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Veronese transform and Castelnuovo–Mumford regularity of modules

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Abstract: Veronese rings, Segre embeddings, or more generally Segre–Veronese embeddings are very important rings in algebraic geometry. In this paper we present an original, elementary way to compute the Hilbert–Poincaré series of these rings; as a consequence we compute their Castelnuovo–Mumford regularity and also the highest graded Betti number. Moreover, using the Castelnuovo–Mumford regularity of a Cohen–Macaulay finitely generated graded module, we compute that of its Veronese transforms.

Key words: Castelnuovo–Mumford regularity, Veronese ring, Segre ring, Hilbert Series

1. Introduction

Veronese rings, Segre embeddings, or more generally Segre–Veronese embeddings are very important rings in algebraic geometry. It is well known that these rings are arithmetically Cohen–Macaulay; hence their Hilbert–Poincaré series can be written: $P_R(t) = \frac{Q_R(t)}{(1-t)^{\dim R}}$, where $Q_R(t)$ is a polynomial on t with $Q_R(1) \neq 0$ having positive integer coefficients; the sequence of the coefficients of $Q_R(t)$ is also called the $h-$vector of R. The degree of $Q_R(t)$ is the Castelnuovo–Mumford regularity (c.f.[5][Chapter 4]), and the coefficient of the leading term of $Q_R(t)$ is the highest graded Betti number of R. By using very original and elementary methods we are able to compute the leading term of $Q_R(t)$. Our results allow to compute the Castelnuovo–Mumford regularity of the n Veronese module of any finitely generated Cohen–Macaulay graded module, and the rings called of Veronese type. Note that this result can be proved easily by using local cohomology, but our purpose is to give a very elementary proof.

Our main results improve partially [1] and [4].

Theorem. *Let consider the Segre–Veronese ring* $R_{\underline{b},\underline{n}}$, $\dim R_{\underline{b},\underline{n}} = b_1 + \cdots + b_m + 1$. *Let* $P_{R_{\underline{b},\underline{n}}}(t) = \frac{Q_{R_{\underline{b},\underline{n}}}}{(1-t)^{\dim R_{\underline{b},\underline{n}}}}$ *be the Hilbert–Poincaré series of* $R_{\underline{b},\underline{n}}$, *with* $Q_{R_{\underline{b},\underline{n}}} = h_0 + h_1 t + \ldots + h_{r_{\underline{b},\underline{n}}} t^{r_{\underline{b},\underline{n}}}$, *where* $r_{\underline{b},\underline{n}}$ *is the Castelnuovo–Mumford regularity of* $R_{\underline{b},\underline{n}}$. *We set* $\alpha_{\underline{b},\underline{n}} = \dim R_{\underline{b},\underline{n}} - r_{\underline{b},\underline{n}}$. *After a permutation of* $b_1, \ldots, b_m$, *we can assume that for all* $i = 1, \ldots, m$, $\lceil \frac{b_1+1}{n_1} \rceil > \frac{b_i}{n_i}$. *Then*

$$\alpha_{\underline{b},\underline{n}} = \lceil \frac{b_1+1}{n_1} \rceil ,$$

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and the highest graded Betti number of $R_{\underline{b},\underline{n}}$ *is*

$$\beta_{r_{\underline{b},\underline{n}}} = h_{r_{\underline{b},\underline{n}}} = \binom{n_1\alpha_{\underline{b},\underline{n}} - 1}{b_1} \cdots \binom{n_m\alpha_{\underline{b},\underline{n}} - 1}{b_m}.$$

In fact we get a more general statement about a class of formal powers series:

Theorem. *Fix integers* $d, n \in \mathbb{N}^*, \tau \in \mathbb{Z}$. *Let* $(a_l)_{l\in\mathbb{Z}}$ *be a sequence of complex numbers, such that* $a_l = 0$ *for* $l << 0$, *set:*

$$f(t) = \sum_{l\in\mathbb{Z}} a_l t^l, \quad f^{<n,\tau>}(t) = \sum_{l\in\mathbb{Z}} a_{nl+\tau} t^l.$$

If $f(t) = \frac{h(t)}{(1-t)^d}$ *with* $h(t) \in \mathbb{C}[t, t^{-1}]$ *then* $f^{<n,\tau>}(t) = \frac{h^{<n>}(t)}{(1-t)^d}$ *for some* $h^{<n>}(t) \in \mathbb{C}[t, t^{-1}]$ *such that:*

- $\deg h^{<n,\tau>}(t) \leq d - \lceil \frac{d - \deg h(t) + \tau}{n} \rceil$,
- *If all the coefficients of* $h(t)$ *are positive real numbers then* $\deg h^{<n,\tau>}(t) = d - \lceil \frac{d-\deg h(t)+\tau}{n} \rceil$,
- *If* $\deg h(t) = d$ *then* $\deg h^{<n>}(t) = d$.

2. Preliminaries on toric rings and Hilbert–Poincaré series

Let $R = K[x_0, \ldots, x_b, x_0^{-1}, \ldots, x_b^{-1}]$ be a Laurent polynomial ring over a field K on a finite set of variables. For any finite set $\mathcal{M}$ of monomials in R, let $K[\mathcal{M}] \subset R$ be the subring of R generated by the set $\mathcal{M}$. It is the toric ring defined by the semigroup generated by $\mathcal{M}$. In what follows we consider the special case where $R = K[x_0, \ldots, x_b]$ is a polynomial ring over the field K and all the monomials in $\mathcal{M}$ are of the same degree.

Example 2.1. *Let* $n \in \mathbb{N}^*$, $R = K[x_0, \ldots, x_b] = \oplus_{l\in\mathbb{N}} R_l$, *and* $\mathcal{M} = \{x_0^{\alpha_0} \ldots x_b^{\alpha_b} \mid \alpha_0 + \ldots + \alpha_b = n\}$. *So that*

$$R_{b,n} := K[\mathcal{M}] = \oplus_{l\in\mathbb{N}} R_{nl}.$$

This toric ring is known as the $n-$ *Veronese embedding of* R.

