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A lower bound for Stanley depth of squarefree monomial ideals

Guangjun ZHU*
School of Mathematical Sciences, Soochow University, Suzhou, P.R. China

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Abstract: Let $S = K[x_1, \ldots, x_n]$ be a polynomial ring over a field K in n variables and I a squarefree monomial ideal of S with Schmitt–Vogel number $sv(I)$. In this paper, we show that $\operatorname{sdepth}(I) \geq \max\{1, n-1-\lfloor\frac{sv(I)}{2}\rfloor\}$, which improves the lower bound obtained by Herzog, Vladoiu, and Zheng. As some applications, we show that Stanley's conjecture holds for the edge ideals of some special n-cyclic graphs with a common edge.

Key words: Stanley depth, Stanley conjecture, monomial ideal, Schmitt–Vogel number, n-cyclic graph

1. Introduction

Let $S = K[x_1, \ldots, x_n]$ be a polynomial ring over a field K in n variables and M a finitely generated $\mathbb{Z}^n$-graded S-module. For a homogeneous element $u \in M$ and a subset $Z \subseteq \{x_1, \ldots, x_n\}$, $uK[Z]$ denotes the K-subspace of M generated by all the homogeneous elements of the form uv, where v is a monomial in $K[Z]$. The $\mathbb{Z}^n$-graded K-subspace $uK[Z]$ is said to be a Stanley space of dimension $|Z|$ if it is a free $K[Z]$-module, where $|Z|$ denotes the cardinality of Z. A Stanley decomposition of M is a decomposition of M as a finite direct sum of $\mathbb{Z}^n$-graded K-vector spaces

$$\mathcal{D} : M = \bigoplus_{i=1}^{r} u_i K[Z_i]$$

where each $u_iK[Z_i]$ is a Stanley space of M. The number $\operatorname{sdepth}_S(\mathcal{D}) = \min\{|Z_i| : i = 1, \ldots, r\}$ is called the Stanley depth of decomposition $\mathcal{D}$ and the number

$$\operatorname{sdepth}_S(M) := \max\{\operatorname{sdepth}(\mathcal{D}) : \mathcal{D} \text{ is a Stanley decomposition of } M\}.$$

is called the Stanley depth of M.

In [4], Schmitt and Vogel introduced the Schmitt–Vogel number, which is given in the following definition.

Definition 1.1 *Let I be a monomial ideal and $G(I)$ the set of its minimal monomial generators. The Schmitt–Vogel number of I, denoted by $sv(I)$, is the smallest integer t for which there exist subsets $P_1, \ldots, P_t$ of $G(I)$ such that*

(i) $\bigcup_{i=1}^{t} P_i = G(I)$;

*Correspondence: zhuguangjun@suda.edu.cn
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(ii) P_1 *has exactly one element;*

(iii) if p *and* p' *are different elements of* $P_i\,(2 \le i \le t)$*, then there is an integer* i' *with* $1 \le i' < i$ *and an element in* $P_{i'}$ *that divides* pp'*.*

They proved that for any monomial ideal I, the Schmitt–Vogel number $sv(I)$ is an upper bound for the arithmetical rank of I. It is clear that $sv(I) \le |G(I)|$, and this inequality is strict in general. Herzog et al. [1] proved the following result:

Lemma 1.2 *([1, Proposition 3.4]) Let* $I \subset S$ *be a monomial ideal with* $|G(I)| = m$*. Then* $sdepth_S\,(I) \ge max\{1, n-m+1\}$*.*

Recall that a monomial $v \in S$ is said to be squarefree if the exponent of each x_i in v is less than or equal to 1, and a monomail ideal I is said to be squarefree if it is generated by some squarefree monomials. The main result in this paper is the following: for a squarefree monomial ideal I, we have that

$$\mathrm{sdepth}_S\,(I) \ge \max\{1, n-1-\lfloor \frac{sv(I)}{2} \rfloor\}.$$

Our result improves the lower bound obtained by Herzog et al. stated above. As some applications, we show that Stanley's conjecture holds for the edge ideals of some special n-cyclic graphs with a common edge.

