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Fourth-order Birkhoff regular problems with eigenvalue parameter dependent boundary conditions

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Abstract: A regular fourth-order differential equation that depends quadratically on the eigenvalue parameter λ is considered with classes of separable boundary conditions independent of λ or depending on λ linearly. Conditions are given for the problems to be Birkhoff regular.

Key words: Fourth order, Birkhoff regular problems, Birkhoff matrices, eigenvalue, boundary conditions

1. Introduction

The spectral theory of Sturm–Liouville operators is well developed due to their intrinsic mathematical challenges and their applications in physics and engineering. Apart from classical Sturm–Liouville problems, also higher order linear differential equations occur in applications, with or without the eigenvalue parameter in the boundary conditions. Such problems are realized as operator polynomials, also called operator pencils. Some recent developments of higher order differential operators whose boundary conditions depend on the eigenvalue parameter, including spectral asymptotics and basis properties, were investigated in [2–4, 11, 12].

The generalized Regge problem [13], the small transversal vibrations of a homogeneous beam compressed or stretched problems investigated in [6–10], the sixth-order problem investigated in [11], and the self-adjoint higher order problem investigated in [12] have boundary conditions with partial first-order derivatives with respect to the time variable t or whose mathematical model leads to an eigenvalue problem with the eigenvalue parameter λ occurring linearly in the boundary conditions. Such problems have an operator representation of the form

$$L(\lambda) = \lambda^2 M - i\lambda K - A \tag{1.1}$$

in the Hilbert space $H = L_2(0,a) \oplus \mathbb{C}^l$, where l is the number of eigenvalue dependent boundary conditions.

Spectral theory of differential operators originated from the works of Birkhoff, who defined Birkhoff regular problems by providing conditions for eigenvalue problems to be regular [1]. Birkhoff also proved in [1] an expansion theorem for eigenvalue functions and a theorem for the distribution of the eigenvalues of a Birkhoff regular eigenvalue problem. Stone [15] showed that Birkhoff expansions are equivalent to Fourier series. Salaff considered in [14] an arbitrary mth order linear differential expression and m linearly independent, homogeneous, two-point boundary conditions and proved that if the problem is self-adjoint, then it is Birkhoff regular.

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Separation of variables leads the vibration beam problems investigated in [6–10] to fourth-order boundary problems with eigenvalue parameter dependent boundary conditions, where the differential equation

$$y^{(4)} - (gy')' = \lambda^2 y \tag{1.2}$$

depends on the eigenvalue parameter. Mennicken and Möller [5] developed useful tools for the identification of the Birkhoff regularity of boundary eigenvalue problems. These tools had been used to prove that the problems under consideration in [6, 8–10] were Birkhoff regular. It follows that the eigenvalues for general g of these problems are small perturbations of the eigenvalues for $g = 0$. Hence, the asymptotics of the eigenvalues for general g were obtained from those of $g = 0$. Note that the main operator coefficients A in the operator pencils $L(\lambda)$ defined in (1.1) and investigated in [6, 8–10] are self-adjoint; see [7].

In this paper we consider eigenvalue problems with the operator representation given in (1.1), where the main operator A is not necessary self-adjoint, consisting of the fourth-order differential equation (1.2) with separated boundary conditions $B_j(\lambda)y = 0$, $j = 1, 2, 3, 4$. The boundary conditions $B_j(\lambda)y = 0$, $j = 1, 2, 3, 4$ are independent of λ or depend on λ linearly. We derive conditions for the problems to be Birkhoff regular. For the definition of the boundary terms $B_j(\lambda)y$, we refer the reader to (2.3). In a forthcoming paper we will investigate the asymptotics of the eigenvalues for which the problems are Birkhoff regular.

We introduce the eigenvalue problem under consideration in Section 2, while in Section 3 we give conditions for the fourth-order eigenvalue problems under consideration to be Birkhoff regular.

