü.

Key words: Meromorphic functions, small functions, sharing of values, differential polynomials, Nevanlinna theory

1. Introduction

Let f be a nonconstant meromorphic function in the complex plane $\mathbb{C}$. We assume that the reader is familiar with the standard notions of the Nevanlinna value distribution theory such as $T(r,f)$, $m(r,f)$, $N(r,f)$ (see e.g., [4, 5, 7]). By $S(r,f)$, as usual, we shall mean a quantity that satisfies

$$S(r,f)=\circ(T(r,f)) \text{ as } r\to\infty,$$

possibly outside an exceptional set of finite logarithmic measure. A meromorphic function a is said to be a small function of f, if $T(r,a)=S(r,f)$ as $r\to\infty$.

For a small function a of f, we write $E(a,f)=\{z\in\mathbb{C}: f(z)-a(z)=0\}$, where a zero of $f-a$ is counted according to its multiplicity. Also by $\overline{E}(a,f)$, we denote the zeros of $f-a$, where a zero is counted only once. Let g be another nonconstant meromorphic function. We say that f and g share the function a CM(counting multiplicity) if $E(a,f)=E(a,g)$. Further, if $\overline{E}(a,f)=\overline{E}(a,g)$, then we say that f and g share the function a IM (ignoring multiplicity). Note that a can be a value in $\mathbb{C}\cup\{\infty\}$.

A more general concept is the weighted sharing of meromorphic functions. For a nonnegative integer k, we denote by $E_k(a,f)$ the set of all zeros of $f-a$, where a zero of multiplicity m is counted m times if $m\leq k$ and $k+1$ times if $m>k$. If $E_k(a,f)=E_k(a,g)$, then f and g are said to share the function a with weight k. We write "f and g share (a,k)" to mean that "f and g share the function a with weight k". Since $E_k(a,f)=E_k(a,g)$ implies that $E_p(a,f)=E_p(a,g)$ for any integer $p(0\leq p<k)$, if f and g share (a,k), then f and g share (a,p), $0\leq p<k$. Moreover, we note that f and g share the function a IM (ignoring multilicity) or CM (counting multiplicity) if and only if f and g share $(a,0)$ or (a,∞), respectively. In particular, the small function a can be a value in $\mathbb{C}\cup\{\infty\}$.

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For notational purposes, let f and g share 1 IM, and let z_0 be a zero of $f-1$ with multiplicity p and a zero of $g-1$ with multiplicity q. We denote by $N_E^{1)}(r,1/(f-1))$ the counting function of the zeros of $f-1$ when $p=q=1$. By $\overline{N}_E^{(2}(r,1/(f-1))$ we denote the counting function of the zeros of $f-1$ when $p=q\geq 2$ and by $\overline{N}_L(r,1/(f-1))$ we denote the counting function of the zeros of $f-1$ when $p>q\geq 1$; each point in these counting functions is counted only once; similarly, the terms $N_E^{1)}(r,1/(g-1))$, $\overline{N}_E^{(2}(r,1/(g-1))$, and $\overline{N}_L(r,1/(g-1))$. In addition, we denote by $\overline{N}_{f>k}(r,1/(g-1))$ the reduced counting function of those zeros of $f-1$ and $g-1$ such that $p>q=k$, and similarly the term $\overline{N}_{g>k}(r,1/(f-1))$.

A *differential polynomial* $P[f]$ of a nonconstant meromorphic function f is defined as

$$P[f] := \sum_{i=1}^{m} M_i[f],$$

where $M_i[f] = a_i.\prod_{j=0}^{k}(f^{(j)})^{n_{ij}}$ with $n_{i0},\ n_{i1},\ldots,n_{ik}$ as nonnegative integers and $a_i(\not\equiv 0)$ are meromorphic functions satisfying $T(r,a_i)=o(T(r,f))$ as $r\to\infty$. The numbers $\overline{d}(P)=\max_{1\leq i\leq m}\sum_{j=0}^{k} n_{ij}$ and $\underline{d}(P)=\min_{1\leq i\leq m}\sum_{j=0}^{k} n_{ij}$ are respectively called the degree and lower degree of $P[f]$. If $\overline{d}(P)=\underline{d}(P)=d$ (say), then we say that $P[f]$ is a homogeneous differential polynomial of degree d.

Inspired by a uniqueness result due to Mues and Steinmetz [6]: "*If f is a non-constant entire function sharing two distinct values ignoring multiplicity with f', then $f\equiv f'$* ", the study of the uniqueness of f and $f^{(k)}$, f^n and $(f^m)^{(k)}$, f and $P[f]$ is carried out by numerous authors. For example, Zhang and Lü [8] proved:

Theorem A. *Let k, n be the positive integers, f be a nonconstant meromorphic function, and $a(\not\equiv 0,\infty)$ be a meromorphic function satisfying $T(r,a)=o(T(r,f))$ as $r\to\infty$. If f^n and $f^{(k)}$ share a IM and*

$$(2k+6)\Theta(\infty,f)+4\Theta(0,f)+2\delta_{2+k}(0,f)>2k+12-n,$$

or f^n and $f^{(k)}$ share a CM and

$$(k+3)\Theta(\infty,f)+2\Theta(0,f)+\delta_{2+k}(0,f)>k+6-n,$$

then $f^n\equiv f^{(k)}$.

