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New oscillation tests and some refinements for first-order delay dynamic equations

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Abstract: In this paper, we present new sufficient conditions for the oscillation of first-order delay dynamic equations on time scales. We also present some examples to which none of the previous results in the literature can apply.

Key words: Oscillation, delay dynamic equations, time scales

1. Introduction

In this paper, we study the oscillation of the solution to the first-order delay dynamic equation

$$x^{\Delta}(t) + p(t)x(\tau(t)) = 0 \quad \text{for } t \in [t_0, \infty)_{\mathbb{T}}, \tag{1}$$

where $\mathbb{T}$ is a time scale unbounded above with $t_0 \in \mathbb{T}$. We discuss (1) under the following assumptions.

(C1) $p \in \mathrm{C}_{\mathrm{rd}}([t_0, \infty)_{\mathbb{T}}, \mathbb{R}^+)$.

(C2) $\tau \in \mathrm{C}_{\mathrm{rd}}([t_0, \infty)_{\mathbb{T}}, \mathbb{T})$ is nondecreasing and satisfies the following conditions:

- (a) $\tau^{\sigma}(t) \le t$ for all $t \in [t_0, \infty)_{\mathbb{T}}$.
- (b) $\lim_{t\to\infty} \tau(t) = \infty$.

Before we proceed, let us recall some basic notions of the time scale concept. A *time scale*, which inherits the standard topology on $\mathbb{R}$, is a nonempty closed subset of reals. Here, and later throughout this paper, a time scale will be denoted by the symbol $\mathbb{T}$, and the intervals with a subscript $\mathbb{T}$ are used to denote the intersection of the usual interval with $\mathbb{T}$. For $t \in \mathbb{T}$, we define the *forward jump operator* $\sigma : \mathbb{T} \to \mathbb{T}$ by $\sigma(t) := \inf(t, \infty)_{\mathbb{T}}$ while the *backward jump operator* $\rho : \mathbb{T} \to \mathbb{T}$ is defined by $\rho(t) := \sup(-\infty, t)_{\mathbb{T}}$, and the *graininess function* $\mu : \mathbb{T} \to \mathbb{R}_0^+$ is defined to be $\mu(t) := \sigma(t) - t$. A point $t \in \mathbb{T}$ is called *right-dense* if $\sigma(t) = t$ and/or equivalently $\mu(t) = 0$ holds; otherwise, it is called *right-scattered*, and similarly *left-dense* and *left-scattered* points are defined with respect to the backward jump operator. A function $f : \mathbb{T} \to \mathbb{R}$ is said to be Δ-*differentiable* at the point $t \in \mathbb{T}$ provided that there exists $\ell \in \mathbb{R}$ such that for every $\varepsilon > 0$ there exists a neighborhood U of t such that

$$|[f^{\sigma}(t) - f(s)] - \ell[\sigma(t) - s]| \le \varepsilon|\sigma(t) - s| \quad \text{for all } s \in U,$$

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where $f^\sigma := f \circ \sigma$. In this case, we denote by $f^\Delta(t)$ the Δ-*derivative* of f at t and define it to be $f^\Delta(t) := \ell$. We shall mean the Δ-derivative of a function when we only say derivative unless otherwise specified. A function f is called *rd-continuous* provided that it is continuous at right-dense points in $\mathbb{T}$, and has finite limit at left-dense points, and the *set of rd-continuous functions* is denoted by $\mathrm{C_{rd}}(\mathbb{T}, \mathbb{R})$. The set of functions $\mathrm{C^1_{rd}}(\mathbb{T}, \mathbb{R})$ consists of functions whose derivative is in $\mathrm{C_{rd}}(\mathbb{T}, \mathbb{R})$ too. For $f \in \mathrm{C^1_{rd}}(\mathbb{T}, \mathbb{R})$, the so-called "simple useful formula" is given by $f^\sigma = f + \mu f^\Delta$ on $\mathbb{T}^\kappa$. For $s, t \in \mathbb{T}$ and a function $f \in \mathrm{C_{rd}}(\mathbb{T}, \mathbb{R})$, the Δ-integral of f is defined by

$$\int_s^t f(\eta)\Delta\eta = F(t) - F(s) \quad \text{for } s, t \in \mathbb{T},$$

where $F \in \mathrm{C^1_{rd}}(\mathbb{T}, \mathbb{R})$ is an antiderivative of f, i.e. $F^\Delta = f$ on $\mathbb{T}^\kappa$. Readers are referred to [5] for further interesting details of the time scale theory.

Now we can return to our discussion on the oscillation of solutions to (1). As is customary, a function $x \in \mathrm{C_{rd}}([\tau(t_0), \infty)_\mathbb{T}, \mathbb{R})$ with $x \in \mathrm{C^1_{rd}}([t_0, \infty)_\mathbb{T}, \mathbb{R})$ is called a *solution* of (1) if it satisfies (1) identically on $[t_0, \infty)_\mathbb{T}$.

