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On the zero-divisor graphs of finite free semilattices

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Abstract: Let SL_X be the free semilattice on a finite nonempty set X. There exists an undirected graph $\Gamma(SL_X)$ associated with SL_X whose vertices are the proper subsets of X, except the empty set, and two distinct vertices A and B of $\Gamma(SL_X)$ are adjacent if and only if $A \cup B = X$. In this paper, the diameter, radius, girth, degree of any vertex, domination number, independence number, clique number, chromatic number, and chromatic index of $\Gamma(SL_X)$ have been established. Moreover, we have determined when $\Gamma(SL_X)$ is a perfect graph and when the core of $\Gamma(SL_X)$ is a Hamiltonian graph.

Key words: Finite free semilattice, zero-divisor graph, clique number, domination number, perfect graph, Hamiltonian graph

1. Introduction

The zero-divisor graph was first introduced by Beck in the study of commutative rings [3], and later studied by Anderson et al. [1, 2]. In [6, 7] DeMeyer et al. considered the zero-divisor graph on a commutative semigroup S with 0. If the set of zero-divisor elements in S is $Z(S)$, then the zero-divisor graph $\Gamma(S)$ is defined as an undirected graph with vertices $Z(S) \setminus \{0\}$ and the vertices x and y are adjacent with a single edge if and only if $xy = 0$. It is known that $\Gamma(S)$ is a connected graph (see [7]).

Let X be a finite nonempty set, and let SL_X be the set consisting of all subsets of X except the empty set. Then SL_X is a commutative semigroup of idempotents with the multiplication $A \cdot B = A \cup B$ for $A, B \in SL_X$ and it is called the free semilattice on X. The zero-divisor graph $\Gamma(SL_X)$ is associated with SL_X and defined by:

- the vertex set of $\Gamma(SL_X)$, denoted by $V(\Gamma(SL_X))$, which is the proper subsets of X except the empty set; and
- the undirected edge set of $\Gamma(SL_X)$, denoted by $E(\Gamma(SL_X))$ and

$$E(\Gamma(SL_X)) = \{A - B \mid A, B \in V(\Gamma(SL_X)); A \cup B = X\}.$$

Moreover, we say that A and B are adjacent or A is adjacent to B if $A - B \in E(\Gamma(SL_X)$. Throughout this paper we suppose that $|X| = n$ and that, without loss of generality, $X = \{1, 2, \ldots, n\}$. Thus, there are $2^n - 2$ vertices in $\Gamma(SL_X)$.

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In this paper, the diameter, radius, girth, degree of any vertex, domination number, independence number, clique number, chromatic number, and chromatic index of $\Gamma(SL_X)$ have been established. Moreover, we have determined when $\Gamma(SL_X)$ is a perfect graph and when the core of $\Gamma(SL_X)$ is a Hamiltonian graph.

For graph theoretical terminology see [8], and for semigroup terminology see [9].

2. Some basic properties of $\Gamma(SL_X)$

For any simple graph G, the length of the shortest path between two vertices u and v of G is denoted by $d_G(u,v)$. The eccentricity of a vertex v in a connected simple graph G is the maximum distance (length of the shortest path) between v and any other vertex u of G and it is denoted by $\mathrm{ecc}(v)$; that is,

$$\mathrm{ecc}(v) = \max\{d_G(u,v) \mid u \in V(G)\}.$$

The diameter of G, denoted by $\mathrm{diam}(G)$, is

$$\mathrm{diam}(G) = \max\{\mathrm{ecc}(v) \mid v \in V(G)\},$$

and it is known that the diameter of the zero-divisor graph of any commutative semigroup with zero is at most 3 (see Theorem 1.2 in [7]). The radius of G, denoted by $\mathrm{rad}(G)$, is

$$\mathrm{rad}(G) = \min\{\mathrm{ecc}(v) \mid v \in V(G)\}.$$

The central vertex set of G, denoted by $C(G)$, is

$$C(G) = \{v \in V(G) \mid \mathrm{ecc}(v) = \mathrm{rad}(G)\}.$$

The girth of G is the length of a shortest cycle contained in G and it is denoted by $\mathrm{gr}(G)$. If G does not contain any cycles, then its girth is defined to be infinity. The degree of a vertex $v \in V(G)$ is the number of vertices adjacent to v and denoted by $\deg_G(v)$. Among all degrees, the maximum degree $\Delta(G)$ (the minimum degree $\delta(G)$) of G is the biggest (the smallest) degree in G. A vertex of maximum degree is called a delta-vertex and we denote the set of delta-vertices of G by Λ_G. An independent set of a graph G is a subset of vertices $V(G)$ such that no two vertices in the subset represent an edge of G. Independence number, denoted by $\alpha(G)$, is defined by