In this paper, we will focus on the case where I is a squarefree monomial ideal in S and let $G(I) = \{v_1, \ldots, v_m\}$ be the set of its minimal squarefree monomial generators.

2. Preliminaries

We first recall some definitions and basic facts about the edge ideal of a graph and the lower bounds for Stanley depth of some special monomial ideals in order to make this paper self-contained. However, for more details on the notions, we refer the reader to [2, 3, 6].

Definition 2.1 *A finite graph* G *is an ordered pair* $G = (V(G), E(G))$ *where* $V(G) = \{x_1, \ldots, x_n\}$ *is the set of vertices of* G*, and* $E(G)$ *is a collection of two-element subsets of* $V(G)$*, usually called the edges of* G*.*

In this case, we may suppose that $x_1, \ldots, x_n$ *are indeterminates over the field* K*. The edge ideal of* G *in the polynomial ring* $S = K[x_1, \ldots, x_n]$ *is the squarefree monomial ideal*

$$I(G) = (x_i x_j \mid \{x_i, x_j\} \in E(G)).$$

Definition 2.2 *Let* $G_i = (V(G_i), E(G_i))$ *be some graphs with vertex set* $V(G_i)$ *and edge set* $E(G_i)$*, for* $i = 1, \ldots, k$*. The union of the graphs* $G_1, G_2, \ldots, G_k$*, written* $\bigcup_{i=1}^{k} G_i$*, is the graph with vertex set* $\bigcup_{i=1}^{k} V(G_i)$ *and edge set* $\bigcup_{i=1}^{k} E(G_i)$*.*

Definition 2.3 *Let* $G = (V(G), E(G))$ *be a graph. A walk of length* m *in* G *is an alternating sequence of vertices and edges* $w = \{x_1, y_1, x_2, \ldots, x_m, y_m, x_{m+1}\}$*, where* $y_i = \{x_i, x_{i+1}\}$ *is the edge joining* x_i *and* x_{i+1}*. If* $x_1 = x_{m+1}$*, we call this walk closed.*

A cycle of length m $(m \geq 3)$ *is a closed walk in which the vertices* $x_1, \ldots, x_m$ *are distinct. We denote by* C_m *the graph consisting of a cycle with* m *vertices. An* n*-cyclic graph with a common edge is a graph consisting of the union of* n *cycles* $C_{3r_1}, \ldots, C_{3r_{k_1}}$, $C_{3s_1+1}, \ldots, C_{3s_{k_2}+1}$, $C_{3t_1+2}, \ldots, C_{3t_{k_3}+2}$ *connected through a common edge, where* $k_1 + k_2 + k_3 = n$, *and* r_i, s_j, t_l *are positive integers for any* $1 \leq i \leq k_1$, $1 \leq j \leq k_2$ *and* $1 \leq l \leq k_3$.

The Stanley depth of the complete intersection monomial ideal is completely computed by Shen.

Lemma 2.4 *([5, Theorem 2.4]) Let* $I \subset S$ *be a complete intersection monomial ideal with* $|G(I)| = m$. *Then* $sdepth_S(I) = n - \lfloor \frac{m}{2} \rfloor$.

Keller and Young [2] and Okazaki [3] independently improved this lower bound stated above; they showed that:

Lemma 2.5 *Let* $I \subset S$ *be a monomial ideal with* $|G(I)| = m$. *Then* $sdepth_S(I) \geq max\{1, n - \lfloor \frac{m}{2} \rfloor\}$.

Let $\mathrm{mod}_{\mathbb{Z}}^{n}(S)$ denote the category whose objects are finitely generated $\mathbb{Z}^n$-graded S-modules and morphisms are degree-preserving S-homomorphisms, that is, S-homomorphisms $f : M \to N$ such that $f(M_a) \subseteq N_a$ for $a \in \mathbb{Z}^n$. Clearly, the following lemma holds.