In the same paper, Zhang and Lü asked the following question:

Question 1: *What will happen if f^n and $(f^{(k)})^m$ share a meromorphic function $a(\not\equiv 0,\infty)$ satisfying $T(r,a)=o(T(r,f))$ as $r\to\infty$?*

Bhoosnurmath and Kabbur [3] proved:

Theorem B. *Let f be a nonconstant meromorphic function and $a(\not\equiv 0,\infty)$ be a meromorphic function satisfying $T(r,a)=o(T(r,f))$ as $r\to\infty$. Let $P[f]$ be a nonconstant differential polynomial of f. If f and $P[f]$ share a IM and*

$$(2Q+6)\Theta(\infty,f)+(2+3\underline{d}(P))\delta(0,f)>2Q+2\underline{d}(P)+\overline{d}(P)+7,$$

or if f and $P[f]$ share a CM and

$$3\Theta(\infty,f)+(\underline{d}(P)+1)\delta(0,f)>4,$$

then $f \equiv P[f]$.

Banerjee and Majumder [2] considered the weighted sharing of f^n and $(f^m)^{(k)}$ and proved the following result:

Theorem C. *Let* f *be a nonconstant meromorphic function,* $k, n, m \in \mathbb{N}$ *and* l *be a nonnegative integer. Suppose* $a(\not\equiv 0, \infty)$ *is a meromorphic function satisfying* $T(r, a) = o(T(r, f))$ *as* $r \to \infty$ *such that* f^n *and* $(f^m)^{(k)}$ *share* (a, l). *If* $l \geq 2$ *and*

$$(k+3)\Theta(\infty, f) + (k+4)\Theta(0, f) > 2k + 7 - n,$$

or $l = 1$ *and*

$$\left(k + \frac{7}{2}\right)\Theta(\infty, f) + \left(k + \frac{9}{2}\right)\Theta(0, f) > 2k + 8 - n,$$

or $l = 0$ *and*

$$(2k+6)\Theta(\infty, f) + (2k+7)\Theta(0, f) > 4k + 13 - n,$$

then $f^n \equiv (f^m)^{(k)}$.

Motivated by such uniqueness investigations, it is natural to consider the problem in a more general setting: *Let* f *be a nonconstant meromorphic function,* $P[f]$ *be a nonconstant differential polynomial of* f, $p(z)$ *be a polynomial of degree* $n \geq 1$, *and* $a(\not\equiv 0, \infty)$ *be a meromorphic function satisfying* $T(r, a) = o(T(r, f))$ *as* $r \to \infty$. *If* $p(f)$ *and* $P[f]$ *share* (a, l), $l \geq 0$, *then is it true that* $p(f) \equiv P[f]$?

Generally this is not true, but under certain essential conditions, we prove the following result:

Theorem 1.1 *Let* f *be a nonconstant meromorphic function,* $a(\not\equiv 0, \infty)$ *be a meromorphic function satisfying* $T(r, a) = o(T(r, f))$ *as* $r \to \infty$, *and* $p(z)$ *be a polynomial of degree* $n \geq 1$ *with* $p(0) = 0$. *Let* $P[f]$ *be a nonconstant differential polynomial of* f. *Suppose* $p(f)$ *and* $P[f]$ *share* (a, l) *with one of the following conditions:*
(i) $l \geq 2$ *and*

$$(Q+3)\Theta(\infty, f) + 2n\Theta(0, p(f)) + \overline{d}(P)\delta(0, f) > Q + 3 + 2\overline{d}(P) - \underline{d}(P) + n, \tag{1.1}$$

(ii) $l = 1$ *and*

$$\left(Q + \frac{7}{2}\right)\Theta(\infty, f) + \frac{5n}{2}\Theta(0, p(f)) + \overline{d}(P)\delta(0, f) > Q + \frac{7}{2} + 2\overline{d}(P) - \underline{d}(P) + \frac{3n}{2}, \tag{1.2}$$

(iii) $l = 0$ *and*

$$(2Q+6)\Theta(\infty, f) + 4n\Theta(0, p(f)) + 2\overline{d}(P)\delta(0, f) > 2Q + 6 + 4\overline{d}(P) - 2\underline{d}(P) + 3n. \tag{1.3}$$

Then $p(f) \equiv P[f]$.

Example 1.2. Consider the function $f(z) = cos\alpha z + 1 - 1/\alpha^4$, where $\alpha \neq 0, \pm 1, \pm i$ and $p(z) = z$. Then $p(f)$ and $P[f] \equiv f^{(iv)}$ share $(1, l)$, $l \geq 0$ and none of the inequalities (1.1), (1.2), and (1.3) is satisfied, and $p(f) \neq P[f]$. Thus the conditions in Theorem 1.1 cannot be removed.