Example 2.2. *More generally, let* $X_1, \ldots, X_m$, m *sets of independent disjoint variables, with* $\mathrm{Card}(X_i) = b_i + 1$. *Let* $R(i) = K[X_i]$ *for* $i = 1, \ldots, m$, $R = K[X_1 \cup X_2 \cup \ldots \cup X_m]$, *and* $\mathcal{M} = \{x_1 x_2 \ldots x_m \mid x_i \in X_i\}$. *So that*

$$R_{b_1,\ldots,b_m} := K[\mathcal{M}] = \oplus_{l\in\mathbb{N}} (R(1))_l \otimes \ldots \otimes (R(m))_l.$$

This toric ring is known as the Segre embedding of the m *polynomial rings* $R(1), \ldots, R(m)$.

Example 2.3. *Let* $X_1, \ldots, X_m$, *sets of independent disjoint variables such that* $X_i = \{x_{i,0}, \ldots, x_{i,b_i}\}$, $R(i) = K[X_i]$ *for* $i = 1, \ldots, m$, *and* $n_1, \ldots, n_m \in \mathbb{N}$. *Let* $R = K[X_1 \cup X_2 \cup \ldots \cup X_m]$, *and*

$$\mathcal{M} = \{\underline{x}_1^{\alpha_1} \ldots \underline{x}_m^{\alpha_m} \mid\mid \alpha_i \mid= n_i\},$$

where $\alpha_i = (\alpha_{i,0}, \ldots, \alpha_{i,b_i})$, $\underline{x}_i^{\alpha_i} = x_{i,0}^{\alpha_{i,0}} \ldots x_{i,b_i}^{\alpha_{i,b_i}}$, *and* $\mid \alpha_i \mid= \alpha_{i,0} + \ldots + \alpha_{i,b_i}$ *The Segre–Veronese embedding is defined by:*

$$R_{\underline{b},\underline{n}} = K[\mathcal{M}] = \oplus_{l\in\mathbb{N}} (R(1))_{n_1 l} \otimes \ldots \otimes (R(m))_{n_m l},$$

where $\underline{b} = (b_1, \ldots, b_m), \underline{n} = (n_1, \ldots, n_m)$.

Let $R = K[x_0, \ldots, x_s]$ be a polynomial ring over the field K, graded by the standard graduation, that is $\deg x_i = 1$, for all i. Let $T := R/I$, where $I \subset R$ is a graded ideal, and let $M = \oplus_{l\in\mathbb{Z}} M_l$ be a finitely generated graded T-module; hence M is also an $R-$ module. The Hilbert function of M is defined by $H_M(l) = \dim_K M_l$, for all $l \in \mathbb{Z}$, and the Hilbert–Poincaré series of M:

$$P_M(t) = \sum_{l\in\mathbb{Z}} H_M(l)t^l.$$

It is well known that

$$P_M(t) = \frac{Q_M(t)}{(1-t)^{\dim M}}$$

where $Q_M(t)$ is a Laurent polynomial on t, t^{-1} with $Q_M(1) \neq 0$. Moreover, if M is a Cohen–Macaulay R-module, all the coefficients of $Q_M(t)$ are natural integers, and the Castelnuovo–Mumford regularity of M is the degree of $Q_M(t)$. For more details on Hilbert–Poincaré series see [7], [3][Chapter 4], [5][Chapter 4].

Theorem 2.4. *(Hilbert's Theorem) let $M = \oplus_{l\in\mathbb{Z}} M_l$ be a finitely generated graded R-module. There exists a polynomial with integer coefficients $\Phi_{H_M}(l)$ such that $H_M(l) = \Phi_{H_M}(l)$, for l large enough. Moreover, the leading term of $\Phi_{H_M}(l)$ can be written as: $\frac{\deg(M)}{d!} l^d$, where $d+1$ is the dimension of M and $\deg(M)$ is the degree or multiplicity of M.*

Remark 2.5. *The postulation number of the Hilbert function is the biggest integer l such that $H_M(l) \neq \Phi_{H_M}(l)$. It is well known ([7], [3][Chapter 4]) that the postulation number equals the degree of the rational fraction defining the Poincaré series.*

Remark 2.6. *We recall that binomial coefficients can be defined in a more general setting than natural numbers; indeed for $k \in \mathbb{N}$, binomial coefficients are polynomial functions in the variable n. More precisely:*

(1) *If $k = 0$ then let $\binom{n}{0} = 1$, for all $n \in \mathbb{C}$.*

(2) *If $k > 0$ then let $\binom{n}{k} = \frac{n(n-1)\ldots(n-k+1)}{k!}$, for all $n \in \mathbb{C}$.*

Note that for all $n \in \mathbb{C}$, $\binom{n}{k} = (-1)^k \binom{k-n-1}{k}$ and if $n \in \mathbb{N}, n < k$, then $\binom{n}{k} = 0$.

Example 2.7. *Let $R = K[x_0, \ldots, x_b]$, be a polynomial ring. Then*

$$H_R(l) = \begin{cases} \binom{l+b}{b} & \text{if } l \geq 0 \\ 0 & \text{if } l < 0 \end{cases}, \quad P_R(t) = \frac{1}{(1-t)^{b+1}}.$$

Note that in fact $\forall l \geq -b$, $H_R(l) = \binom{l+b}{b}$ and $0 = H_R(-b-1) \neq \binom{-b-1+b}{b} = (-1)^b$, and so the postulation number of R is $-(b+1)$.

Example 2.8. *Let $R = K[x_0, \ldots, x_b]$, $\mathcal{M} = \{x_0^{\alpha_0} \ldots x_b^{\alpha_b} \mid \alpha_0 + \ldots + \alpha_b = n\}$, and $R_{b,n} = K[\mathcal{M}]$ the $n-$ Veronese embedding. Then*

$$H_{R_{b,n}}(l) = H_R(nl) = \begin{cases} \binom{nl+b}{b} & \text{if } l \geq 0 \\ 0 & \text{if } l < 0 \end{cases}.$$

Note that $\binom{nl+b}{b} = \frac{(nl+1)(nl+2)\dots(nl+b)}{b!}$ *is a polynomial on* l *with leading term* $\frac{n^b l^b}{b!}$, *so that* $\deg(R_{b,n}) = n^b, \dim R_{b,n} = b+1$. *Note also that* $\forall l > -\lceil \frac{b+1}{n} \rceil$, $H_{R_{b,n}}(l) = \binom{nl+b}{b}$ *and* $0 = H_{R_{b,n}}(-\lceil \frac{b+1}{n} \rceil) \neq \binom{-\lceil \frac{b+1}{n} \rceil n+b}{b} = (-1)^b \binom{\lceil \frac{b+1}{n} \rceil n-1}{b}$, *and so the postulation number of* $R_{b,n}$ *is* $-\lceil \frac{b+1}{n} \rceil$. *More generally, let* $R_{b,n}[\tau] := \oplus_{l\in\mathbb{N}} R_{nl+\tau}$. *Note that* $H_{R_{b,n}[\tau]}(l) = \binom{nl+\tau+b}{b}$ *for* $nl+\tau+b \geq 0$, *and* $H_{R_{b,n}[\tau]}(l) = 0 \neq \binom{nl+\tau+b}{b}$ *for* $nl+\tau+1+b \leq 0$. *Hence the postulation number of* $R_{b,n}[\tau]$ *is* $-\lceil \frac{b+1+\tau}{n} \rceil$.