Lemma 2.6 *Let* $0 \to L \to M \to N \to 0$ *be a short exact sequence in* $mod_{\mathbb{Z}}^{n}(S)$. *Then* $sdepth_S(M) \geq min\{sdepth_S(L), sdepth_S(N)\}$.

Let $R = K[x_1, \ldots, x_{n-1}]$. We consider the natural map $\varphi : S \to R$ via $\varphi(x_i) = x_i$ for any $1 \leq i \leq n-1$ and $\varphi(x_n) = 1$. Thus, any $\mathbb{Z}^{n-1}$ graded R-module has a structure of $\mathbb{Z}^n$-graded S-modules by the map φ. We need the following lemma.

Lemma 2.7 *([3, Lemma 2.5]) Let* $v_1, \ldots, v_m$ *be monomials in* S *such that* $x_n | v_i$ *for* $i = 1, \ldots, r$ *and* $x_n \nmid v_i$ *for* $i = r+1, \ldots, m$, *where* $1 \leq r \leq m-1$. *Let* $\mathfrak{a} = (v_1, \ldots, v_r)$, $\mathfrak{b} = (v_{r+1}, \ldots, v_m)$ *be monomial ideals generated by* $v_1, \ldots, v_r$ *and* $v_{r+1}, \ldots, v_m$, *respectively. Let* $I = \mathfrak{a} + \mathfrak{b}$ *and* $I' = \mathfrak{a} + x_n\mathfrak{b}$. *Then*

$$I/I' \cong \mathfrak{b} \cap R$$

as $\mathbb{Z}^n$*-graded* S*-modules, where the structure of* $\mathbb{Z}^n$*-graded* S*-modules* $\mathfrak{b} \cap R$ *is given as above.*

3. Main results

In this section we provide a lower bound for the Stanley depth of squarefree monomial ideals. This lower bound is given in terms of the Schmitt–Vogel number $sv(I)$. In the following three propositions, we consider the behavior of the Schmitt–Vogel number of an arbitrary monomial ideal under the elimination of variables.

Proposition 3.1 *Let* $I \subset S$ *be a squarefree monomial ideal with* $G(I) = \{v_1, \ldots, v_m\}$ *such that* $x_n | v_i$ *for any* $1 \leq i \leq m$. *Let* $v_i' = v_i/x_n$, *and let* I' *be a monomial ideal of* $R = K[x_1, \ldots, x_{n-1}]$ *generated by* $\{v_1', \ldots, v_m'\}$. *If* $sv(I) \geq 2$, *then* $sv(I') = sv(I)$.

Proof Let $sv(I)=t$. Then $v_i \neq x_n$ for any $1 \leq i \leq m$. Otherwise, $m=t=1$, which contradicts with $t \geq 2$. We can assume that $P_1=\{v_1\}, P_2=\{v_2,\dots,v_{s_2}\},\dots,P_t=\{v_{s_{t-1}+1},\dots,v_m\}$ are the subsets of $G(I)$. Then $P_1'=\{v_1'\}, P_2'=\{v_2',\dots,v_{s_2}'\},\dots,P_t'=\{v_{s_{t-1}+1}',\dots,v_m'\}$ are the subsets of $G(I')$. Since I is a squarefree monomial ideal, it is obvious that $P_1,\dots,P_t$ satisfy the conditions of Definition 1.1 if and only if $P_1',\dots,P_t'$ satisfy the conditions of Definition 1.1. Therefore, $sv(I')=sv(I)$. This completes the proof. □

Proposition 3.2 *Let $I \subset S$ be a squarefree monomial ideal with $G(I)=\{v_1,\dots,v_m\}$ such that $x_n|v_i$ for $i=1,\dots,r$ and $x_n \nmid v_i$ for $i=r+1,\dots,m$, where $2 \leq r \leq m-1$. Let $v_i'=v_i/x_n$ for any $1 \leq i \leq r$, and let I' be a squarefree monomial ideal of $R=K[x_1,\dots,x_{n-1}]$ generated by $\{v_1',\dots,v_r',v_{r+1},\dots,v_m\}$. If $sv(I) \geq 2$, then $sv(I') \leq sv(I)$.*