Next let us recall some known oscillation results on this subject. For $\mathbb{T} = \mathbb{R}$ and $\mathbb{T} = \mathbb{Z}$, (1) reduces to

$$x'(t) + p(t)x(\tau(t)) = 0 \quad \text{for } t \in \mathbb{R}_0^+ \tag{2}$$

and

$$\Delta x(n) + p(n)x(\tau(n)) = 0 \quad \text{for } n \in \mathbb{N}_0, \tag{3}$$

respectively. In the literature, (3) is mostly considered with a constant delay, i.e.

$$\Delta x(n) + p(n)x(n - \tau_0) = 0 \quad \text{for } n \in \mathbb{N}_0. \tag{4}$$

Let us proceed by quoting some well-known results on the equations (2), (3), and (4), which are particular cases of the equation (1).

In 1972, Ladas et al. obtained the following result.

Theorem A *([15, Corollary 2.1]) If*

$$\limsup_{t\to\infty} \int_{\tau(t)}^t p(\eta)\mathrm{d}\eta > 1,$$

then every solution of (2) *is oscillatory.*

Then, in 1989, Erbe and Zhang gave the discrete analogue of the result due to Ladas et al.

Theorem B *([12, Theorem 2.5]) If*

$$\limsup_{n\to\infty} \sum_{\ell=n-\tau_0}^{n} p(\ell) > 1,$$

then every solution of (4) *is oscillatory.*

Moreover, in 2008, Chatzarakis et al. obtained this result for (3) with monotone arbitrary delays.

Theorem C *([8, Theorem 2.1]) If*

$$\limsup_{n\to\infty} \sum_{\ell=\tau(n)}^{n} p(\ell) > 1,$$

then every solution of (3) *is oscillatory.*

In 2006, Şahiner and Stavroulakis presented the dynamic unification of the upper limit condition for (1), which exactly covers Theorem A, Theorem B, and Theorem C.

Theorem D *([18, Theorem 2.4]) If*

$$\limsup_{t\to\infty} \int_{\tau(t)}^{\sigma(t)} p(\eta)\Delta\eta > 1,$$

then every solution of (1) *is oscillatory.*

On the other hand, these tests are improved by also considering the lower limit together with the upper limit. In 1988, Erbe and Zhang made the successful first attempt.

Theorem E *([11, Theorem 2.2]) If*

$$\liminf_{t\to\infty} \int_{\tau(t)}^{t} p(\eta)\mathrm{d}\eta > \alpha \tag{5}$$

and

$$\limsup_{t\to\infty} \int_{\tau(t)}^{t} p(\eta)\mathrm{d}\eta > 1 - \frac{\alpha^2}{4},$$

then every solution of (2) *is oscillatory.*

However, there appeared some mistakes in obtaining the discrete version of this result (see the discussion in [9, 10]), and, in 2004, Stavroulakis gave the following result with a correct proof.

Theorem F *([19, Theorem 2.6]) If*

$$\liminf_{n\to\infty} \sum_{\ell=n-\tau_0}^{n-1} p(\ell) > \alpha \tag{6}$$

and

$$\limsup_{n\to\infty} \sum_{\ell=n-\tau_0}^{n-1} p(\ell) > 1 - \frac{\alpha^2}{4},$$

then every solution of (4) *is oscillatory.*

In 2006, Şahiner and Stavroulakis also gave the dynamic unification of this condition for (1).

Theorem G *([18, Theorem 2.5]) If*

$$\liminf_{t\to\infty} \int_{\tau(t)}^{t} p(\eta)\Delta\eta > \alpha \tag{7}$$

and

$$\limsup_{t\to\infty}\int_{\tau(t)}^{t}p(\eta)\Delta\eta>1-\frac{\alpha^2}{4},\tag{8}$$

then every solution of (1) *is oscillatory.*

In [1], Agarwal and Bohner showed that for any $\lambda\in(0,1)_{\mathbb{R}}$ the right-hand side of (8) can be replaced with

$$1-\frac{\lambda(1-\lambda)\alpha^2}{1-\lambda\alpha}.$$

Minimizing this for λ gives the following result.

Theorem H *([1, Theorem 3]) If* (7) *and*

$$\limsup_{t\to\infty}\int_{\tau(t)}^{t}p(\eta)\Delta\eta>1-\left(1-\sqrt{1-\alpha}\right)^2,\tag{9}$$

then every solution of (1) *is oscillatory.*

It can be seen from Figure 1 that the right-hand side of (9) is smaller than that of (8). Hence, Theorem H improves Theorem G.

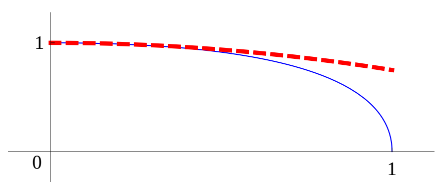

Figure 1. The solid and the dashed lines denote the graphics of the curves $1-(1-\sqrt{1-\alpha})^2$ and $1-\frac{\alpha^2}{4}$ for $\alpha\in[0,1]_{\mathbb{R}}$, respectively.

In 2006, Chatzarakis et al. gave a very similar result for the discrete case.