Remark 1.3. Theorem 1.1 generalizes Theorem A, Theorem B, Theorem C (and also generalizes Theorem 1.1 and Theorem 1.2 of [2]) and provides an answer to Question 1 asked by Zhang and Lü [8].

The main tool of our investigations in this paper is Nevanlinna value distribution theory ([4, 5, 7]).

2. Proof of the main result

We shall use the following results in the proof of Theorem 1.1:

Lemma 2.1 *[3] Let f be a nonconstant meromorphic function and $P[f]$ be a differential polynomial of f. Then*

$$m\left(r, \frac{P[f]}{f^{\overline{d}(P)}}\right) \leq (\overline{d}(P) - \underline{d}(P))m\left(r, \frac{1}{f}\right) + S(r, f), \tag{2.1}$$

$$N\left(r, \frac{P[f]}{f^{\overline{d}(P)}}\right) \leq (\overline{d}(P) - \underline{d}(P))N\left(r, \frac{1}{f}\right) + Q\left[\overline{N}(r, f) + \overline{N}\left(r, \frac{1}{f}\right)\right] + S(r, f), \tag{2.2}$$

$$N\left(r, \frac{1}{P[f]}\right) \leq Q\overline{N}(r, f) + (\overline{d}(P) - \underline{d}(P))m\left(r, \frac{1}{f}\right) + N\left(r, \frac{1}{f^{\overline{d}(P)}}\right) + S(r, f), \tag{2.3}$$

where $Q = max_{1\leq i\leq m}\{n_{i0} + n_{i1} + 2n_{i2} + ... + kn_{ik}\}$.

Lemma 2.2 *[1] Let f and g be two nonconstant meromorphic functions.*
(i) If f and g share $(1,0)$, then

$$\overline{N}_L\left(r, \frac{1}{f-1}\right) \leq \overline{N}\left(r, \frac{1}{f}\right) + \overline{N}(r, f) + S(r), \tag{2.4}$$

where $S(r) = o(T(r))$ as $r \to \infty$ with $T(r) = max\{T(r, f); T(r, g)\}$.
(ii) If f and g share $(1,1)$, then

$$\begin{aligned} 2\overline{N}_L\left(r, \frac{1}{f-1}\right) + 2\overline{N}_L\left(r, \frac{1}{g-1}\right) + \overline{N}_E^{(2}\left(r, \frac{1}{f-1}\right) - \overline{N}_{f>2}\left(r, \frac{1}{g-1}\right) \\ \leq N\left(r, \frac{1}{g-1}\right) - \overline{N}\left(r, \frac{1}{g-1}\right). \end{aligned} \tag{2.5}$$

Proof of Theorem 1.1: Let $p(z) = z^n + a_{n-1}z^{n-1} + a_{n-2}z^{n-2} + ... + a_1 z$, where a_1, a_2, ..., a_{n-1} are constants, $F = p(f)/a$ and $G = P[f]/a$. Then

$$F - 1 = \frac{p(f) - a}{a} \text{ and } G - 1 = \frac{P[f] - a}{a}. \tag{2.6}$$

Since $p(f)$ and $P[f]$ share (a, l), it follows that F and G share $(1, l)$ except at the zeros and poles of a. Also note that

$$\overline{N}(r, F) = \overline{N}(r, f) + S(r, f) \text{ and } \overline{N}(r, G) = \overline{N}(r, f) + S(r, f).$$

Define

$$\psi = \left(\frac{F''}{F'} - \frac{2F'}{F-1}\right) - \left(\frac{G''}{G'} - \frac{2G'}{G-1}\right). \tag{2.7}$$

Claim: $\psi \equiv 0$.

Suppose on the contrary that $\psi \not\equiv 0$. Then from (2.7) we have

$$m(r, \psi) = S(r, f).$$

By the second fundamental theorem of Nevanlinna we have

$$T(r,F)+T(r,G)\le 2\overline{N}(r,f)+\overline{N}\left(r,\frac{1}{F}\right)+\overline{N}\left(r,\frac{1}{F-1}\right)+\overline{N}\left(r,\frac{1}{G}\right)$$
$$+\overline{N}\left(r,\frac{1}{G-1}\right)-N_0\left(r,\frac{1}{F'}\right)-N_0\left(r,\frac{1}{G'}\right)+S(r,f), \tag{2.8}$$

where $N_0(r,1/F')$ denotes the counting function of the zeros of F', which are not the zeros of $F(F-1)$, and $N_0(r,1/G')$ denotes the counting function of the zeros of G', which are not the zeros of $G(G-1)$.

Case 1. When $l\ge 1$.