$$\alpha(G) = \max\{|I| \mid I \text{ is an independent set of } G\}.$$

Let D be a nonempty subset of the vertex set $V(G)$ of G. If, for each $u \in V(G)\setminus D$, there exists $v_u \in D$ such that $u - v_u \in E(G)$, then D is called a dominating set. The domination number of G, denoted by $\gamma(G)$, is

$$\gamma(G) = \min\{|D| \mid D \text{ is a dominating set of } G\}.$$

The open neighborhood of a vertex $v \in V(G)$, denoted by $N_G(v)$, is the set of vertices that are adjacent to v and the closed neighborhood of v is $N_G[v] = N_G(v) \cup \{v\}$. For a nonempty subset Z of $V(G)$, the closed neighborhood of Z in G, denoted by $N_G[Z]$, is $N_G[Z] = \bigcup\limits_{v \in Z} N_G[v]$. It is clear that $|N_G[v] \cap D| \geq 1$ for each dominating set D, and for each $v \in V(G)$.

In this section, we mainly deal with some graph properties of $\Gamma(SL_X)$, namely the diameter, radius, girth, degree of any vertex, domination number, and independence number of $\Gamma(SL_X)$.

For convenience, we use the notation $\overline{A} = (X \setminus A)$ for each $A \subseteq X$, Γ instead of $\Gamma(SL_X)$ and $d(A, B)$ instead of $d_{\Gamma(SL_X)}(A, B)$. For each pair $A, B \in V(\Gamma)$, notice that

$$A - B \in E(\Gamma) \Leftrightarrow \overline{A} \subseteq B \Leftrightarrow \overline{B} \subseteq A.$$

Theorem 2.1. *If $|X| = n \geq 3$ then we have:*

(i) $gr(\Gamma) = 3$,

(ii) $rad(\Gamma) = 2$ *and* $diam(\Gamma) = 3$.

Proof (i) Since Γ is a simple graph and from the definiton of Γ it is clear that $\text{gr}(\Gamma) \geq 3$, let $|X| \geq 3$ and $A \in V(\Gamma)$ with $|A| \geq 2$. We consider any 2-partition A_1 and A_2 of A, $B = \overline{A} \cup A_1$ and $C = \overline{A} \cup A_2$. Thus, we have a cycle $A - B - C - A$ in Γ.

(ii) Let $|X| \geq 3$; for proof we show that show that the eccentricity of a vertex $A \in V(\Gamma)$ is either 2 or 3. Let $A \in V(\Gamma)$ with $|A| = n - 1$, and $B \in V(\Gamma)$. If $A \cap B = \varnothing$ then it is clear that $\overline{B} \subseteq A$ and so $d(A, B) = 1$ or if $A \cap B \neq \varnothing$ and $\overline{B} \subseteq A$ then $d(A, B) = 1$. If $A \cap B \neq \varnothing$ and $\overline{B} \not\subseteq A$ it is clear that $d(A, B) \geq 2$ and we have a path $A - C - B$ where $C = \overline{A \cap B}$, and so $d(A, B) = 2$. Thus, $\text{ecc}(A) = 2$.

Let $A \in V(\Gamma)$ with $|A| < n - 1$. Then there exists a vertex $D \in V(\Gamma)$ such that $A \cap D = \varnothing$ and $A \cup D \neq X$, and it is clear that $d(A, B) \geq 2$. Assume that there is a vertex $E \in V(\Gamma)$ such that $A - E - D$ in Γ. Then $\overline{A} \subseteq E$ and $\overline{D} \subseteq E$, and so $E \supseteq \overline{A} \cup \overline{D} = \overline{A \cap D} = X$, which is a contradiction. Thus, we have $d(A, D) \geq 3$ and so $\text{ecc}(A) \geq 3$. As we said before, since the diameter of the zero-divisor graph of any commutative semigroup with zero is at most 3 (see Theorem 1.2 in [7]), it follows that $\text{ecc}(A) = 3$. Thus, $\text{rad}(\Gamma) = 2$ and $\text{diam}(\Gamma) = 3$. □

Moreover, we have the following immediate corollary.