3. Veronese of generating series

In a recent paper [2], Brenti and Welker prove that taking the $n-$Veronese transform of the h polynomial is a linear function; in this section we improve this result, giving an elementary proof of the fact that taking the shifted $n-$ Veronese transform of the h polynomial is a linear function on h.

Let us recall the following fact:

Theorem 3.1. *Let* $(a_l)_{l\in\mathbb{Z}}$ *be a sequence of complex numbers, such that* $a_l = 0$ *for* $l << 0$, *set:* $f(t) = \sum_{l\in\mathbb{Z}} a_l t^l$, *TFAE:*

- *There exists* $h(t) \in \mathbb{C}[t, t^{-1}]$ *and a natural integer* d *such that* $f(t) = \frac{h(t)}{(1-t)^d}$.
- *There exists* $\Phi(t) \in \mathbb{C}[t, t^{-1}]$ *of degree* $d-1$ *with leading coefficient* $e_0/(d-1)!$, *such that* $\Phi(l) = a_l$ *for* l *large enough.*

Moreover, $h(1) = e_0$.

Let us introduce some notations.

Notation 3.2. *Fix integers* $d, n \in \mathbb{N}^*, \tau \in \mathbb{Z}$. *Let* $(a_l)_{l\in\mathbb{Z}}$ *be a sequence of complex numbers, such that* $a_l = 0$ *for* $l << 0$, *set:*

$$f(t) = \sum_{l\in\mathbb{Z}} a_l t^l, \quad f^{<n,\tau>}(t) = \sum_{l\in\mathbb{Z}} a_{nl+\tau} t^l.$$

By the Theorem 3.1 if $f(t) = \frac{h(t)}{(1-t)^d}$ *with* $h(t) \in \mathbb{C}[t, t^{-1}]$ *then* $f^{<n,\tau>}(t) = \frac{h^{<n,\tau>}(t)}{(1-t)^d}$ *for some* $h^{<n,\tau>}(t) \in \mathbb{C}[t, t^{-1}]$. *In Theorem 3.5 we will prove that* $h^{<n,\tau>}(t)$ *can be written in terms of* $h(t)$. *To any nonzero polynomial* $h(t) = h_\sigma t^\sigma + \dots + h_0 + h_1 t + \dots + h_s t^s \in \mathbb{C}[t, t^{-1}]$ *we associate the* h*-vector* $\vec{h} = (\dots, 0, h_\sigma, \dots, h_s, 0, \dots)$, *and we set* $\deg \overrightarrow{h} = \deg h(t)$. *For* $j \in \mathbb{Z}$, *let* $\overrightarrow{\varepsilon_j}$ *be the* h*-vector of the polynomial* t^j. *Let us denote by* $[t^k]h(t)$ *the coefficient of* t^k *in the polynomial* $h(t)$. *For any* $i, j \in \mathbb{Z}$ *define* $\mathcal{D}_{i,j}$ *by*

$$\mathcal{D}_{i,j} = [t^{in-j}]\left(\frac{(1-t^n)^d}{(1-t)^d}\right) = [t^{in-j}]((1+t+\dots+t^{n-1})^d).$$

Note that

$$\mathcal{D}_{i,j} = \mathrm{Card}\{(x_1, \dots, x_d) \in \mathbb{N}^d \mid \forall l, x_l \leq n-1; x_1 + \dots + x_d = in - j\}.$$

Finally let $\mathcal{D}[\sigma, \tau]$ *be the infinite square matrix* $\mathcal{D}[\sigma, \tau] = (\mathcal{D}_{i+\sigma, j+\tau})$. *For* $\sigma = \tau = 0$ *we write* $\mathcal{D}$ *instead of* $\mathcal{D}[0, 0]$.

We can give some properties of the numbers $\mathcal{D}_{i,j}$.

Lemma 3.3. *Let $i,j,k \in \mathbb{Z}$; then we have:*

$\mathcal{D}_{i,j} = 0$ *if either* $in - j < 0$ *or* $in - j > d(n-1)$.

- *For any i,j, $\mathcal{D}_{i,j} = \mathcal{D}_{d-i,d-j}$. That is $\mathcal{D}$ is symmetrical around the point $(d/2, d/2)$.*
- *For $0 \le k \le n-1$, $\mathcal{D}_{d,d+k} = \binom{k+d-1}{d-1}$.*
- *$\mathcal{D}_{1,0} = \binom{n+d-1}{d-1} - d$, and for $1 \le k \le n$, $\mathcal{D}_{1,k} = \binom{n-k+d-1}{d-1}$.*
- *For any integers q,k, $\mathcal{D}_{d+q,nq+k} = \mathcal{D}_{d,k}$.*
- *For any i, let $d - i = nq - k$ with $q = \lceil \frac{d-i}{n} \rceil, 0 \le k < n$; then*

$$\mathcal{D}_{d-\lceil \frac{d-i}{n} \rceil, i} = \binom{k+d-1}{d-1} = \binom{n\lceil \frac{d-i}{n} \rceil + i - 1}{d-1}.$$

Proof The first claim is trivial. In order to prove the other claims, let us remark that the map $(x_1, \ldots, x_d) \mapsto (y_1, \ldots, y_d)$, where $y_l = (n-1) - x_l$ for $l = 1, \ldots, d$, establishes a bijection between

$$\{(x_1, \ldots, x_d) \in \mathbb{N}^d \mid x_l \le n-1 \text{ for } l = 1, \ldots, d; \quad x_1 + \ldots + x_d = in - j\}$$

and

$$\{(y_1, \ldots, y_d) \in \mathbb{N}^d \mid y_l \le n-1 \text{ for } l = 1, \ldots, d; \quad y_1 + \ldots + y_d = (d-i)n - (d-j)\}.$$

The third claim follows from the second claim, because if $0 \le k \le n-1$, then the sets

$$\{(x_1, \ldots, x_d) \in \mathbb{N}^d \mid x_l \le n-1 \text{ for } l = 1, \ldots, d; \quad x_1 + \ldots + x_d = dn - d - k\}$$

and

$$\{(y_1, \ldots, y_d) \in \mathbb{N}^d \mid \quad y_1 + \ldots + y_d = k\}$$

are in bijection.