Then from (2.7) we have

$$\begin{aligned}N_E^{1)}\left(r,\frac{1}{F-1}\right)&\le N\left(r,\frac{1}{\psi}\right)+S(r,f)\\&\le T(r,\psi)+S(r,f)\\&=N(r,\psi)+S(r,f)\\&\le \overline{N}(r,F)+\overline{N}_{(2}\left(r,\frac{1}{F}\right)+\overline{N}_{(2}\left(r,\frac{1}{G}\right)+\overline{N}_L\left(r,\frac{1}{F-1}\right)\\&+\overline{N}_L\left(r,\frac{1}{G-1}\right)+N_0\left(r,\frac{1}{F'}\right)+N_0\left(r,\frac{1}{G'}\right)+S(r,f),\end{aligned}$$

and so

$$\begin{aligned}\overline{N}\left(r,\frac{1}{F-1}\right)+\overline{N}\left(r,\frac{1}{G-1}\right)&=N_E^{1)}\left(r,\frac{1}{F-1}\right)+\overline{N}_E^{(2}\left(r,\frac{1}{F-1}\right)+\overline{N}_L\left(r,\frac{1}{F-1}\right)\\&+\overline{N}_L\left(r,\frac{1}{G-1}\right)+\overline{N}\left(r,\frac{1}{G-1}\right)+S(r,f)\\&\le \overline{N}(r,f)+\overline{N}_{(2}\left(r,\frac{1}{F}\right)+\overline{N}_{(2}\left(r,\frac{1}{G}\right)+2\overline{N}_L\left(r,\frac{1}{F-1}\right)\\&+2\overline{N}_L\left(r,\frac{1}{G-1}\right)+\overline{N}_E^{(2}\left(r,\frac{1}{F-1}\right)+\overline{N}\left(r,\frac{1}{G-1}\right)\\&+N_0\left(r,\frac{1}{F'}\right)+N_0\left(r,\frac{1}{G'}\right)+S(r,f).\end{aligned} \tag{2.9}$$

Subcase 1.1: When $l=1$.
In this case we have

$$\overline{N}_L\left(r,\frac{1}{F-1}\right)\le\frac{1}{2}N\left(r,\frac{1}{F'}|F\ne 0\right)\le\frac{1}{2}\overline{N}(r,F)+\frac{1}{2}\overline{N}\left(r,\frac{1}{F}\right), \tag{2.10}$$

where $N\left(r,\frac{1}{F'}|F\ne 0\right)$ denotes the zeros of F', which are not the zeros of F.

From (2.5) and (2.10) we have

$$\begin{aligned}2\overline{N}_L\left(r,\frac{1}{F-1}\right)+2\overline{N}_L\left(r,\frac{1}{G-1}\right)+\overline{N}_E^{(2}\left(r,\frac{1}{F-1}\right)+\overline{N}\left(r,\frac{1}{G-1}\right)\\ \leq N\left(r,\frac{1}{G-1}\right)+\overline{N}_L\left(r,\frac{1}{F-1}\right)+S(r,f)\\ \leq N\left(r,\frac{1}{G-1}\right)+\frac{1}{2}\overline{N}(r,F)+\frac{1}{2}\overline{N}\left(r,\frac{1}{F}\right)+S(r,f)\\ \leq N\left(r,\frac{1}{G-1}\right)+\frac{1}{2}\overline{N}(r,f)+\frac{1}{2}\overline{N}\left(r,\frac{1}{p(f)}\right)+S(r,f).\end{aligned} \tag{2.11}$$

Thus, from (2.9) and (2.11) we have

$$\begin{aligned}\overline{N}\left(r,\frac{1}{F-1}\right)+\overline{N}\left(r,\frac{1}{G-1}\right)&\leq\overline{N}(r,f)+\overline{N}_{(2}\left(r,\frac{1}{F}\right)+\overline{N}_{(2}\left(r,\frac{1}{G}\right)\\ &+\frac{1}{2}\overline{N}(r,f)+\frac{1}{2}\overline{N}\left(r,\frac{1}{p(f)}\right)+N\left(r,\frac{1}{G-1}\right)\\ &+N_0\left(r,\frac{1}{F'}\right)+N_0\left(r,\frac{1}{G'}\right)+S(r,f)\\ &\leq\overline{N}(r,f)+\overline{N}_{(2}\left(r,\frac{1}{F}\right)+\overline{N}_{(2}\left(r,\frac{1}{G}\right)\\ &+\frac{1}{2}\overline{N}(r,f)+\frac{1}{2}\overline{N}\left(r,\frac{1}{p(f)}\right)+T(r,G)\\ &+N_0\left(r,\frac{1}{F'}\right)+N_0\left(r,\frac{1}{G'}\right)+S(r,f).\end{aligned} \tag{2.12}$$