Theorem I *([8, Theorem 2.2]) If* (6) *and*

$$\limsup_{n\to\infty}\sum_{\ell=\tau(n)}^{n}p(\ell)>1-\left(1-\sqrt{1-\alpha}\right)^2,$$

then every solution of (3) *is oscillatory.*

It should be mentioned that Theorem I for (3) is better than Theorem H since (9) reduces to

$$\limsup_{n\to\infty}\sum_{\ell=\tau(n)}^{n-1}p(\ell)>1-\left(1-\sqrt{1-\alpha}\right)^2.$$

However, in the continuous case, (9) reduces to

$$\limsup_{t\to\infty}\int_{\tau(t)}^{t}p(\eta)\mathrm{d}\eta>1-\left(1-\sqrt{1-\alpha}\right)^2,$$

which is new for (2).

Our aim here is to present the dynamic generalization of Theorem I, which also coincides with the continuous case of Theorem G (see Theorem 2 below). Further, we present some new results that improve the upper limit test Theorem D. Our results improve the results in the papers [1, 8, 18].

We refer the readers to [2–4, 6, 7, 13, 14, 20] for some other oscillation results on (1). It should be mentioned that if (7) holds with $\alpha \in (\frac{1}{\mathrm{e}}, \infty)_{\mathbb{R}}$, then every solution of (1) oscillates (see [3]). Hence, we will assume $\alpha \in [0, \frac{1}{\mathrm{e}}]_{\mathbb{R}}$ starting from our main results section. On the other hand, if $\tau(t) < t$ for all $t \in [t_0, \infty)_{\mathbb{T}}$ (which is assumed to hold in [1, 8, 18]), then (C2a) holds. However, (C2a) is weaker than this condition since it allows $\tau(t) = t$ when t is right-dense.

Readers are also referred to [17, 21, 22] for some other interesting results/discussions on this topic.

The paper is structured as follows. In Section 2, we present and prove our preparatory results. In Section 3, we state our main results by using the results in Section 2. Section 4 includes some illustrative examples. In Section 5, we finalize the paper with some remarks.

2. Auxiliary lemmas

In this section, we establish four lemmas to be used in the next section.

Lemma 1 *If (1) admits a nonoscillatory solution, then*

$$\frac{x(t)}{x(\tau(t))} > \mu(t)p(t) \quad \textit{for all large } t \in [t_0, \infty)_{\mathbb{T}}.$$

Proof Without loss of generality, we may suppose that x is eventually positive. Then we may find $t_1 \in [t_0, \infty)_{\mathbb{T}}$ such that $x(t), x(\tau(t)) > 0$ for all $t \in [t_1, \infty)_{\mathbb{T}}$. It follows from (1) that x is nonincreasing on $[t_1, \infty)_{\mathbb{T}}$. By using the "simple useful formula" and (1), we have for all $t \in [t_1, \infty)_{\mathbb{T}}$ that

$$\begin{aligned} 0 =& \mu(t)x^{\Delta}(t) + \mu(t)p(t)x(\tau(t)) \\ =& x^{\sigma}(t) - x(t) + \mu(t)p(t)x(\tau(t)), \end{aligned}$$

which yields

$$x(t) > \mu(t)p(t)x(\tau(t)).$$

This completes the proof. □

Remark 1 *Let x be a nonoscillatory solution of (1). Then $\mu(t)p(t) < 1$ for all large $t \in [t_0, \infty)_{\mathbb{T}}$.*

The conclusion of Remark 1 follows from the monotonicity property of nonoscillatory solutions (see [3]).

Lemma 2 *(See the proof of [18, Lemma 2.3]) Let x be a nonoscillatory solution of (1), then*

$$\int_{\tau(t)}^{\sigma(t)} p(\eta)\Delta\eta \leq 1 - \frac{x^{\sigma}(t)}{x(\tau(t))} \quad \textit{for all large } t \in [t_0, \infty)_{\mathbb{T}}. \tag{10}$$

Proof Without loss of generality, we may suppose that x is eventually positive. Then we may find $t_1 \in [t_0, \infty)_{\mathbb{T}}$ such that $x(t), x(\tau(t)) > 0$ for all $t \in [t_1, \infty)_{\mathbb{T}}$. It follows from (1) that x is nonincreasing on $[t_1, \infty)_{\mathbb{T}}$. Integrating (1) from $\tau(t)$ to $\sigma(t)$, we get

$$0 = x^{\sigma}(t) - x(\tau(t)) + \int_{\tau(t)}^{\sigma(t)} p(\eta)x(\tau(\eta))\Delta\eta$$

where $t \in [t_2, \infty)_{\mathbb{T}}$ for some $t_2 \in [t_1, \infty)_{\mathbb{T}}$ with $\tau^2(t_2) \geq t_1$. Since the integral variable η above satisfies $\tau(t) \leq \eta \leq t$ (see also [18, Lemma 2.1]), we obtain

$$0 \geq x^{\sigma}(t) + \left[\int_{\tau(t)}^{\sigma(t)} p(\eta)\Delta\eta - 1\right] x(\tau(t))$$

for all $t \in [t_2, \infty)_{\mathbb{T}}$. This completes the proof. □

Although the following new result gives us nothing interesting at right-dense points, it provides a natural lower bound (we call this natural since there are no additional assumptions) for the quotient $\frac{x^{\sigma}}{x \circ \tau}$ at right-scattered points.