Proof Note that for any $1 \leq i \leq r$, $v_i \neq x_n$. Otherwise, $r=1$, and this contradicts with $r \geq 2$. Let π be a permutation of the set $\{1,\dots,m\}$ and $sv(I)=t$, and let $P_1=\{v_{\pi(1)}\}, P_2=\{v_{\pi(2)},\dots,v_{\pi(s_2)}\},\dots,P_t=\{v_{\pi(s_{t-1}+1)},\dots,v_{\pi(m)}\}$ be the subsets of $G(I)$. Then $P_1'=\{v_{\pi(1)}'\}, P_2'=\{v_{\pi(2)}',\dots,v_{\pi(s_2)}'\},\dots,P_t'=\{v_{\pi(s_{t-1}+1)}',\dots,v_{\pi(m)}'\}$ are the subsets of $G(I')$ such that $\bigcup_{i=1}^{t} P_i'=G(I')$, where v_i', for $i=1,\dots,m$ is the monomial obtained by substitution of 1 to x_n in v_i. Hence, in order to prove the assertion, it is enough to prove that the sets $P_1',\dots,P_t'$ satisfy conditions (ii) and (iii) of Definition 1.1. It is clear that $P_1' \neq \emptyset$. Assume that $v_{\pi(i)}'$ and $v_{\pi(j)}'$ are different elements of P_k' for some k with $2 \leq k \leq t$. Then $v_{\pi(i)}$ and $v_{\pi(j)}$ are different elements of P_k. Since $P_1,\dots,P_t$ satisfy condition (iii) of Definition 1.1, it follows that there exists an integer s with $1 \leq s < k$ and some monomial $v_{\pi(l)} \in P_s$ such that $v_{\pi(l)}|v_{\pi(i)}v_{\pi(j)}$. Since $v_{\pi(l)}$, $v_{\pi(i)}$ and $v_{\pi(j)}$ are squarefree, we have that $v_{\pi(l)}'|v_{\pi(i)}'v_{\pi(j)}'$. Thus, $v_{\pi(l)}' \in P_s'$. Therefore, $sv(I') \leq sv(I)$. This completes the proof. □

Proposition 3.3 *Let $I \subset S$ be a squarefree monomial ideal with $G(I)=\{v_1,\dots,v_m\}$ such that $x_n|v_i$ for $i=1,\dots,r$ and $x_n \nmid v_i$ for $i=r+1,\dots,m$, where $2 \leq r \leq m-1$. Let I' be a squarefree monomial ideal of $K[x_1,\dots,x_{n-1}]$ generated by $\{v_{r+1},\dots,v_m\}$. If $sv(I) \geq 2$, then $sv(I') \leq sv(I)$.*

Proof Let $R=K[x_1,\dots,x_{n-1}]$; then $G(I')=G(I)\cap R$. Note that for any $1 \leq i \leq r$, $v_i \neq x_n$ from the proof of Proposition 3.2. Let $sv(I)=t$, and $P_1,\dots,P_t$ be the subsets of $G(I)$ that satisfy the conditions of Definition 1.1. Set $P_i'=P_i\cap R$ for any $1 \leq i \leq t$ and $P_1=\{u\}$. We distinguish two cases:

(1) If $x_n \nmid u$, then $P_1' \neq \emptyset$ and it is obviously seen that $P_1',\dots,P_t'$ are the subsets of $G(I')$ that satisfy the conditions of Definition 1.1. Thus, $sv(I') \leq sv(I)$.

(2) If $x_n|u$, then $P_1'=\emptyset$. Thus, there exist integers $2 \leq i_1 < i_2 < \cdots < i_l \leq t$ such that $P_{i_k}' \neq \emptyset$ for any $1 \leq k \leq l$ and $P_j'=\emptyset$ for any $j \notin \{i_1,\dots,i_l\}$. It is clear that $G(I')=\bigcup_{k=1}^{l} P_{i_k}'$. Since $i_1 \geq 2$, it follows that $l \leq t-1$. We claim that the sets $P_{i_1}',\dots,P_{i_l}'$ satisfy conditions (ii) and (iii) of Definition 1.1.