From (2.3), (2.8), and (2.12) we obtain

$$\begin{aligned}T(r,F)&\leq 3\overline{N}(r,f)+\overline{N}\left(r,\frac{1}{F}\right)+\overline{N}_{(2}\left(r,\frac{1}{F}\right)+\overline{N}\left(r,\frac{1}{G}\right)+\overline{N}_{(2}\left(r,\frac{1}{G}\right)\\ &+\frac{1}{2}\overline{N}(r,f)+\frac{1}{2}\overline{N}\left(r,\frac{1}{p(f)}\right)+S(r,f)\\ &\leq\frac{7}{2}\overline{N}(r,f)+2\overline{N}\left(r,\frac{1}{F}\right)+N\left(r,\frac{1}{G}\right)+\frac{1}{2}\overline{N}\left(r,\frac{1}{p(f)}\right)+S(r,f)\\ &\leq\frac{7}{2}\overline{N}(r,f)+\frac{5}{2}\overline{N}\left(r,\frac{1}{p(f)}\right)+N\left(r,\frac{1}{P[f]}\right)+S(r,f)\\ &\leq\left(Q+\frac{7}{2}\right)\overline{N}(r,f)+\frac{5}{2}\overline{N}\left(r,\frac{1}{p(f)}\right)+(\overline{d}(P)-\underline{d}(P))T(r,f)+\overline{d}(P)N\left(r,\frac{1}{f}\right)+S(r,f)\\ &\leq\left[\left(Q+\frac{7}{2}\right)\{1-\Theta(\infty,f)\}+\frac{5n}{2}\{1-\Theta(0,p(f))\}+\overline{d}(P)\{1-\delta(0,f)\}\right]T(r,f)\\ &+(\overline{d}(P)-\underline{d}(P))T(r,f)+S(r,f).\end{aligned}$$

That is,

$$\begin{aligned} nT(r,f) &= T(r,F) + S(r,f) \\ &\leq \left[\left(Q+\frac{7}{2}\right)\{1-\Theta(\infty,f)\} + \frac{5n}{2}\{1-\Theta(0,p(f))\} + \overline{d}(P)\{1-\delta(0,f)\}\right] T(r,f) \\ &+ (\overline{d}(P) - \underline{d}(P))T(r,f) + S(r,f), \end{aligned}$$

which yields that

$$\left[\left(Q+\frac{7}{2}\right)\Theta(\infty,f) + \frac{5n}{2}\Theta(0,p(f)) + \overline{d}(P)\delta(0,f) - Q - \frac{7}{2} - 2\overline{d}(P) + \underline{d}(P) - \frac{3n}{2}\right] T(r,f)$$

$\leq S(r,f)$.

That is,

$$\left(Q+\frac{7}{2}\right)\Theta(\infty,f) + \frac{5n}{2}\Theta(0,p(f)) + \overline{d}(P)\delta(0,f) \leq Q + \frac{7}{2} + 2\overline{d}(P) - \underline{d}(P) + \frac{3n}{2},$$

which violates (1.2).

Subcase 1.2: When $l \geq 2$.
In this case we have

$$2\overline{N}_L\left(r,\frac{1}{F-1}\right) + 2\overline{N}_L\left(r,\frac{1}{G-1}\right) + \overline{N}_E^{(2}\left(r,\frac{1}{F-1}\right) + \overline{N}\left(r,\frac{1}{G-1}\right) \leq N\left(r,\frac{1}{G-1}\right) + S(r,f).$$

Thus from (2.9) we obtain

$$\begin{aligned} \overline{N}\left(r,\frac{1}{F-1}\right) + \overline{N}\left(r,\frac{1}{G-1}\right) &\leq \overline{N}(r,f) + \overline{N}_{(2}\left(r,\frac{1}{F}\right) + \overline{N}_{(2}\left(r,\frac{1}{G}\right) + N\left(r,\frac{1}{G-1}\right) \\ &+ N_0\left(r,\frac{1}{F'}\right) + N_0\left(r,\frac{1}{G'}\right) + S(r,f) \\ &\leq \overline{N}(r,f) + \overline{N}_{(2}\left(r,\frac{1}{F}\right) + \overline{N}_{(2}\left(r,\frac{1}{G}\right) + T(r,G) \\ &+ N_0\left(r,\frac{1}{F'}\right) + N_0\left(r,\frac{1}{G'}\right) + S(r,f). \end{aligned} \tag{2.13}$$

Now from (2.3), (2.8), and (2.13) we obtain

$$\begin{aligned}
T(r,F) &\leq 3\overline{N}(r,f) + \overline{N}\left(r,\frac{1}{F}\right) + \overline{N}_{(2}\left(r,\frac{1}{F}\right) + \overline{N}\left(r,\frac{1}{G}\right) + \overline{N}_{(2}\left(r,\frac{1}{G}\right) + S(r,f)\\
&\leq 3\overline{N}(r,f) + 2\overline{N}\left(r,\frac{1}{F}\right) + N\left(r,\frac{1}{G}\right) + S(r,f)\\
&\leq 3\overline{N}(r,f) + 2\overline{N}\left(r,\frac{1}{p(f)}\right) + N\left(r,\frac{1}{P[f]}\right) + S(r,f)\\
&\leq (Q+3)\overline{N}(r,f) + 2\overline{N}\left(r,\frac{1}{p(f)}\right) + (\overline{d}(P) - \underline{d}(P))T(r,f) + \overline{d}(P)N\left(r,\frac{1}{f}\right) + S(r,f)\\
&\leq [(Q+3)\{1-\Theta(\infty,f)\} + 2n\{1-\Theta(0,p(f))\} + \overline{d}(P)\{1-\delta(0,f)\}]T(r,f)\\
&+ (\overline{d}(P) - \underline{d}(P))T(r,f) + S(r,f).
\end{aligned}$$