The fourth claim follows trivially from the previous items.

The fifth claim follows from the equality: $(d+q)n - (nq+k) = dn - k$.

Finally the sixth claim follows from the third claim, since, if $d - i = nq - k$ with $0 \le k < n$, then $(d-q)n - i = dn - (d+k)$; hence $\mathcal{D}_{d-q,i} = \mathcal{D}_{d,d+k}$, and $n\lceil \frac{d-i}{n} \rceil + i - 1 = k + d - 1$. $\square$

Remark 3.4. *Let $(a_l)_{l \in \mathbb{Z}}$ be a sequence of complex numbers, such that $a_l = 0$ for $l << 0$, set:*

$$f(t) = \sum_{l \in \mathbb{Z}} a_l t^l.$$

Fix integers $d,k,n \in \mathbb{N}^*, \tau \in \mathbb{Z}$. With the notations introduced in 3.2, it is clear that $f^{<n,kn+\tau>}(t) = t^{-k} f^{<n,\tau>}(t)$, which implies $h^{<n,kn+\tau>}(t) = t^{-k} h^{<n,\tau>}(t)$ for any integer numbers k, τ.

The following theorem improves and has a simpler proof than that of [2, Theorem 1.1]:

Theorem 3.5. *Fix integers $d,k,n \in \mathbb{N}^*, \tau \in \mathbb{Z}$. Let $(a_l)_{l\in\mathbb{Z}}$ be a sequence of complex numbers, such that $a_l = 0$ for $l << 0$, set:*

$$f(t) = \sum_{l\in\mathbb{Z}} a_l t^l = \frac{h(t)}{(1-t)^d},$$

$$f^{<n,\tau>}(t) = \sum_{l\in\mathbb{Z}} a_{nl+\tau} t^l = \frac{h^{<n,\tau>}(t)}{(1-t)^d},$$

where $h(t), h^{<n,\tau>}(t) \in \mathbb{C}[t,t^{-1}]$. Then

$$\overrightarrow{h^{<n,kn+\tau>}} = \mathcal{D}[-k,-\tau]\vec{h}.$$

Proof Because of Remark 3.4 we have to compute $h^{<n,\tau>}(t)$ only for $0 \le \tau \le n-1$. The following formula is clear:

$$f^{<n,0>}(t^n) + tf^{<n,1>}(t^n) + \ldots + t^{n-1}f^{<n,n-1>}(t^n) = f(t);$$

hence

$$\frac{h^{<n,0>}(t^n) + th^{<n,1>}(t^n) + \ldots + t^{n-1}h^{<n,n-1>}(t^n)}{(1-t^n)^d} = \frac{h(t)}{(1-t)^d},$$

and

$$h^{<n,0>}(t^n) + th^{<n,1>}(t^n) + \ldots + t^{n-1}h^{<n,n-1>}(t^n) = h(t)\frac{(1-t^n)^d}{(1-t)^d},$$

$t^\tau h^{<n,\tau>}(t^n)$ equals the sum of all the terms $A_\beta t^\beta$ of $h(t)\frac{(1-t^n)^d}{(1-t)^d}$ with $\beta \equiv \tau \mod n$. In particular, $h^{<n,\tau>}(t)$ is a linear function of $h(t)$. Therefore, it is enough to compute $h^{<n,\tau>}(t)$ for the canonical basis $\{\varepsilon_j := t^j, j \in \mathbb{Z}\}$ of $\mathbb{C}[t,t^{-1}]$. We have

$$[t^i](h^{<n,\tau>}(t)) = [t^{ni+\tau}](h(t)\frac{(1-t^n)^d}{(1-t)^d});$$

hence

$$\forall j \in \mathbb{Z}; [t^i](\varepsilon_j^{<n,\tau>}(t)) = [t^{ni+\tau}](t^j)\frac{(1-t^n)^d}{(1-t)^d} = [t^{ni+\tau-j}](\frac{(1-t^n)^d}{(1-t)^d}),$$

which proves our statement. □

Corollary 3.6. *Fix an integer $d \in \mathbb{N}^*$. For $j \in \mathbb{Z}$, let $\vec{\varepsilon_j}$ be the h-vector of the polynomial t^j. Then for any $n \in \mathbb{N}^*$, we have $\deg \overrightarrow{\varepsilon_j^{<n>}} = d - \lceil \frac{d-j}{n} \rceil$. Moreover, the leading coefficient of $\overrightarrow{\varepsilon_j^{<n>}}$ is $\binom{n\lceil \frac{d-j}{n}\rceil + j - 1}{d-1}$.*

Proof Let us remark that the set of $t^j, j \in \mathbb{Z}$ is the canonical basis of $\mathbb{C}[t,t^{-1}]$. We have by Theorem 3.5 that $\mathcal{D}\vec{\varepsilon_j} = \overrightarrow{\varepsilon_j^{<n>}}$; hence $\overrightarrow{\varepsilon_j^{<n>}}$ is the j column vector of $\mathcal{D}$. By Example 2.8, we have that $\deg \overrightarrow{\varepsilon_j^{<n>}} = d - \lceil \frac{d-j}{n} \rceil$.