Corollary 2.2. *If $|X| = n \geq 3$ then*

$$C(\Gamma) = \{A \in V(\Gamma) \mid |A| = n - 1\}.$$

Lemma 2.3. *Let $|X| = n \geq 2$ and $A \in V(\Gamma)$. If $|A| = r$ $(1 \leq r \leq n - 1)$ then* $\deg_\Gamma(A) = 2^r - 1$.

Proof Let $|X| \geq 2$ and $A \in V(\Gamma)$ with $|A| = r$. For $B \in V(\Gamma)$, since $A - B \in E(\Gamma)$ if and only if $\overline{A} \subseteq B \subsetneq X$, there exists a proper subset Y of A such that $B = \overline{A} \cup Y$, and so $\deg_\Gamma(A) = 2^r - 1$. □

Corollary 2.4. *Let $|X| = n \geq 2$ and $1 \leq r \leq n - 1$. In Γ there are $\binom{n}{r}$ vertices whose vertex degrees are $2^r - 1$. Moreover, $\Delta(\Gamma) = 2^{n-1} - 1$ and $\delta(\Gamma) = 1$.* □

Theorem 2.5. *(i) If $|X| = 2$ then $\gamma(\Gamma) = 1$ and if $|X| = n \geq 3$ then $\gamma(\Gamma) = n$.*

(ii) If $|X| = n \geq 2$ then $\alpha(\Gamma) = 2^{n-1} - 1$.

Proof (i) It is clear that $\gamma(\Gamma) = 1$ when $|X| = 2$. Let $|X| = n \geq 3$ and D be a dominating set of Γ. For each $k \in X$ since the vertex degree of $\{k\}$ is 1, equivalently $N_\Gamma[\{k\}] = \{X \setminus \{k\}, \{k\}\}$, and since $|N_\Gamma[\{k\}] \cap D| \geq 1$, either $\{k\} \in D$ or $X \setminus \{k\} \in D$. Moreover, for any $i, j \in X$ with $i \neq j$, since $|X| \geq 3$, we have $N_\Gamma[\{i\}] \cap N_\Gamma[\{j\}] = \varnothing$. Thus, $|D| \geq n$. Now we consider the set

$$D = \{X \setminus \{k\} \mid k \in X\}.$$

It is clear that $|D| = n$ and D is a dominating set, and so $\gamma(\Gamma) = n$.

(ii) Let $|X| = n \geq 2$, $i \in X$ and let $B = X \setminus \{i\}$. Then consider the subsets

$$P(B) = \{Y \mid \varnothing \neq Y \subseteq B\} \text{ and } Q(B) = \{X \setminus Y \mid Y \in P(B)\}$$

of $V(\Gamma)$. Notice that $i \notin Y$ for each $Y \in P(B)$, and it follows that $i \in Z$ for each $Z \in Q(B)$. Thus, $P(B) \cap Q(B) = \varnothing$ and $|P(B)| = |Q(B)| = 2^{n-1} - 1$, and it follows that $P(B) \cup Q(B) = V(\Gamma)$. If $A \subseteq V(\Gamma)$ is an independent set, then from the pigeonhole principle, $|A| \leq 2^{n-1} - 1$. (Otherwise, A must contain both Y and $X \setminus Y$ for some Y in $P(B)$, which contradicts the independence of A.) Moreover, since $P(B)$ is an independent set in Γ, then $\alpha(\Gamma) = 2^{n-1} - 1$. □

3. Perfectness of $\Gamma(SL_X)$

Let G be a graph. Each of the maximal complete subgraphs of G is called a clique. The number of all the vertices in any clique of G, denoted by $\omega(G)$, is called a clique number. There exists another graph parameter, namely the chromatic number. It is the minimum number of colors needed to assign the vertices of a graph G such that no two adjacent vertices have the same color and it is denoted by $\chi(G)$. It is well known that

$$\chi(G) \geq \omega(G) \tag{1}$$

for any graph G (see Corollary 6.2 in [4]). Moreover, let $V^{'} \subseteq V(G)$. Then the induced subgraph $G^{'} = (V^{'}, E^{'})$ is a subgraph of G such that $E^{'}$ consists of those edges whose endpoints are in $V^{'}$. For each induced subgraph H of G, if $\chi(H) = \omega(H)$, then G is called a perfect graph.