That is,

$$\begin{aligned}
nT(r,f) &= T(r,F) + S(r,f)\\
&\leq [(Q+3)\{1-\Theta(\infty,f)\} + 2n\{1-\Theta(0,p(f))\} + \overline{d}(P)\{1-\delta(0,f)\}]T(r,f)\\
&+ (\overline{d}(P) - \underline{d}(P))T(r,f) + S(r,f),
\end{aligned}$$

which implies that

$$[\{(Q+3)\Theta(\infty,f) + 2n\Theta(0,p(f)) + \overline{d}(P)\delta(0,f)\} - \{(Q+3+2\overline{d}(P) - \underline{d}(P) + n\}]T(r,f) \leq S(r,f).$$

That is,

$$(Q+3)\Theta(\infty,f) + 2n\Theta(0,p(f)) + \overline{d}(P)\delta(0,f) \leq Q+3+2\overline{d}(P) - \underline{d}(P) + n,$$

which violates (1.1).

Case 2. When $l = 0$.

Then we have

$$\begin{aligned}
N_E^{1)}\left(r,\frac{1}{F-1}\right) &= N_E^{1)}\left(r,\frac{1}{G-1}\right) + S(r,f),\\
\overline{N}_E^{(2}\left(r,\frac{1}{F-1}\right) &= \overline{N}_E^{(2}\left(r,\frac{1}{G-1}\right) + S(r,f),
\end{aligned}$$

and also from (2.7) we have

$$\begin{aligned}\overline{N}\left(r,\frac{1}{F-1}\right)+\overline{N}\left(r,\frac{1}{G-1}\right) &\leq N_E^{1)}\left(r,\frac{1}{F-1}\right)+\overline{N}_E^{(2}\left(r,\frac{1}{F-1}\right)+\overline{N}_L\left(r,\frac{1}{F-1}\right)\\ &+\overline{N}_L\left(r,\frac{1}{G-1}\right)+\overline{N}\left(r,\frac{1}{G-1}\right)+S(r,f)\\ &\leq N_E^{1)}\left(r,\frac{1}{F-1}\right)+\overline{N}_L\left(r,\frac{1}{F-1}\right)+N\left(r,\frac{1}{G-1}\right)+S(r,f)\\ &\leq \overline{N}(r,F)+\overline{N}_{(2}\left(r,\frac{1}{F}\right)+\overline{N}_{(2}\left(r,\frac{1}{G}\right)+2\overline{N}_L\left(r,\frac{1}{F-1}\right)\\ &+\overline{N}_L\left(r,\frac{1}{G-1}\right)+N\left(r,\frac{1}{G-1}\right)+N_0\left(r,\frac{1}{F'}\right)\\ &+N_0\left(r,\frac{1}{G'}\right)+S(r,f). \qquad (2.14)\end{aligned}$$

From (2.3), (2.4), (2.8), and (2.14) we obtain

$$\begin{aligned}T(r,F) &\leq 3\overline{N}(r,f)+\overline{N}\left(r,\frac{1}{F}\right)+\overline{N}_{(2}\left(r,\frac{1}{F}\right)+\overline{N}\left(r,\frac{1}{G}\right)+\overline{N}_{(2}\left(r,\frac{1}{G}\right)\\ &+2\overline{N}_L\left(r,\frac{1}{F-1}\right)+\overline{N}_L\left(r,\frac{1}{G-1}\right)+S(r,f)\\ &\leq 3\overline{N}(r,f)+2\overline{N}\left(r,\frac{1}{F}\right)+N\left(r,\frac{1}{G}\right)+2\overline{N}\left(r,\frac{1}{F}\right)\\ &+2\overline{N}(r,F)+\overline{N}\left(r,\frac{1}{G}\right)+\overline{N}(r,G)+S(r,f)\\ &\leq 6\overline{N}(r,f)+4\overline{N}\left(r,\frac{1}{F}\right)+2N\left(r,\frac{1}{G}\right)+S(r,f)\\ &\leq 6\overline{N}(r,f)+4\overline{N}\left(r,\frac{1}{p(f)}\right)+2N\left(r,\frac{1}{P[f]}\right)+S(r,f)\\ &\leq (2Q+6)\overline{N}(r,f)+4\overline{N}\left(r,\frac{1}{p(f)}\right)+2(\overline{d}(P)-\underline{d}(P))T(r,f)+2\overline{d}(P)N\left(r,\frac{1}{f}\right)+S(r,f)\\ &\leq [(2Q+6)\{1-\Theta(\infty,f)\}+4n\{1-\Theta(0,p(f))\}+2\overline{d}(P)\{1-\delta(0,f)\}]T(r,f)\\ &+2(\overline{d}(P)-\underline{d}(P))T(r,f)+S(r,f).\end{aligned}$$