The last claim follows from Lemma 3.3. Indeed for any $j \in \mathbb{Z}$, we have $\mathcal{D}_{d-\lceil \frac{d-j}{n}\rceil, j} = \binom{n\lceil \frac{d-j}{n}\rceil + j - 1}{d-1}$. This proves that the leading coefficient of $\overrightarrow{\varepsilon_j^{<n>}}$ is $\binom{n\lceil \frac{d-j}{n}\rceil + j - 1}{d-1}$. □

Example 3.7. *Let* $d = 2$ *and* $n \in \mathbb{N}^*$*; we can describe the matrix* $\mathcal{D}$

i \ j	-(n+1)	...	-1	0	1	2	3	...	n	n+1	n+2	...	2n
-1	2	...	0	0	0	0	0	...	0	0	0	....	0
0	n-2	...	2	1	0	0	0	...	0	0	0	...	0
1	0	...	n-2	n-1	n	n-1	n-2	...	1	0	0	...	0
2	0	...	0	0	0	1	2	...	n-1	n	n-1	...	1
3	0	...	0	0	0	0	0	...	0	0	1	...	n-1

Theorem 3.8. *Fix integers* $d, n \in \mathbb{N}^*, \tau \in \mathbb{Z}$*. Let* $(a_l)_{l\in\mathbb{Z}}$ *be a sequence of complex numbers, such that* $a_l = 0$ *for* $l << 0$*, set :*

$$f(t) = \sum_{l\in\mathbb{Z}} a_l t^l, \quad f^{<n,\tau>}(t) = \sum_{l\in\mathbb{Z}} a_{nl+\tau} t^l.$$

If $f(t) = \frac{h(t)}{(1-t)^d}$ *with* $h(t) \in \mathbb{C}[t, t^{-1}]$ *then* $f^{<n,\tau>}(t) = \frac{h^{<n>}(t)}{(1-t)^d}$ *with* $h^{<n>}(t) \in \mathbb{C}[t, t^{-1}]$ *such that:*

- $\deg h^{<n,\tau>}(t) \le d - \lceil \frac{d-\deg h(t)+\tau}{n} \rceil$,
- *If all the coefficients of* $h(t)$ *are positive real numbers then* $\deg h^{<n,\tau>}(t) = d - \lceil \frac{d-\deg h(t)+\tau}{n} \rceil$,
- *If* $\deg h(t) = d$ *then* $\deg h^{<n>}(t) = d$.

Proof Let $f(t) = \sum_{l\in\mathbb{Z}} a_l t^l = \frac{h(t)}{(1-t)^d}$, where $h(t) \in \mathbb{C}[t, t^{-1}]$ $h(t) = \gamma_\sigma t^\sigma + \ldots + \gamma_s t^s$ with $\deg h(t) = s, \gamma_s \neq 0$. It follows that $\overrightarrow{h} = \sum_{l=\sigma}^{s} \gamma_l \overrightarrow{\varepsilon l}$. We multiply this relation on the left by $\mathcal{D}[-\tau]$, and so Theorem 3.5 implies $\overrightarrow{h^{<n,\tau>}} = \sum_{l=\sigma}^{s} \gamma_l \overrightarrow{\varepsilon^{<n>}_{l-\tau}}$. Since $\deg \overrightarrow{\varepsilon^{<n>}_{\sigma-\tau}} \le \deg \overrightarrow{\varepsilon^{<n>}_{\sigma-\tau+1}} \le \ldots \le \deg \overrightarrow{\varepsilon^{<n>}_{s-\tau}}$, we have, $\deg \overrightarrow{h^{<n,\tau>}} \le \deg \overrightarrow{\varepsilon^{<n>}_{s-\tau}}$. It is clear that if all the coefficients of $h(t)$ are positive real numbers then $\deg \overrightarrow{h^{<n,\tau>}} = \deg \overrightarrow{\varepsilon^{<n>}_{s-\tau}}$.

In the special case $s = d$, we have seen that for $0 \le l \le d-1$ and any $n \in \mathbb{N}^*$, $\deg \overrightarrow{\varepsilon^{<n>}_l} = d - \lceil \frac{d-l}{n} \rceil \le d-1$, and $\deg \overrightarrow{\varepsilon^{<n>}_d} = d$, which implies $\deg \overrightarrow{h^{<n>}} = d$. □

As an application of the main results of this section, the following relates invariants of a module and that of its Veronese transform.

Theorem 3.9. *Let* $n \in \mathbb{N}^*$*,* R *be a standard graded polynomial ring,* $M = \oplus_{l\in\mathbb{Z}} M_l$ *be a finitely generated Cohen–Macaulay graded* R*-module of dimension* $d \ge 1$*, and* $M^{<n>} = \oplus_{l\in\mathbb{Z}} M_{nl}$*. Let* $\dfrac{Q(t)}{(1-t)^d}$ *be the Hilbert–Poincaré series of* M*, where* $Q(t) = \gamma_\sigma t^\sigma + \ldots + \gamma_s t^s \in \mathbb{C}[t, t^{-1}]$ *is the* $h-$*polynomial of* M*, with* $\operatorname{reg}(M) = \deg Q(t) = s$*. Then*

- $\operatorname{reg} M^{<n>} = d - \lceil \frac{d-\operatorname{reg} M}{n} \rceil$. *Moreover, by taking the sum over all index* l *such that* $\lceil \frac{d-l}{n} \rceil = \lceil \frac{d-\operatorname{reg} M}{n} \rceil$*, we will get the leading coefficient of* $Q^{<n>}(t)$*:*

$$\sum_{l \,\mid\, \lceil \frac{d-l}{n} \rceil = \lceil \frac{d-\operatorname{reg} M}{n} \rceil} \gamma_l \binom{n\lceil \frac{d-l}{n} \rceil + l - 1}{d-1}.$$

- *If* $\operatorname{reg} M \leq d-1$ *and* $n \geq d$ *then* $\operatorname{reg} M^{<n>} = d-1$, *and the leading coefficient of* $Q^{<n>}(t)$ *is*

$$\sum_{l=0}^{d-1} \gamma_l \binom{n\lceil \frac{d-l}{n} \rceil + l - 1}{d-1}.$$

- *If* $n > \operatorname{reg} M \geq d$ *then* $\operatorname{reg} M^{<n>} = d$, *and the leading coefficient of* $Q^{<n>}(t)$ *is*

$$\sum_{l=d}^{\operatorname{reg} M} \gamma_l \binom{l-1}{d-1}.$$

Proof We have $\overrightarrow{Q} = \sum_{l=\sigma}^{s} \gamma_l \overrightarrow{\varepsilon_l}$. We multiply this relation on the left by $\mathcal{D}$, and so Theorem 3.5 implies that for any $n \in \mathbb{N}^*$, $\overrightarrow{Q^{<n>}} = \sum_{l=\sigma}^{s} \gamma_l \overrightarrow{\varepsilon_l^{<n>}}$. Since $\gamma_l \geq 0$ for all l, $\gamma_s > 0$, and $\deg \overrightarrow{\varepsilon_\sigma^{<n>}} \leq \deg \overrightarrow{\varepsilon_{\sigma+1}^{<n>}} \leq \ldots \leq \deg \overrightarrow{\varepsilon_s^{<n>}}$, we have $\deg \overrightarrow{Q^{<n>}} = \deg \overrightarrow{\varepsilon_s^{<n>}} = d - \lceil \frac{d - \operatorname{reg} M}{n} \rceil$; this number is $\operatorname{reg}(M^{<n>})$ since $M^{<n>}$ is a Cohen–Macaulay R-module. The computation of the leading coefficient of $Q^{<n>}(t)$ is immediate from Corollary 3.6. □