Lemma 3 *Let x be a nonoscillatory solution of (1); then*

$$\frac{x^{\sigma}(t)}{x(\tau(t))} > \frac{\mu^{\sigma}(t)p^{\sigma}(t)}{1 - \mu^{\sigma}(t)p^{\sigma}(t)} \int_{\tau^{\sigma}(t)}^{\sigma(t)} p(\eta)\Delta\eta \quad \textit{for all large } t \in [t_0, \infty)_{\mathbb{T}}. \tag{11}$$

Proof Without loss of generality, we may suppose that x is eventually positive. Then we may find $t_1 \in [t_0, \infty)_{\mathbb{T}}$ such that $x(t), x(\tau(t)) > 0$ for all $t \in [t_1, \infty)_{\mathbb{T}}$. It follows from (1) that x is nonincreasing on $[t_1, \infty)_{\mathbb{T}}$. Integrating (1) from $\tau^{\sigma}(t)$ to $\sigma(t)$, we get

$$\begin{aligned} 0 =& x^{\sigma}(t) - x(\tau^{\sigma}(t)) + \int_{\tau^{\sigma}(t)}^{\sigma(t)} p(\eta)x(\tau(\eta))\Delta\eta \\ \geq& x^{\sigma}(t) - x(\tau^{\sigma}(t)) + \left[\int_{\tau^{\sigma}(t)}^{\sigma(t)} p(\eta)\Delta\eta\right] x(\tau(t)) \end{aligned}$$

and

$$x(\tau^{\sigma}(t)) \geq x^{\sigma}(t) + \left[\int_{\tau^{\sigma}(t)}^{\sigma(t)} p(\eta)\Delta\eta\right] x(\tau(t)). \tag{12}$$

for all $t \in [t_2, \infty)_{\mathbb{T}}$, where $t_2 \in [t_1, \infty)_{\mathbb{T}}$ with $\tau^2(t_2) \geq t_1$. Combining Lemma 1 and (12), we see for all $t \in [t_2, \infty)_{\mathbb{T}}$ that

$$x^{\sigma}(t) > \mu^{\sigma}(t)p^{\sigma}(t)x(\tau^{\sigma}(t)) \geq \mu^{\sigma}(t)p^{\sigma}(t)\left\{x^{\sigma}(t) + \left[\int_{\tau^{\sigma}(t)}^{\sigma(t)} p(\eta)\Delta\eta\right] x(\tau(t))\right\}$$

or simply

$$x^{\sigma}(t) > \frac{\mu^{\sigma}(t)p^{\sigma}(t)}{1 - \mu^{\sigma}(t)p^{\sigma}(t)} \left[\int_{\tau^{\sigma}(t)}^{\sigma(t)} p(\eta)\Delta\eta\right] x(\tau(t)).$$

□

The following lemma plays the major role in our oscillation test, which uses the lower limit condition.

Lemma 4 *Assume that there exists a constant* $\alpha \in [0,1]_{\mathbb{R}}$ *such that*

$$\int_{\tau(t)}^{t} p(\eta)\Delta\eta \geq \alpha \quad \textit{for all large } t \in [t_0,\infty)_{\mathbb{T}}. \tag{13}$$

Then every nonoscillatory x *solution of* (1) *satisfies*

$$\frac{x^{\sigma}(t)}{x(\tau(t))} > \left(1-\sqrt{1-\alpha}\right)^2 \quad \textit{for all large } t \in [t_0,\infty)_{\mathbb{T}}. \tag{14}$$

Proof The claim is trivial if $\alpha = 0$ since the right-hand side of (14) becomes 0. Thus, we consider below the case where $\alpha \in (0,1]_{\mathbb{R}}$. Without loss of generality, we may suppose that x is eventually positive. Then we may find $t_1 \in [t_0,\infty)_{\mathbb{T}}$ such that $x(t), x(\tau(t)) > 0$ and (13) hold for all $t \in [t_1,\infty)_{\mathbb{T}}$. It follows from (1) that x is nonincreasing on $[t_1,\infty)_{\mathbb{T}}$. Fix $\lambda \in (0,1)_{\mathbb{R}}$. Now we claim that for any $t \in [t_1,\infty)_{\mathbb{T}}$ there exists $s \geq t$ with $t > \tau(s)$ such that

$$\int_{t}^{\sigma(s)} p(\eta)\Delta\eta \geq \lambda\alpha \quad \text{and} \quad \int_{\tau(s)}^{t} p(\eta)\Delta\eta \geq (1-\lambda)\alpha. \tag{15}$$