That is,

$$\begin{aligned}nT(r,f) &= T(r,F)+S(r,f)\\ &\leq [(2Q+6)\{1-\Theta(\infty,f)\}+4n\{1-\Theta(0,p(f))\}+2\overline{d}(P)\{1-\delta(0,f)\}]T(r,f)\\ &+2(\overline{d}(P)-\underline{d}(P))T(r,f)+S(r,f),\end{aligned}$$

which implies that

$$[\{(2Q+6)\Theta(\infty,f)+4n\Theta(0,p(f))+2\overline{d}(P)\delta(0,f)\}-\{2Q+6+4\overline{d}(P)-2\underline{d}(P)+3n\}]T(r,f)\leq S(r,f).$$

That is,

$$(2Q+6)\Theta(\infty,f)+4n\Theta(0,p(f))+2\overline{d}(P)\delta(0,f)\leq 2Q+6+4\overline{d}(P)-2\underline{d}(P)+3n,$$

which violates (1.3).

This proves the claim and thus $\psi\equiv 0$. Therefore, (2.7) implies that

$$\frac{F''}{F'}-\frac{2F'}{F-1}=\frac{G''}{G'}-\frac{2G'}{G-1},$$

and so we obtain

$$\frac{1}{F-1}=\frac{C}{G-1}+D, \tag{2.15}$$

where $C\neq 0$ and D are constants.

Here, the following three cases can arise:

Case (i) : When $D\neq 0,\ -1$. Rewriting (2.15) as

$$\frac{G-1}{C}=\frac{F-1}{D+1-DF},$$

we have

$$\overline{N}(r,G)=\overline{N}\left(r,\frac{1}{F-(D+1)/D}\right).$$

In this case, the second fundamental theorem of Nevanlinna yields

$$\begin{aligned}
nT(r,f)&=T(r,F)+S(r,f)\\
&\leq \overline{N}(r,F)+\overline{N}\left(r,\frac{1}{F}\right)+\overline{N}\left(r,\frac{1}{F-(D+1)/D}\right)+S(r,f)\\
&\leq \overline{N}(r,F)+\overline{N}\left(r,\frac{1}{F}\right)+\overline{N}(r,G)+S(r,f)\\
&\leq 2\overline{N}(r,f)+\overline{N}\left(r,\frac{1}{p(f)}\right)+S(r,f)\\
&=[2\{1-\Theta(\infty,f)\}+n\{1-\Theta(0,p(f))\}]T(r,f)+S(r,f).
\end{aligned}$$

$\Rightarrow$

$$\{2\Theta(\infty,f)+n\Theta(0,p(f))\}-2]T(r,f)\leq S(r,f).$$

That is,

$$2\Theta(\infty,f)+n\Theta(0,p(f))\leq 2,$$

which contradicts (1.1), (1.2), and (1.3).

Case (ii) : When $D=0$. Then from (2.15) we have

$$G=CF-(C-1). \tag{2.16}$$

Therefore, if $C\neq 1$, then

$$\overline{N}\left(r,\frac{1}{G}\right)=\overline{N}\left(r,\frac{1}{F-(C-1)/C}\right).$$

Now the second fundamental theorem of Nevanlinna and (2.3) gives

$$\begin{aligned}
nT(r,f) &= T(r,F) + S(r,f)\\
&\le \overline{N}(r,F) + \overline{N}\left(r,\frac{1}{F}\right) + \overline{N}\left(r,\frac{1}{F-(C-1)/C}\right) + S(r,f)\\
&\le \overline{N}(r,F) + \overline{N}\left(r,\frac{1}{F}\right) + \overline{N}\left(r,\frac{1}{G}\right) + S(r,f)\\
&\le \overline{N}(r,f) + \overline{N}\left(r,\frac{1}{p(f)}\right) + \overline{N}\left(r,\frac{1}{P[f]}\right) + S(r,f)\\
&\le \overline{N}(r,f) + \overline{N}\left(r,\frac{1}{p(f)}\right) + Q\overline{N}(r,f) + (\overline{d}(P) - \underline{d}(P))m\left(r,\frac{1}{f}\right)\\
&+ N\left(r,\frac{1}{f^{\overline{d}(P)}}\right) + S(r,f)\\
&\le (Q+1)\overline{N}(r,f) + \overline{N}\left(r,\frac{1}{p(f)}\right) + (\overline{d}(P) - \underline{d}(P))T(r,f)\\
&+ \overline{d}(P)N\left(r,\frac{1}{f}\right) + S(r,f)\\
&\le [(Q+1)\{1-\Theta(\infty,f)\} + n\{1-\Theta(0,p(f))\} + \overline{d}(P)\{1-\delta(0,f)\}]T(r,f)\\
&+ (\overline{d}(P) - \underline{d}(P))T(r,f) + S(r,f).
\end{aligned}$$

That is,

$$[\{(Q+1)\Theta(\infty,f) + n\Theta(0,p(f)) + \overline{d}(P)\delta(0,f)\} - \{Q+1+2\overline{d}(P) - \underline{d}(P)\}]T(r,f) \le S(r,f),$$

which implies that

$$(Q+1)\Theta(\infty,f) + n\Theta(0,p(f)) + \overline{d}(P)\delta(0,f) \le Q+1+2\overline{d}(P) - \underline{d}(P),$$

which contradicts (1.1), (1.2), and (1.3).