We first verify condition (ii). Assume that $|P_{i_1}'| \geq 2$. This implies that there exist two different monomials μ_1,μ_2 in P_{i_1} that are not divisible by x_n. Thus, by condition (iii) of Definition 1.1, there exists an integer $q < i_1$ and some monomial $\mu_3 \in P_q$ with $\mu_3|\mu_1\mu_2$. However, this is not possible because $P_q'=\emptyset$ and therefore every element of P_q and in particular μ_3 is divisible by x_n. This proves condition (ii).

Now we verify condition (iii). Let ν_1, ν_2 be two different monomials in P'_{i_k} for some k with $2 \le k \le l$. Then $\nu_1, \nu_2 \in P_{i_k}$ and since $P_1, \dots, P_t$ satisfy condition (iii) of Definition 1.1, it follows that there exists an integer s with $1 \le s < i_k$ and some monomial $\nu_3 \in P_s$, such that $\nu_3 | \nu_1\nu_2$. Since ν_1 and ν_2 are not divisible by x_n, we conclude that $x_n \nmid \nu_3$. Thus, $s \in \{i_1, \dots, i_l\}$ and $\nu_3 \in P'_s$. This verifies condition (iii) of Definition 1.1. Thus, $sv(I') \le sv(I) - 1$. This completes the proof. □

Now we state and prove the main result of this section.

Theorem 3.4 *Let I be a squarefree monomial ideal of S with Schmitt–Vogel number $sv(I)$. Then:*

$$sdepth_S(I) \ge max\{1, n-1-\lfloor \frac{sv(I)}{2} \rfloor\}.$$

Proof It suffices to show that $\operatorname{sdepth}_S(I) \ge n-1-\lfloor \frac{sv(I)}{2} \rfloor$ by Lemma 1.2. Let $G(I) = \{v_1, \dots, v_m\}$. We use induction on n. If $n = 1$ or $sv(I) = 1$, then I is a principal ideal, so we have $\operatorname{sdepth}_S(I) = n$. Thus, the assertion holds. Now we assume that $n \ge 2$ and the assertion holds for $n-1$. It suffices to consider only the case $sv(I) \ge 2$. For $i = 1, \dots, n$, we set $t_i(I) = |\{v_j \in G(I) \mid x_i \text{ divides } v_j\}|$. If $t_i(I) \le 1$ for any $1 \le j \le m$, then I is a complete intersection and $sv(I) = |G(I)| = m$, and hence we obtain that the assertion holds by Lemma 2.4. Thus, we may assume that $t_i(I) \ge 2$ for some i, and hence, without loss of generality, that $t_n(I) \ge 2$. We distinguish the following two cases:

(1) If $t_n(I) = m$, then $x_n | v_i$ for any $1 \le i \le m$. Set $v'_i = v_i/x_n$, and let I' be a squarefree monomial ideal of S generated by $v'_1, \dots, v'_m$. It is readily seen that I' is naturally isomorphic to I in $\operatorname{mod}^n_{\mathbb{Z}}(S)$ up to degree shifting, and it follows that $\operatorname{sdepth}_S(I) = \operatorname{sdepth}_S(I')$. Note that I' is also a squarefree monomial ideal of $R = K[x_1, \dots, x_{n-1}]$. By inductive hypothesis, Proposition 3.1, and [1, Lemma 3.6], we have

$$\operatorname{sdepth}_S(I) = \operatorname{sdepth}_S(I') = \operatorname{sdepth}_R(I') + 1 \ge (n-1) - \lfloor \frac{sv(I')}{2} \rfloor + 1 > n-1-\lfloor \frac{sv(I)}{2} \rfloor.$$