Assume the contrary that

$$\int_{t_2}^{\sigma(s)} p(\eta)\Delta\eta < \lambda\alpha \quad \text{or} \quad \int_{\tau(s)}^{t_2} p(\eta)\Delta\eta < (1-\lambda)\alpha \tag{16}$$

for some $t_2 \in [t_1,\infty)_{\mathbb{T}}$ and any $s \geq t_2$ with $t_2 > \tau(s)$. If we define $\Gamma : [t_2,\infty)_{\mathbb{T}} \to \mathbb{R}$ by

$$\Gamma(t) := \int_{t_2}^{t} p(\eta)\Delta\eta - \lambda\alpha,$$

then we see that Γ is nondecreasing on $[t_2,\infty)_{\mathbb{T}}$. Moreover, $\Gamma(t_2) < 0$ and

$$\Gamma(t_3) = \int_{t_2}^{\tau(t_3)} p(\eta)\Delta\eta + \int_{\tau(t_3)}^{t_3} p(\eta)\Delta\eta - \lambda\alpha \geq \alpha - \lambda\alpha = (1-\lambda)\alpha > 0,$$

where $t_3 \in [t_2,\infty)_{\mathbb{T}}$ satisfies $\tau(t_3) \geq t_2$. Without loss of generality, we let $t_3 \in [t_2,\infty)_{\mathbb{T}}$ be the minimal of such points, i.e. $\tau(\rho(t_3)) \leq t_2$. Hence, if $t_2 \leq t < t_3$, then $t \leq \rho(t_3)$ and thus $\tau(t) \leq \tau(\rho(t_3)) \leq t_2$. Due to the intermediate value theorem [5, Theorem 1.115], there exists $s \in [t_2,t_3)_{\mathbb{T}}$ such that $\Gamma^{\sigma}(s) \geq 0$ and $\Gamma(s) \leq 0$. Then we have

$$\int_{t_2}^{\sigma(s)} p(\eta)\Delta\eta = \lambda\alpha + \Gamma^{\sigma}(s) \geq \lambda\alpha$$

and

$$\begin{aligned}\int_{\tau(s)}^{t_2} p(\eta)\Delta\eta &= \int_{\tau(s)}^{s} p(\eta)\Delta\eta - \int_{t_2}^{s} p(\eta)\Delta\eta = \int_{\tau(s)}^{s} p(\eta)\Delta\eta - [\Gamma(s)+\lambda\alpha] \\ &\geq \alpha - \lambda\alpha = (1-\lambda)\alpha,\end{aligned}$$

which contradicts (16). Hence, (15) holds for some $s \geq t$ with $t > \tau(s)$. Thus, for all $t \in [t_2, \infty)_{\mathbb{T}}$, where $t_2 \in [t_1, \infty)_{\mathbb{T}}$ with $\tau^2(t_2) \geq t_1$, we have from (1) that

$$\begin{aligned}
x^{\sigma}(t) =& x^{\sigma}(r) + [x^{\sigma}(t) - x^{\sigma}(r)] = x^{\sigma}(r) + \int_{\sigma(t)}^{\sigma(r)} p(\eta)x(\tau(\eta))\Delta\eta \\
\geq & x^{\sigma}(r) + \left[\int_{\sigma(t)}^{\sigma(r)} p(\eta)\Delta\eta\right] x(\tau(r)) \\
=& x^{\sigma}(r) + \big\{x^{\sigma}(t) + [x(\tau(r)) - x^{\sigma}(t)]\big\}\left[\int_{\sigma(t)}^{\sigma(r)} p(\eta)\Delta\eta\right] \\
=& x^{\sigma}(r) + \left\{x^{\sigma}(t) + \left[\int_{\tau(r)}^{\sigma(t)} p(\eta)x(\tau(\eta))\Delta\eta\right]\right\}\left[\int_{\sigma(t)}^{\sigma(r)} p(\eta)\Delta\eta\right] \\
\geq & x^{\sigma}(r) + \left\{x^{\sigma}(t) + \left[\int_{\tau(r)}^{\sigma(t)} p(\eta)\Delta\eta\right] x(\tau(t))\right\}\left[\int_{\sigma(t)}^{\sigma(r)} p(\eta)\Delta\eta\right] \\
\geq & x^{\sigma}(r) + \big\{x^{\sigma}(t) + (1-\lambda)\alpha x(\tau(t))\big\}\lambda\alpha,
\end{aligned}$$

where r is the point that corresponds to $\sigma(t)$ in (15). This yields

$$x^{\sigma}(t) > \frac{\lambda(1-\lambda)\alpha^2}{1-\lambda\alpha} x(\tau(t)) \quad \text{for all } t \in [t_1, \infty)_{\mathbb{T}}.$$

Since the left-hand side is independent of λ, we can maximize the right-hand side for λ to get the best upper bound, i.e.

$$\max_{\substack{\lambda\in(0,1)_{\mathbb{R}},\\ \alpha\in(0,1]_{\mathbb{R}}}} \left\{\frac{\lambda(1-\lambda)\alpha^2}{1-\lambda\alpha}\right\} = \left(1-\sqrt{1-\alpha}\right)^2.$$

This completes the proof. $\square$

3. Main results

Below, we give our first oscillation test.