4. h-vector of the Segre–Veronese embedding

The next Theorem improves partially [1] and [4].

Theorem 4.1. *Let us consider the Segre–Veronese ring* $R_{\underline{b},\underline{n}}$, $\dim R_{\underline{b},\underline{n}} = b_1 + \ldots + b_m + 1$. *Let* $P_{R_{\underline{b},\underline{n}}}(t) = \dfrac{Q_{R_{\underline{b},\underline{n}}}(t)}{(1-t)^{\dim R_{\underline{b},\underline{n}}}}$ *be the Hilbert-Poincaré series of* $R_{\underline{b},\underline{n}}$, *with* $Q_{R_{\underline{b},\underline{n}}}(t) = h_0 + h_1 t + \ldots + h_{r_{\underline{b},\underline{n}}} t^{r_{\underline{b},\underline{n}}}$, *where* $r_{\underline{b},\underline{n}} = \deg Q_{R_{\underline{b},\underline{n}}}(t)$ *is the Castelnuovo–Mumford regularity of* $R_{\underline{b},\underline{n}}$. *We set* $\alpha_{\underline{b},\underline{n}} = \dim R_{\underline{b},\underline{n}} - r_{\underline{b},\underline{n}}$. *After a permutation of* $b_1, \ldots, b_m$, *we can assume that* $\lceil \frac{b_1+1}{n_1} \rceil > \frac{b_i}{n_i}$ $\forall i$; *then*

$$\alpha_{\underline{b},\underline{n}} = \lceil \frac{b_1+1}{n_1} \rceil, \quad r_{\underline{b},\underline{n}} = (b_1 + \ldots + b_m + 1) - \lceil \frac{b_1+1}{n_1} \rceil,$$

and the highest graded Betti number of $R_{\underline{b},\underline{n}}$ *is*

$$\beta_{r_{\underline{b},\underline{n}}} = h_{r_{\underline{b},\underline{n}}} = \binom{n_1 \alpha_{\underline{b},\underline{n}} - 1}{b_1} \cdots \binom{n_m \alpha_{\underline{b},\underline{n}} - 1}{b_m}.$$

Proof The proof is by double induction on m and b_m. The case $m=1$ is given by Example 2.8 and Corollary 3.6, and so we can assume $m \geq 2$. We have that $\lceil \frac{b_1+1}{n_1} \rceil > \frac{b_m}{n_m} > \frac{b_m - 1}{n_m}$, and so by induction hypothesis the theorem is true for $R_{\underline{b}-\epsilon_m,\underline{n}}$, where $\underline{b} - \epsilon_m = (b_1, \ldots, b_{m-1}, b_m - 1)$. On the other hand, the Hilbert function of $R_{\underline{b},\underline{n}}$ is $H_{R_{\underline{b},\underline{n}}}(l) = \binom{n_1 l + b_1}{b_1} \cdots \binom{n_m l + b_m}{b_m}$, and so

$$H_{R_{\underline{b},\underline{n}}}(l) = (1 + \frac{n_m}{b_m} l) H_{R_{\underline{b}-\epsilon_m,\underline{n}}}(l). \tag{1}$$

Let $P_{R_{\underline{b}-\epsilon_m,\underline{n}}}(t) = \dfrac{Q_{R_{\underline{b}-\epsilon_m,\underline{n}}}(t)}{(1-t)^{b_1+\ldots+b_m}}$ be the Hilbert–Poincaré series of $R_{\underline{b}-\epsilon_m,\underline{n}}$, where $Q_{R_{\underline{b}-\epsilon_m,\underline{n}}}(t) = h_0 + h_1 t + \ldots + h_{r_{\underline{b}-\epsilon_m,\underline{n}}} t^{r_{\underline{b}-\epsilon_m,\underline{n}}}$, with $h_{r_{\underline{b}-\epsilon_m,\underline{n}}} \neq 0$. In order to avoid any confusion we also set: $P_{R_{\underline{b},\underline{n}}}(t) =$

$\dfrac{Q_{R_{\underline{b},\underline{n}}}(t)}{(1-t)^{b_1+\ldots+b_m+1}}$ to be the Hilbert–Poincaré series of $R_{\underline{b},\underline{n}}$, where

$$Q_{R_{\underline{b},\underline{n}}}(t) = \hat{h}_0 + \ldots + \hat{h}_{r_{\underline{b},\underline{n}}} t^{r_{\underline{b},\underline{n}}},$$

with $\hat{h}_{r_{\underline{b},\underline{n}}} \neq 0$.

By simple calculations from (1) we get:

$$P_{R_{\underline{b},\underline{n}}}(t) = P_{R_{\underline{b}-\epsilon_m,\underline{n}}}(t) + \frac{n_m}{b_m} t P'_{R_{\underline{b}-\epsilon_m,\underline{n}}}(t). \tag{2}$$

Hence $\dim R_{\underline{b},\underline{n}} = \dim R_{\underline{b}-\epsilon_m,\underline{n}} + 1$, and $Q_{R_{\underline{b},\underline{n}}}(t)$ equals

$$Q_{R_{\underline{b}-\epsilon_m,\underline{n}}}(t) + t[Q_{R_{\underline{b}-\epsilon_m,\underline{n}}}(t)(\frac{n_m}{b_m} R_{\underline{b}-\epsilon_m,\underline{n}} - 1) + \frac{n_m}{b_m} Q'_{R_{\underline{b}-\epsilon_m,\underline{n}}}(t) - \frac{n_m}{b_m} t Q'_{R_{\underline{b}-\epsilon_m,\underline{n}}}(t)];$$

note that $Q_{R_{\underline{b},\underline{n}}}(1) = \frac{n_m}{b_m} \dim R_{\underline{b}-\epsilon_m,\underline{n}} Q_{R_{\underline{b}-\epsilon_m,\underline{n}}}(1) \neq 0$.