Thus, $C=1$ and so in this case from (2.16) we obtain $F \equiv G$ and so

$$p(f) \equiv P[f].$$

Case (iii) : When $D=-1$. Then from (2.15) we have

$$\frac{1}{F-1} = \frac{C}{G-1} - 1. \tag{2.17}$$

Therefore, if $C \ne -1$, then

$$\overline{N}\left(r,\frac{1}{G}\right) = \overline{N}\left(r,\frac{1}{F-C/(C+1)}\right),$$

and as in case (ii) we find that

$$\begin{aligned}
nT(r,f) \le (Q+1)\overline{N}(r,f) + \overline{N}\left(r,\frac{1}{p(f)}\right) + (\overline{d}(P) - \underline{d}(P))T(r,f)\\
+ \overline{d}(P)N\left(r,\frac{1}{f}\right) + S(r,f),
\end{aligned}$$

which implies that

$$[\{(Q+1)\Theta(\infty,f)+n\Theta(0,p(f))+\overline{d}(P)\delta(0,f)\}-\{Q+1+2\overline{d}(P)-\underline{d}(P)\}]T(r,f)\le S(r,f).$$

That is,

$$(Q+1)\Theta(\infty,f)+n\Theta(0,p(f))+\overline{d}(P)\delta(0,f)\le Q+1+2\overline{d}(P)-\underline{d}(P),$$

which contradicts (1.1), (1.2), and (1.3).

Therefore, $C=-1$ and so in this case from (2.17) we obtain $FG\equiv 1$ and so $p(f)P[f]=a^2$. Thus, in this case $\overline{N}(r,f)+\overline{N}\left(r,1/f\right)=S(r,f)$.

Now, by using (2.1) and (2.2), we have

$$\begin{aligned}(n+\overline{d}(P))T(r,f) &\le T\left(r,\frac{a^2}{f^{n+\overline{d}(P)}}\right)+S(r,f)\\ &\le T\left(r,\left[1+\frac{a_{n-1}}{f}+---+\frac{a_1}{f^{n-1}}\right].\frac{P[f]}{f^{\overline{d}(P)}}\right)+S(r,f)\\ &\le (n-1)T(r,f)+T\left(r,\frac{P[f]}{f^{\overline{d}(P)}}\right)+S(r,f)\\ &= (n-1)T(r,f)+m\left(r,\frac{P[f]}{f^{\overline{d}(P)}}\right)+N\left(r,\frac{P[f]}{f^{\overline{d}(P)}}\right)+S(r,f)\\ &\le (n-1)T(r,f)+(\overline{d}(P)-\underline{d}(P))m\left(r,\frac{1}{f}\right)+(\overline{d}(P)-\underline{d}(P))N\left(r,\frac{1}{f}\right)\\ &+Q\left[\overline{N}(r,f)+\overline{N}\left(r,\frac{1}{f}\right)\right]+S(r,f)\\ &\le (n-1)T(r,f)+(\overline{d}(P)-\underline{d}(P))T(r,f)+S(r,f).\end{aligned}$$

Thus

$$(1+\underline{d}(P))T(r,f)\le S(r,f),$$

which is a contradiction. □

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References

[1] Banerjee A. Meromorphic functions sharing one value. Int J Math Math Sci 2005; 22: 3587-3598.

[2] Banerjee A, Majumder S. Some uniqueness results related to meromorphic function that share a small function with its derivative. Math Reports 2014; 66: 95-111.

[3] Bhoosnurmath S, Kabbur SR. On entire and meromorphic functions that share one small function with their differential polynomial, Hindawi Publishing Corporation. Intl J Analysis 2013; Article ID 926340.

[4] Goldberg AA, Ostrovskii IV. Value Distribution of Meromorhic Functions. Translated from the 1970 Russian original by Mikhail Ostrovskii. With an appendix by Alexandre Eremenko and James K. Langley. Translations of Mathematical Monographs, 236. American Mathematical Society, Providence, RI, USA, 2008.

[5] Hayman WK. Meromorphic Functions. Oxford, UK: Clarendon Press, 1964.

[6] Mues E, Steinmetz N. Meromorphe funktionen die unit ihrer ableitung werte teilen. Manuscripta Math 1979; 29: 195-206 (in German).

[7] Yang CC, Yi HX. Uniqueness Theory of Meromorphic Functions. Dordrecht, the Netherlands: Kluwer Academic Publishers, 2003.

[8] Zhang T, Lü W. Notes on a meromorphic function sharing one small function with its derivative. Complex Variables and Elliptic Equations 2008; 53: 857-867.