Theorem 1 *Assume that there exists an increasing unbounded sequence* $\{\xi_n\}_{n\in\mathbb{N}} \subset [t_0, \infty)_{\mathbb{T}}$ *such that*

$$\mu^{\sigma}(\xi_n)p^{\sigma}(\xi_n) \geq 1 \tag{17}$$

or

$$\mu(\tau(\xi_n))p(\tau(\xi_n)) \geq 1 \tag{18}$$

or

$$\frac{\int_{\tau(\xi_n)}^{\sigma(\xi_n)} p(\eta)\Delta\eta}{1-[1-\mu(\tau(\xi_n))p(\tau(\xi_n))]\mu^{\sigma}(\xi_n)p^{\sigma}(\xi_n)} \geq 1 \tag{19}$$

for all $n \in \mathbb{N}$. *Then every solution of* (1) *is oscillatory.*

Proof Assume the contrary that (1) admits a nonoscillatory solution x. It is obvious that if (17) or (18) holds, then we arrive at a contradiction (see Remark 1). Hence, we only consider the case where (19) holds but (17) and (18) do not hold. It follows from Lemma 2 and Lemma 3 that

$$\int_{\tau(t)}^{\sigma(t)} p(\eta)\Delta\eta + \frac{\mu^\sigma(t)p^\sigma(t)}{1-\mu^\sigma(t)p^\sigma(t)}\left[\int_{\tau^\sigma(t)}^{\sigma(t)} p(\eta)\Delta\eta\right] < 1 \tag{20}$$

for all $t \in [t_1,\infty)_{\mathbb{T}}$, where $t_1 \in [t_0,\infty)_{\mathbb{T}}$ is sufficiently large. Simply, we have

$$\begin{aligned}\int_{\tau(t)}^{\sigma(t)} p(\eta)\Delta\eta &+ \frac{\mu^\sigma(t)p^\sigma(t)}{1-\mu^\sigma(t)p^\sigma(t)}\left[\int_{\tau(t)}^{\sigma(t)} p(\eta)\Delta\eta - \mu(\tau(t))p(\tau(t))\right] \\ &= \frac{1}{1-\mu^\sigma(t)p^\sigma(t)}\left[\int_{\tau(t)}^{\sigma(t)} p(\eta)\Delta\eta - \mu^\sigma(t)p^\sigma(t)\mu(\tau(t))p(\tau(t))\right]\end{aligned} \tag{21}$$

for all $t \in [t_1,\infty)_{\mathbb{T}}$. Using (20) and (21), we get for all $t \in [t_1,\infty)_{\mathbb{T}}$ that

$$\frac{\int_{\tau(t)}^{\sigma(t)} p(\eta)\Delta\eta}{1-\mu^\sigma(t)p^\sigma(t)+\mu^\sigma(t)p^\sigma(t)\mu(\tau(t))p(\tau(t))} < 1$$

or equivalently

$$\frac{\int_{\tau(t)}^{\sigma(t)} p(\eta)\Delta\eta}{1-[1-\mu(\tau(t))p(\tau(t))]\mu^\sigma(t)p^\sigma(t)} < 1.$$

This contradicts (19) and thus every solution of (1) is oscillatory. □

As an immediate consequence of Theorem 1, we can give the following corollary, which improves Theorem D.

Corollary 1 *If*

$$\limsup_{t\to\infty} \frac{\int_{\tau(t)}^{\sigma(t)} p(\eta)\Delta\eta}{1-[1-\mu(\tau(t))p(\tau(t))]\mu^\sigma(t)p^\sigma(t)} > 1,$$

then every solution of (1) *is oscillatory.*

Next we state the main result of this paper.

Theorem 2 *Assume that there exists a constant* $\alpha \in [0,1]_{\mathbb{R}}$ *such that* (13) *holds. Assume further that there exists an increasing unbounded sequence* $\{\xi_n\}_{n\in\mathbb{N}} \subset [t_0,\infty)_{\mathbb{T}}$ *such that*

$$\int_{\tau(\xi_n)}^{\sigma(\xi_n)} p(\eta)\Delta\eta \geq 1-\left(1-\sqrt{1-\alpha}\right)^2 \quad \text{for all } n \in \mathbb{N}. \tag{22}$$

Then every solution of (1) *is oscillatory.*

Proof Assume the contrary that (1) admits a nonoscillatory solution x. Without loss of generality, we may suppose that x is eventually positive. Then we may find $t_1 \in [t_0,\infty)_{\mathbb{T}}$ such that $x(t), x(\tau(t)) > 0$ for all $t \in [t_1,\infty)_{\mathbb{T}}$. By Lemma 2 and Lemma 4, we respectively have (10) and (14). Combining (10) and (14), we get

$$\int_{\tau(t)}^{\sigma(t)} p(\eta)\Delta\eta \leq 1-\frac{x^\sigma(t)}{x(\tau(t))} < 1-\left(1-\sqrt{1-\alpha}\right)^2 \quad \text{for all } t \in [t_1,\infty)_{\mathbb{T}},$$

which contradicts (22). Therefore, every solution of (1) is oscillatory. $\square$

As an immediate consequence of Theorem 2, we have the following result.

Corollary 2 *If there exists* $\alpha \in [0,1]_{\mathbb{R}}$ *such that*

$$\liminf_{t\to\infty} \int_{\tau(t)}^{t} p(\eta)\Delta\eta > \alpha \quad \text{and} \quad \limsup_{t\to\infty} \int_{\tau(t)}^{\sigma(t)} p(\eta)\Delta\eta > 1 - \left(1 - \sqrt{1-\alpha}\right)^2,$$

then every solution of (1) *is oscillatory.*

4. Some applications

In this section, we present two examples to show the significance of our new results.