In particular, we have $r_{\underline{b},\underline{n}} \leq r_{\underline{b}-\epsilon_m,\underline{n}} + 1$ and for all $k = 0, \ldots, r_{\underline{b}-\epsilon_m,\underline{n}} + 1$ we have

$$\hat{h}_k = h_{k-1}(\frac{n_m}{b_m} \dim R_{\underline{b}-\epsilon_m,\underline{n}} - (k-1)\frac{n_m}{b_m} - 1) + h_k(k\frac{n_m}{b_m} + 1). \tag{3}$$

By induction hypothesis we have $\alpha_{\underline{b}-\epsilon_m,\underline{n}} = \lceil \frac{b_1+1}{n_1} \rceil \neq \frac{b_m}{n_m}$, and so we put $k = r_{\underline{b}-\epsilon_m,\underline{n}} + 1$ in equality (3), and we get:

$$\hat{h}_{r_{\underline{b}-\epsilon_m,\underline{n}}+1} = h_{r_{\underline{b}-\epsilon_m,\underline{n}}}(\frac{n_m \alpha_{\underline{b}-\epsilon_m,\underline{n}} - b_m}{b_m}) \neq 0.$$

Hence $\hat{h}_{r_{\underline{b}-\epsilon_m,\underline{n}}+1}$ is the leading coefficient of $Q_{R_{\underline{b},\underline{n}}}$ and $r_{\underline{b},\underline{n}} = r_{\underline{b}-\epsilon_m,\underline{n}} + 1$ and $\alpha_{\underline{b},\underline{n}} = \alpha_{\underline{b}-\epsilon_m,\underline{n}} = \lceil \frac{b_1+1}{n_1} \rceil$. By induction hypothesis

$$h_{r_{\underline{b}-\epsilon_m,\underline{n}}} = \binom{n_1\alpha_{\underline{b},\underline{n}} - 1}{b_1} \ldots \binom{n_{m-1}\alpha_{\underline{b},\underline{n}} - 1}{b_{m-1}} \binom{n_m\alpha_{\underline{b},\underline{n}} - 1}{b_m},$$

so that

$$\begin{aligned} \hat{h}_{r_{\underline{b},\underline{n}}} &= \binom{n_1\alpha_{\underline{b},\underline{n}} - 1}{b_1} \ldots \binom{n_{m-1}\alpha_{\underline{b},\underline{n}} - 1}{b_{m-1}} \binom{n_m\alpha_{\underline{b},\underline{n}} - 1}{b_m - 1} (\frac{n_m\alpha_{\underline{b},\underline{n}} - b_m}{b_m}) \\ &= \binom{n_1\alpha_{\underline{b},\underline{n}} - 1}{b_1} \ldots \binom{n_{m-1}\alpha_{\underline{b},\underline{n}} - 1}{b_{m-1}} \binom{n_m\alpha_{\underline{b},\underline{n}} - 1}{b_m}. \end{aligned}$$

□

5. Rings of Veronese type

Let $b, n \in \mathbb{N}^*, \underline{a} = (a_0, \ldots, a_b) \in \mathbb{N}^{b+1}$ such that $1 \leq a_i \leq n, a_0 + \ldots + a_b > n$, and $\mathcal{M}_{b,n,\underline{a}}$ be the following subset of the polynomial ring $K[x_0, \ldots, x_b]$:

$$\mathcal{M}_{b,n,\underline{a}} = \{x_0^{\alpha_0} \ldots x_b^{\alpha_b} \mid \alpha_0 + \ldots + \alpha_b = n, \alpha_i \leq a_i, \forall i = 0, \ldots, b\}.$$

Let us denote by $R_{b,n,\underline{a}}$ the toric subring of $K[x_0, \ldots, x_b]$ generated by $\mathcal{M}_{b,n,\underline{a}}$. It is well known that $R_{b,n,\underline{a}}$ is a Cohen–Macaulay ring. Let $\mathcal{S}$ be the collection of subsets S of $\{0, \ldots, b\}$ such that $\Sigma S := \sum_{i \in S} a_i < n$.

Theorem 5.1. *([6]) With the above notations the Hilbert function of* $R_{b,n,\underline{a}}$ *is*

$$\forall l \geq 0; \quad H_{b,n,\underline{a}}(l) = \sum_{S\in\mathcal{S}} (-1)^{|S|} \binom{l(n-\Sigma S)-\mid S\mid +b}{b}.$$

We have $\dim(R_{b,n,\underline{a}}) = b+1$, and its degree or multiplicity is

$$\deg(R_{b,n,\underline{a}}) = \sum_{S\in\mathcal{S}} (-1)^{|S|}(n-\Sigma S)^b.$$

Our aim is to study the Hilbert–Poincaré series of $R_{b,n,\underline{a}}$:

$$P_{R_{b,n,\underline{a}}} = \sum_{S\in\mathcal{S}} (-1)^{|S|} \sum_{l\geq 0} \binom{l(n-\Sigma S)-\mid S\mid +b}{b} t^l. \tag{4}$$

The following corollary follows immediately from Corollary 3.6 by setting $d = b+1, n = k$, and $j = -|S|$. Note that $\varepsilon_j^{<n>}(t) = \varepsilon_0^{<n,j>}(t)$.

Corollary 5.2. *For any* $S \in \mathcal{S}$ *and* $k \in \mathbb{N}^*$*, we have:*

$$\sum_{l\geq 0} \binom{kl-\mid S\mid +b}{b} t^l = \frac{Q_{S,k}(t)}{(1-t)^{b+1}},$$

where $Q_{S,k}(t)$ *is a polynomial with* $Q_{S,k}(1) \neq 0$*, with leading term*

$$\binom{k\alpha_{S,k}+\mid S\mid -1}{b} t^{b+1-\alpha_{S,k}},$$

with $\alpha_{S,k} = \lceil \frac{b+1-|S|}{k} \rceil$.