The complement or inverse of a simple graph G is a simple graph on the same vertices such that two distinct vertices are adjacent with a single edge if and only if they are not adjacent in G and it is denoted by G^c. A graph G is called Berge if no induced subgraph of G is an odd cycle of length of at least five or the complement of one.

The edges are called adjacent if they share a common end vertex. An edge coloring of a graph is an assignment of colors to the edges of G such that no two adjacent edges have the same color. The minimum required number of colors for and the edge coloring of G is called the chromatic index of G and is denoted by $\chi^{'}(G)$. A fundamental theorem due to Vizing states that, for any graph G, we have

$$\Delta(G) \leq \chi^{'}(G) \leq \Delta(G) + 1$$

(see [11]). Graph G is called class-1 if $\Delta(G) = \chi^{'}(G)$ and class-2 if $\chi^{'}(G) = \Delta(G) + 1$.

The core of a graph G is defined to be the largest induced subgraph of G such that each edge in the core is part of a cycle and it is denoted by G_{Δ}. Finally, let M be a subset of $E(G)$ for a graph G; if there are no two edges in M that are adjacent, then M is called a matching.

Theorem 3.1. *If* $|X| = n \geq 2$ *then* $\omega(\Gamma) = \chi(\Gamma) = n$.

Proof Without loss of generality suppose that $X = \{1, 2, \ldots, n\}$. Let $A_i = X \setminus \{i\}$ for each $i \in X$, and let Π be the induced subgraph by the subset $\{A_i \mid i \in X\} \subseteq V(\Gamma)$. Then it is clear that Π is a complete graph with n vertices, and so $\omega(\Gamma) \geq n$.

On the other hand, let

$$\begin{aligned}\mathcal{P}_1 &= \{B \mid \varnothing \neq B \subseteq A_1\},\\ \mathcal{P}_2 &= \{B \mid \varnothing \neq B \subseteq A_2 \text{ and } B \notin \mathcal{P}_1\},\\ &\vdots\\ \mathcal{P}_n &= \{B \mid \varnothing \neq B \subseteq A_n \text{ and } B \notin \bigcup_{i=1}^{n-1} \mathcal{P}_i\}.\end{aligned}$$

Then it is easy to see that $\bigcup_{i=1}^{n} \mathcal{P}_i = V(\Gamma)$. It is also easy to see that $B \in \mathcal{P}_1$ if and only if $1 \notin B$, and for each $2 \leq k \leq n$, $B \in \mathcal{P}_k$ if and only if $1, \ldots, k-1 \in B$, but $k \notin B$. Thus, $\mathcal{P}_i \neq \varnothing$ for each $1 \leq i \leq n$ and $\mathcal{P}_i \cap \mathcal{P}_j = \varnothing$ for each $1 \leq i \neq j \leq n$.

For each $1 \leq k \leq n$, if we choose a different color for each $\mathcal{P}_k$ and assign the chosen color to the all vertices in $\mathcal{P}_k$, there are no two adjacent vertices that have the same color, and so $\chi(\Gamma) \leq n$.

Since $n \geq \chi(\Gamma)$ and $\omega(\Gamma) \geq n$, it follows from equation (1) that

$$\chi(\Gamma) = \omega(\Gamma) = n,$$

as required. □

Lemma 3.2. *[5] A graph is perfect if and only if it is Berge.* □

Therefore, a graph G is perfect if and only if neither G nor G^c contains an odd cycle of length of at least 5 as an induced subgraph.

Theorem 3.3. *Γ is a perfect graph if $|X| = 2, 3$, or 4, but Γ is not a perfect graph if $|X| \geq 5$.*

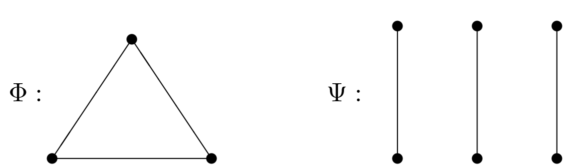
Proof For $|X| = 2$, it is clear.