Example 1 *Let* $\mathbb{T} = \mathbb{Z}$ *and consider the following difference equation*

$$\Delta x(t) + p(t)x(t-2) = 0 \quad \text{for } t \in [0,\infty)_{\mathbb{Z}},$$

where

$$p(t) := \begin{cases} 0.5, & t \bmod 3 = 1 \\ 0.125, & \text{otherwise.} \end{cases}$$

Let us first show that the tests mentioned in the introduction fail for this equation. Clearly,

$$\limsup_{t\to\infty} \sum_{\ell=t-2}^{t} p(\ell) = 0.5 + 2 \times 0.125 = 0.75 \not> 1$$

shows that Theorem D fails. The well-known oscillation test [16, Theorem 1] $\liminf_{t\to\infty} \sum_{\ell=t-\tau_0}^{t-1} p(\ell) > \left(\frac{\tau_0}{\tau_0+1}\right)^{\tau_0+1}$ *fails since*

$$\liminf_{t\to\infty} \sum_{\ell=t-2}^{t-1} p(\ell) = 2 \times 0.125 = 0.25 \not> \left(\frac{2}{3}\right)^3.$$

Moreover,

$$0.75 \not> 1 - \left(1 - \sqrt{1 - 0.25}\right)^2$$

shows that Theorem I (thus Theorem H also) fails. Further, [8, Theorem 2.2] does not apply because of $0.125 \not\geq 1 - \sqrt{1-0.25}$. *However, letting* $\xi_n = 3n$ *for* $n \in \mathbb{N}_0$, *we have*

$$\frac{\sum_{\ell=3n-2}^{3n} p(\ell)}{1 - [1 - p(3n-2)]p(3n+1)} = \frac{0.75}{1 - (1-0.5)0.5} = 1$$

and therefore every solution oscillates by Theorem 1.

The graphic of an oscillating solution is given in Figure 2.

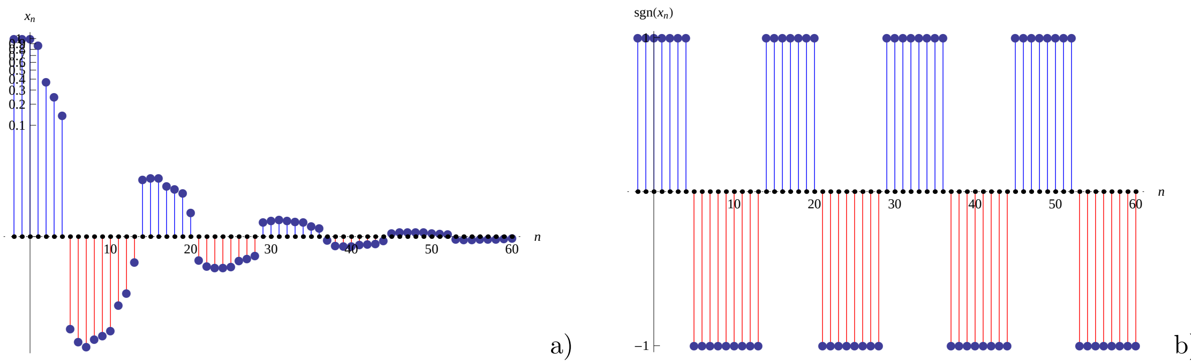

Figure 2. The graphs of the solution x and $\mathrm{sgn}(x)$ of 60 iterates. (**a**) The graph of the solution x with the initial conditions $x(-2) = x(-1) = x(0) = 1$. (**b**) The graph of $\mathrm{sgn}(x)$.

Example 2 *Let* $\mathbb{T} = \mathbb{P}_{1,2} = \cup_{\ell\in\mathbb{Z}}[3\ell, 3\ell+1]_{\mathbb{R}}$ *and consider*

$$x^{\Delta}(t) + p(t)x(t-3) = 0 \quad \text{for } t \in [0,\infty)_{\mathbb{P}_{a,b}12}, \tag{23}$$

where

$$p(t) := \begin{cases} 0.33, & t \text{ is right-dense} \\ 0.01, & t \text{ is right-scattered and } t \bmod 6 = 1 \\ 0.31, & t \text{ is right-scattered and } t \bmod 6 = 4. \end{cases} \tag{24}$$

The graphics of the coefficient and the delay function are given in Figure 3.