(2) If $2 \le t_n(I) \le m-1$, we set $r = t_n(I)$. Without loss of generality, we may assume that $x_n | v_i$ for $i = 1, \dots, r$ and $x_n \nmid v_i$ for $i = r+1, \dots, m$. Let $\mathfrak{a} = (v_1, \dots, v_r)$, $\mathfrak{b} = (v_{r+1}, \dots, v_m)$ be squarefree monomial ideals generated by $v_1, \dots, v_r$ and $v_{r+1}, \dots, v_m$, respectively. Then $I = \mathfrak{a} + \mathfrak{b}$. Set $I' = \mathfrak{a} + x_n\mathfrak{b}$; thus, each minimal generator of I' can be divided by x_n. Set $v'_i = v_i/x_n$ for $1 \le i \le r$, and let I'' be the squarefree monomial ideal generated by $\{v'_1, \dots, v'_r, v_{r+1}, \dots, v_m\}$. By the same argument as in case (1), we have that $\operatorname{sdepth}_S(I'') = \operatorname{sdepth}_S(I')$. Applying our inductive hypothesis and Proposition 3.2, we have

$$\operatorname{sdepth}_S(I') = \operatorname{sdepth}_S(I'') \ge n-1-\lfloor \frac{sv(I'')}{2} \rfloor \ge n-1-\lfloor \frac{sv(I)}{2} \rfloor.$$

We consider the exact sequence

$$0 \to I' \to I \to I/I' \to 0.$$

It follows from Lemma 2.6 that

$$\operatorname{sdepth}_S(I) \ge \min\{\operatorname{sdepth}_S(I'), \operatorname{sdepth}_S(I/I')\}.$$

As for $\operatorname{sdepth}_S(I/I')$, we can apply Lemma 2.7, and it follows that

$$\operatorname{sdepth}_S(I/I') = \operatorname{sdepth}_S(\mathfrak{b} \cap R).$$

Note that $\mathfrak{b} \cap R$ is minimally generated by $v_{r+1}, \ldots, v_m$ as an ideal of R. By inductive hypothesis, Proposition 3.3, and [1, Lemma 3.6], we have

$$\begin{aligned} \operatorname{sdepth}_S(\mathfrak{b} \cap R) &= \operatorname{sdepth}_R(\mathfrak{b} \cap R) + 1 \\ &\geq (n-2) - \lfloor \frac{sv(\mathfrak{b} \cap R)}{2} \rfloor + 1 \\ &= n - 1 - \lfloor \frac{sv(\mathfrak{b} \cap R)}{2} \rfloor \\ &\geq n - 1 - \lfloor \frac{sv(I)}{2} \rfloor. \end{aligned}$$

Summing up, we conclude that $\operatorname{sdepth}_S(I) \geq n - 1 - \lfloor \frac{sv(I)}{2} \rfloor$, which completes the proof.

□

Lemma 3.5 *(Auslander–Buchsbaum). Let M be a finitely generated graded S-module. Then*

$$pd_S(M) + depth(M) = dim(S),$$

where $pd_S(M)$ is the projective dimension of M.

Zhu et al. [6] provided some upper bounds for Schmitt–Vogel number $sv(I(G))$ of the edge ideals $I(G)$ of some special graphs G with a common edge and the lower bounds for the projective dimensions of their quotient ring $S/I(G)$.

Lemma 3.6 *(1) Let G be a graph consisting of the union of k_1 cycles $C_{3r_1}, \ldots, C_{3r_{k_1}}$ with a common edge. Then $pd_S(S/I(G)) = 1 + \sum_{i=1}^{k_1}(2r_i - 1)$ and $sv(I(G)) \leq 1 + \sum_{i=1}^{k_1}(2r_i - 1)$.*

(2) Let G be a graph consisting of the union of k_2 cycles $C_{3s_1+1}, \ldots, C_{3s_{k_2}+1}$ with a common edge. Then $pd_S(S/I(G)) \geq 2 - k_2 + 2\sum_{i=1}^{k_2} s_i$ and $sv(I(G)) \leq 1 + 2\sum_{i=1}^{k_2} s_i$.

(3) Let G be a graph consisting of the union of k_3 cycles $C_{3t_1+2}, \ldots, C_{3t_{k_3}+2}$ with a common edge. Then $pd_S(S/I(G)) = 1 + 2\sum_{i=1}^{k_3} t_i$ and $sv(I(G)) \leq 1 + 2\sum_{i=1}^{k_3} t_i$.