For $|X| = 3$ or 4, we assume that there exists an induced subgraph of Γ that is an odd cycle with $2m-1$ vertices where $m \geq 3$, say

$$C_1 - C_2 - \cdots - C_{2m-1} - C_1.$$

Since $C_i \neq X$, it is clear that $|C_i| = 2$ for each $1 \leq i \leq 2m-1$ for $|X| = 3$. Similarly for $|X| = 4$, it is clear that $|C_i| \geq 2$ for each $1 \leq i \leq 2m-1$. Moreover, if $|C_i| = 3$ for any $1 \leq i \leq 2m-1$, without loss of generality, say $|C_1| = 3$, then neither C_3 nor C_4 must include $X \setminus C_1$. On the other hand, since C_3 and C_4 are adjacent vertices, one of them must contain $X \setminus C_1$, which is a contradiction. Thus, $|C_i| = 2$ for each $1 \leq i \leq 2m-1$.

Suppose that the subgraphs of Γ induced by the set of all the vertices whose cardinality is 2 are Φ and Ψ for $|X| = 3$ and $|X| = 4$, respectively. Then we have



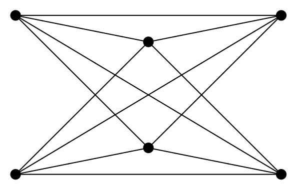
so the result is clear.

Similarly, for $|X|=3$ or 4, we assume that there exists an induced subgraph of Γ^c that is an odd cycle with $2m-1$ vertices where $m\geq 3$, say

$$C_1 - C_2 - \cdots - C_{2m-1} - C_1.$$

For $|X|=3$ or 4, $|C_i|\geq 2$ for each $1\leq i\leq 2m-1$; otherwise, if $|C_i|=1$ for any $1\leq i\leq 2m-1$, then all other vertices are adjacent to C_i except $X\setminus C_i$ in Γ^c. Thus, $|C_i|=2$ for each $1\leq i\leq 2m-1$ for $|X|=3$. Now we show that $|C_i|=2$ for each $1\leq i\leq 2m-1$ for $|X|=4$. If $|C_i|=3$ for any $1\leq i\leq 2m-1$, without loss of generality, say $|C_1|=3$. Then C_2 and C_{2m-1} must be subsets of C_1. It follows that C_2 and C_{2m-1} are adjacent vertices in Γ^c, which is a contradiction. Thus, $|C_i|=2$ for each $1\leq i\leq 2m-1$. For $|X|=3$ it is clear that the subgraph of Γ^c induced by the set of all the vertices with cardinality 2 is the null graph with 3 vertices. For $|X|=4$, if Ω is the subgraph of Γ^c induced by the set of all the vertices with cardinality 2, then we have

Ω :



Since all the vertices in Ω have degree 4, it follows that there does not exist an induced subgraph that is a cycle with 5 vertices. Therefore, Γ is a perfect graph if $|X|=2,3$, or 4.

For $|X|=n\geq 5$, without loss of generality, suppose that $X=\{1,2,\ldots,n\}$, $Y=X\setminus\{1,2,3,4,5\}$, and H is the subgraph induced by the vertex set

$$\{\{1,2,3\}\cup Y,\{1,4,5\}\cup Y,\{2,3,5\}\cup Y,\{1,3,4\}\cup Y,\{2,4,5\}\cup Y\}.$$

Then it is clear that H is a cycle graph of length 5 with the cycle

$$\{1,2,3\}\cup Y-\{1,4,5\}\cup Y-\{2,3,5\}\cup Y-\{1,3,4\}\cup Y-\{2,4,5\}\cup Y-\{1,2,3\}\cup Y.$$

Thus, Γ is not a perfect graph if $|X|\geq 5$. □

Lemma 3.4. *[10] Consider the graphs $G_1=(V,E_1)$ and $G_2=(V,E_2)$ with the same vertex set. Suppose that E_1 is a matching such that no edge has both end vertices in $N_{G_2}[\Lambda_{G_2}]$. If the union graph $G=G_1\cup G_2$ has maximum degree $\Delta(G)=\Delta(G_2)+1$, then G is class-1.* □

Now we consider the core of Γ. Notice that, from the proof of Theorem 2.1, Γ_Δ is the subgraph of Γ induced by the vertex set $\{A\in V(\Gamma)\mid |A|\geq 2\}$.