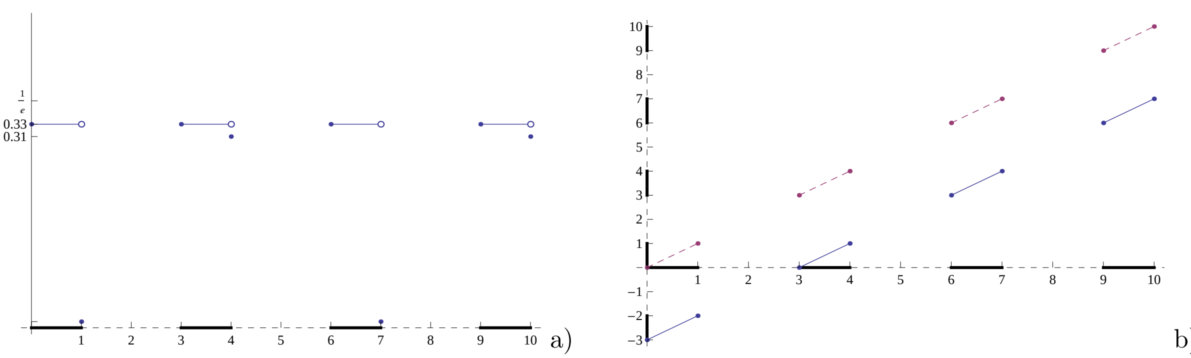

Figure 3. Graphics related to the delay equation (23). (**a**) Graphic of the coefficient (24). (**b**) Graphic of the delay function in (23). Here the solid and the dashed lines denote the delay function $(t-3)$ and the identity function t, respectively.

Since

$$\limsup_{t\to\infty} \int_{t-3}^{\sigma(t)} p(\eta)\Delta\eta = 2 \times 0.01 + 0.33 + 2 \times 0.31 = 0.98 \not> 1$$

Theorem D does not apply. On the other hand, we have

$$\limsup_{t\to\infty}\int_{t-3}^{t}p(\eta)\Delta\eta=0.33+2\times 0.31=0.96$$

and

$$\liminf_{t\to\infty}\int_{t-3}^{t}p(\eta)\Delta\eta=0.33+2\times 0.01=0.36$$

but

$$0.96\not>1-\left(1-\sqrt{1-0.36}\right)^{2}=0.96.$$

This shows that Theorem H fails. Fortunately, we see that

$$0.98>1-\left(1-\sqrt{1-0.36}\right)^{2}$$

and thus due to Corollary 2 every solution of (23) oscillates.

5. Final comments

In the case $\mathbb{T}=\mathbb{R}$, we see that Theorem 1 (or Corollary 1) has no contribution to the literature. However, in the case $\mathbb{T}=\mathbb{Z}$ (or time scales with right-scattered points), these results turn out to give new oscillation tests. These results also improve recent general results (for instance, Theorem D since the left-hand side has a factor not less than 1).

On the other hand, this result also improves some results for difference equations. We explain this fact with the remarks below.

Remark 2 *If (1) is nonoscillatory and (13) holds for some $\alpha\in[0,1]_{\mathbb{R}}$, then $\mu(t)p(t)<1-\alpha$ for all large $t\in[t_0,\infty)_{\mathbb{T}}$. Indeed, we have*

$$1>\int_{\tau^{\sigma}(t)}^{\sigma(t)}p(\eta)\Delta\eta=\int_{\tau^{\sigma}(t)}^{t}p(\eta)\Delta\eta+\mu(t)p(t)\geq\alpha+\mu(t)p(t)\quad\text{for all large } t\in[t_0,\infty)_{\mathbb{T}},$$

from which the claim follows.

Now let us examine the condition (19) (or (20)) of Theorem 1.

Remark 3 *Suppose that (13) holds for some $\alpha\in[0,1]_{\mathbb{R}}$; then by Remark 2 we see that $\mu(t)p(t)<1-\alpha$ for all large $t\in[t_0,\infty)_{\mathbb{T}}$. Suppose in addition that $\mu(t)p(t)\geq\beta$ for all large $t\in[t_0,\infty)_{\mathbb{T}}$, where $\beta\in(0,1-\alpha]_{\mathbb{R}}$. Then we have from (20) that*

$$\int_{\tau(t)}^{\sigma(t)}p(\eta)\Delta\eta<1-\frac{\alpha\beta}{1-\beta}\quad\text{for all large } t\in[t_0,\infty)_{\mathbb{T}}.$$

Hence, if there exists an increasing unbounded sequence $\{\xi_n\}_{n\in\mathbb{N}}\subset[t_0,\infty)_{\mathbb{T}}$ such that

$$\int_{\tau(\xi_n)}^{\sigma(\xi_n)}p(\eta)\Delta\eta\geq1-\frac{\alpha\beta}{1-\beta}\quad\text{for all } n\in\mathbb{N},$$

then every solution of (1) oscillates.

In [8, Theorem 2.2], the authors prove for (3) that if (6) holds for some $\alpha \in (0,1]_{\mathbb{R}}$ and $p(n) \geq 1-\sqrt{1-\alpha}$ for all large $n \in \mathbb{N}_0$. Then

$$\limsup_{n\to\infty} \sum_{\ell=\tau(n)}^{n} p(\ell) > 1-\alpha\frac{1-\sqrt{1-\alpha}}{\sqrt{1-\alpha}},$$

implies oscillation of all solutions of (3). Obviously, this result is a consequence of Remark 3.

It should be mentioned that this remark with $\alpha = 0.25$ and $\beta = 0.125$ also cannot deliver an answer for the oscillation of solutions to the equation in Example 1.

Finally, we conclude the paper by emphasizing that Theorem 2 improves Theorem H as we have mentioned in Section 1.

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