As a consequence of Theorem 3.4 and Lemma 3.6, we have the following results.

Theorem 3.7 *(1) Let G be a graph consisting of the union of k_1 cycles $C_{3r_1}, \ldots, C_{3r_{k_1}}$ with a common edge. Then Stanley's conjecture holds for $I(G)$.*

(2)Let G be a graph consisting of the union of k_2 cycles $C_{3s_1+1}, \ldots, C_{3s_{k_2}+1}$ with a common edge. Then Stanley's conjecture holds for $I(G)$.

(3) Let G be a graph consisting of the union of k_3 cycles $C_{3t_1+2}, \ldots, C_{3t_{k_3}+2}$ with a common edge. Then Stanley's conjecture holds for $I(G)$.

Proof Cases (1) and (3) can be shown by similar arguments, so we only prove case (1). Note that the number of vertices of the graph G is $n=\sum_{i=1}^{k_1}3r_i-2(k_1-1)$. Thus, by Lemma 3.6 (1), we have

$$\begin{aligned} n-1-\lfloor\frac{sv(I)}{2}\rfloor &\geq \sum_{i=1}^{k_1}3r_i-2(k_1-1)-1-\lfloor\frac{1+\sum_{i=1}^{k_1}(2r_i-1)}{2}\rfloor \\ &= 1+\sum_{i=1}^{k_1}r_i-k_1+\lceil\frac{1+\sum_{i=1}^{k_1}(2r_i-1)}{2}\rceil, \end{aligned}$$

and

$$\begin{aligned} \operatorname{depth}(I(G)) &= \operatorname{depth}(S/I(G))+1=n-\operatorname{pd}_S(S/I(G))+1 \\ &\leq \sum_{i=1}^{k_1}3r_i-2(k_1-1)-(1+\sum_{i=1}^{k_1}(2r_i-1))+1 \\ &= 1+\sum_{i=1}^{k_1}r_i-k_1+1. \end{aligned}$$

Since $k_1\geq 2$ and $r_i\geq 1$ for any $1\leq i\leq k_1$, we have that $\lceil\frac{1+\sum_{i=1}^{k_1}(2r_i-1)}{2}\rceil\geq 1$. Therefore, by Theorem 3.4, we have that

$$\operatorname{sdepth}_S(I(G))\geq n-1-\lfloor\frac{sv(I(G))}{2}\rfloor\geq\operatorname{depth}(I(G)).$$

(2) Note that the number of vertices of the graph G is $n=\sum_{i=1}^{k_2}(3s_i+1)-2(k_2-1)$. Thus, by Lemma 3.6 (2), we have

$$\begin{aligned} n-1-\lfloor\frac{sv(I)}{2}\rfloor &\geq \sum_{i=1}^{k_2}(3s_i+1)-2(k_2-1)-1-\lfloor\frac{1+2\sum_{i=1}^{k_2}s_i}{2}\rfloor \\ &= 1+\sum_{i=1}^{k_2}s_i+\sum_{i=1}^{k_2}(s_i-1), \end{aligned}$$

and

$$\begin{aligned} \operatorname{depth}(I(G)) &= \operatorname{depth}(S/I(G))+1=n-\operatorname{pd}_S(S/I(G))+1 \\ &\leq \sum_{i=1}^{k_2}(3s_i+1)-2(k_2-1)-(2-k_2+2\sum_{i=1}^{k_2}s_i)+1 \\ &= 1+\sum_{i=1}^{k_2}s_i. \end{aligned}$$

Therefore, by Theorem 3.4, we have that

$$\mathrm{sdepth}_S\,(I(G)) \geq n-1-\lfloor\frac{sv(I(G))}{2}\rfloor = 1+\sum_{i=1}^{k_2} s_i + \sum_{i=1}^{k_2}(s_i-1) \geq \mathrm{depth}\,(I(G)).$$

□

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