Theorem 3.5. *If $|X|\geq 2$ then $\chi'(\Gamma)=2^{n-1}-1$.*

Proof It is clear for $|X|=2$. For $|X|=n\geq 3$, consider the graphs

$$G_1 = (V(\Gamma),\, B) \qquad \text{and}$$
$$G_2 = (V(\Gamma),\, E(\Gamma_\Delta))$$

where $B=\{\{i\}-(X\setminus\{i\})\mid 1\leq i\leq n\}$. Thus, B is a matching such that no edge has both end vertices in $N_{G_2}[\Lambda_{G_2}]=V(\Gamma_\Delta)$. Since $\Gamma=G_1\cup G_2$ and $\Delta(\Gamma)=\Delta(G_2)+1$, it follows from Lemma 3.4 that Γ is class-1. □

4. Hamiltonian subgraphs of $\Gamma(SL_X)$

A cycle that travels exactly once over each vertex in a graph is called a Hamiltonian cycle. A graph is called a Hamiltonian graph if it has a Hamiltonian cycle. Since all degrees of all vertices in a Hamiltonian graph are at least 2, Γ is not a Hamiltonian graph. However, we may consider the Γ_Δ in the following theorem.

Theorem 4.1. *Γ_Δ is a Hamiltonian graph if $|X| = 3, 4$, or 5, but Γ_Δ is not a Hamiltonian graph if $|X| \geq 6$.*

Proof Without loss of generality suppose that $X = \{1, 2, \ldots, n\}$. If $|X| = 3$ then

$$\{1,2\} - \{1,3\} - \{2,3\} - \{1,2\}$$

is Hamiltonian a cycle in Γ_Δ. If $|X| = 4$ then

$$\{1,2\} - \{3,4\} - \{1,2,4\} - \{1,3\} - \{2,4\} - \{1,3,4\} - \{2,3\} -$$
$$\{1,4\} - \{1,2,3\} - \{2,3,4\} - \{1,2\}$$

is a Hamiltonian cycle in Γ_Δ. If $|X| = 5$ then

$$\{3,4\} - \{1,2,5\} - \{3,4,5\} - \{1,2\} - \{1,3,4,5\} - \{2,5\} - \{1,3,4\} -$$
$$\{2,4,5\} - \{1,3\} - \{1,2,4,5\} - \{2,3\} - \{1,4,5\} - \{1,2,3\} - \{4,5\} -$$
$$\{1,2,3,4\} - \{3,5\} - \{1,2,4\} - \{2,3,5\} - \{1,4\} - \{2,3,4,5\} - \{1,5\} -$$
$$\{2,3,4\} - \{1,3,5\} - \{2,4\} - \{1,2,3,5\} - \{3,4\}$$

is a Hamiltonian cycle in Γ_Δ. Therefore, Γ_Δ is a Hamiltonian graph for $|X| = 3, 4$ or 5.

Suppose that $|X| \geq 6$. Then consider the subsets

$$\begin{aligned} \mathcal{A} &= \{U \in V(\Gamma) \mid |U| = 2\}, \\ \mathcal{B} &= \{T \in V(\Gamma) \mid |T| = n-2\}, \qquad \text{and} \\ \mathcal{C} &= \{W \in V(\Gamma) \mid |W| = n-1\} \end{aligned}$$

of $V(\Gamma)$. Notice that, for any $U \in \mathcal{A}$, each adjacent vertex of U must be in $\mathcal{B} \cup \mathcal{C}$, and that, if $T \in \mathcal{B}$ is an adjacent vertex, then $T = X \setminus U$. Now suppose that Γ_Δ is a Hamiltonian graph. Then we have a Hamiltonian cycle in Γ_Δ of the form

$$U_1 - Y_1 - \cdots - Z_1 - U_2 - Y_2 - \cdots - Z_{k-1} - U_k - Y_k - \cdots - Z_k - U_1$$

, where $U_i \in \mathcal{A}$; $Y_i, Z_i \in \mathcal{B} \cup \mathcal{C}$ for $1 \leq i \leq k = \binom{n}{2}$. Since $|\mathcal{C}| = n$, there are at most n pairs (Y_i, Z_i) such that $Y_i = Z_i \in \mathcal{C}$ for $1 \leq i \leq k$. Since we need at least $n + 2\left(\binom{n}{2} - n\right)$ vertices from $\mathcal{B} \cup \mathcal{C}$, then we have

$$n + 2\left(\binom{n}{2} - n\right) \leq n + \binom{n}{2} = |\mathcal{B} \cup \mathcal{C}|,$$

which contradicts $|X| = n \geq 6$. Therefore, Γ_Δ is not a Hamiltonian graph for $|X| \geq 6$. $\square$

Open Problem 4.2. *What is the maximal cycle length in Γ for $|X| \geq 6$?*

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