The following theorem is immediate from (4) and Corollary 5.2. It improves the description of the Hilbert–Poincaré series given in [6].

Theorem 5.3. *With the above notations, let* $\mathcal{S}$ *be the collection of subsets* S *of* $\{0,\ldots,b\}$ *such that* $\Sigma S := \sum_{i\in S} a_i < n$*. Then we can write the Hilbert–Poincaré series of* $R_{b,n,\underline{a}}$*:*

$$P_{R_{b,n,\underline{a}}} = \frac{Q_{b,n,\underline{a}}(t)}{(1-t)^{b+1}},$$

with $Q_{b,n,\underline{a}}(t) = \sum_{S\in\mathcal{S}} (-1)^{|S|} Q_{S,n}(t)$*, where* $Q_{S,n}(t)$ *is a polynomial with* $Q_{S,n}(1) \neq 0$*, with leading term*

$$\binom{(n-\Sigma S)\alpha_{S,n-\Sigma S}+\mid S\mid -1}{b} t^{b+1-\alpha_{S,n-\Sigma S}},$$

where $\alpha_{S,n-\Sigma S} = \lceil \frac{b+1-|S|}{n-\Sigma S} \rceil$.

Part one of the following corollary improves [6][Cor. 2.12].

Corollary 5.4. *With the above notations:*

1. $\operatorname{reg}(R_{b,n,\underline{a}}) \leq b+1-\lceil \frac{b+1}{n} \rceil$, *and the equality is true if and only if*

$$\sum_{S\in\mathcal{S},\alpha_{S,n-\Sigma S}=\lceil \frac{b+1}{n}\rceil} (-1)^{|S|}\binom{(n-\Sigma S)\alpha_{S,n-\Sigma S}+\mid S\mid -1}{b} \neq 0.$$

2. *If* $b+1>n^2$ *then* $\operatorname{reg}(R_{b,n,\underline{a}}) = b+1-\lceil \frac{b+1}{n} \rceil$. *Moreover, the leading term of* $Q_{b,n,\underline{a}}(t)$ *is* $\binom{n\lceil \frac{b+1}{n}\rceil -1}{b} t^{b+1-\lceil \frac{b+1}{n}\rceil}$.

Proof

1. It is enough to prove that $\min_{S\in\mathcal{S}}\lceil \frac{b+1-|S|}{n-\Sigma S}\rceil = \lceil \frac{b+1}{n}\rceil$. We consider two cases:

   - if $b+1<n$ then $\lceil \frac{b+1}{n}\rceil = 1 \leq \lceil \frac{b+1-|S|}{n-\Sigma S}\rceil, \forall S\in\mathcal{S}$.
   - If $b+1\geq n$, then

$$\frac{b+1}{n} \leq \frac{b+1-\mid S\mid}{n-\Sigma S} \Leftrightarrow (b+1)(n-\Sigma S) \leq n(b+1-\mid S\mid)$$
$$\Leftrightarrow (b+1)\Sigma S \geq n\mid S\mid;$$

   this is true since by hypothesis $\frac{b+1}{n} \geq 1 \geq \frac{|S|}{\Sigma S}$.

2. Let $b+1>n^2$ and $S\neq\emptyset$. By definition $\lceil \frac{b+1}{n}\rceil$ is the integer q such that $b+1=qn-r$, with $0\leq r<n$ and $q\geq n+1$. We have

$$b+1-\mid S\mid = qn-r-\mid S\mid = q(n-\Sigma S)-r-\mid S\mid +q\Sigma S,$$

   and $q\Sigma S-\mid S\mid \geq (n+1)\Sigma S-\mid S\mid \geq n\Sigma S > r$, so that $q\Sigma S-\mid S\mid -r>0$; hence $\lceil \frac{b+1-|S|}{n-\Sigma S}\rceil > q = \lceil \frac{b+1}{n}\rceil$.

□

In general leading terms of the alternating sum can cancel, as we can see in the next example.

Example 5.5. *Let us consider the ring* $R_{4,3,(1,1,1,1,1)}$, *the sets* S *can have 0,1 or 2 elements, and we have: If* $S=\emptyset$ *then* $\alpha_{\emptyset,3}=\lceil \frac{5}{3}\rceil = 2$, *if* $\mid S\mid = 1$ *then* $\alpha_{S,3}=\lceil \frac{4}{2}\rceil = 2$, *and finally if* $\mid S\mid = 2$ *then* $\alpha_{S,2}=\lceil \frac{3}{1}\rceil = 3$. *By using Theorem 5.3 we can write*

$$P_{R_{4,3,(1,1,1,1,1)}} = \frac{Q_0(t)-5Q_1(t)+10Q_2(t)}{(1-t)^5},$$

with $Q_0(t)=5t^3+\ldots;Q_1(t)=t^3+\ldots;Q_2(t)=t^2+\ldots$ *Note that in this case* $Q_0(t)-5Q_1(t)+10Q_2(t) = h_0+h_1t+h_2t^2$, *where* $h_0=1,h_1=5$ *and since* $h_0+h_1+h_2=\deg(R_{4,3,(1,1,1,1,1)})=11$, *we get* $h_2=5$, *so that*

$$P_{R_{4,3,(1,1,1,1,1)}} = \frac{1+5t+5t^2}{(1-t)^5}.$$

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References

[1] Barcanescu S, Manolache N. Betti numbers of Segre-Veronese singularities. Rev Roum Math Pure 1981, A. 26, 549-565.

[2] Brenti F, Welker V. The Veronese construction for formal power series and graded algebras. Adv Appl Math 2009, 42, 545-556.

[3] Bruns W, Herzog J. Cohen-Macaulay rings. Cambridge, UK: Cambridge University Press, 1998.

[4] Cox DA, Materov E. Regularity and Segre-Veronese embeddings. P Am Math Soc 2009, 137: 1883-1890.

[5] Eisenbud D. The Geometry of Syzygies. A Second Course in Commutative Algebra and Algebraic Geometry. New York, NY, USA: Springer, 2005.

[6] Katzman M. The Hilbert series of algebras of the Veronese type. Commun Algebra 2005, 33: 1141-1146.

[7] Morales M. Fonctions de Hilbert, genre géométrique d'une singularité quasi-homogéne Cohen-Macaulay. CR Acad Sci I-Math 1985, t.301, série I No.14, 699-702.