2. The eigenvalue problem

On the interval $[0, a]$, we consider the eigenvalue problem

$$y^{(4)} - (gy')' = \lambda^2 y, \tag{2.1}$$

$$B_j(\lambda)y = 0,\ j = 1, 2, 3, 4, \tag{2.2}$$

where $g \in C^1[0, a]$, $a > 0$, is a real-valued function and (2.2) are separated boundary conditions independent of λ or depending on λ linearly. We assume that

$$B_j(\lambda)y = \sum_{k=0}^{p_j} \alpha_{j,k} y^{(k)}(a_j) + i\lambda \sum_{k=0}^{q_j} \beta_{j,k} y^{(k)}(a_j), \tag{2.3}$$

where $j = 1, 2, 3, 4$ and $p_k, q_k \in \{-\infty, 0, 1, 2, 3\}$, at least one of the numbers $p_j, q_j \neq -\infty$, $j \in \{1, 2, 3, 4\}$, and $\alpha_{j,p_j} = 1$ if $p_j \neq -\infty$, $\beta_{j,q_j} \neq 0$ if $q_j \neq -\infty$, while $a_j = 0$ for $j = 1, 2$ and $a_j = a$ for $j = 3, 4$.

We define

$$\Theta_1 = \{s \in \{1, 2, 3, 4\} : B_s(\lambda) \text{ depends on } \lambda\},\ \Theta_0 = \{1, 2, 3, 4\} \backslash \Theta_1,$$

$$\Theta_1^0 = \Theta_1 \cap \{1, 2\}, \quad \Theta_1^a = \Theta_1 \cap \{3, 4\},$$

and

$$\Lambda = \{s \in \{1, 2, 3, 4\} : p_s > -\infty\}, \quad \Lambda^0 = \Lambda \cap \{1, 2\}, \quad \Lambda^a = \Lambda \cap \{3, 4\}. \tag{2.4}$$

Assumption 2.1 *We assume that the numbers p_s for $s \in \Lambda^0$, q_j for $j \in \Theta_1^0$ are distinct and that the numbers p_s for $s \in \Lambda^a$, q_j for $j \in \Theta_1^a$ are distinct.*

Assumption 2.1 means that for any pair (r, a_j) the term $y^{(r)}(a_j)$ occurs at most once in the boundary conditions (2.2).

We denote the collection of boundary conditions (2.2) by U and define the following operators related to U:

$$U_r y = \left(\sum_{k=0}^{p_j} \alpha_{j,k} y^{(k)} \right)_{j \in \Theta_r}, r = 0, 1, \text{ and } V_1 y = \left(\sum_{k=0}^{q_j} \beta_{j,k} y^{(k)} \right)_{j \in \Theta_1}, \tag{2.5}$$
$$y \in W_4^2(0, a),$$

where $W_4^2(0, a)$ is the Sobolev space of order 4 on the interval $(0, a)$.

We put $l = |\Theta_1|$ and consider the linear operators $A(U)$, K, and M in the space $L_2(0, a) \oplus \mathbb{C}^l$ with domains

$$\mathscr{D}(A(U)) = \left\{ \widetilde{y} = \begin{pmatrix} y \\ V_1 y \end{pmatrix} : y \in W_4^2(0, a),\ U_0 y = 0 \right\}$$
$$\mathscr{D}(K) = \mathscr{D}(M) = L_2(0, a) \oplus \mathbb{C}^l,$$

given by

$$A(U))\widetilde{y} = \begin{pmatrix} y^{(4)} - (gy')' \\ U_1 y \end{pmatrix} \text{ for } \widetilde{y} \in \mathscr{D}(A(U)),\ K = \begin{pmatrix} 0 & 0 \\ 0 & I \end{pmatrix} \text{ and } M = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}.$$

It is easy to check that $K \geq 0$, $M \geq 0$, $M + K = I$ and $M|_{\mathscr{D}(A(U))} > 0$. We associate a quadratic operator pencil

$$L(\lambda) = \lambda^2 M - i\lambda K - A(U), \quad \lambda \in \mathbb{C} \tag{2.6}$$

in the space $L_2(0, a) \oplus \mathbb{C}^l$ with the problem (2.1), (2.2). We observe that (2.6) is an operator representation of the eigenvalue problem (2.1), (2.2) in the sense that a function y satisfies (2.1), (2.2) if and only if it satisfies $L(\lambda)\widetilde{y} = 0$.

We will investigate in the next section the Birkhoff regularity of the problems (2.1), (2.2). That investigation can be conducted using the quasi-derivatives associated with the differential equation (2.1). However, we will use definitions and properties from [5], so we will use the normal derivatives for our investigation. Hence, we are going to write the problem (2.1), (2.2) in the form

$$y^{(4)} - (gy')' = \lambda^2 y, \tag{2.7}$$

$$\Big(\sum_{k=1}^{4} \omega_{j,k}^{(0)}(\lambda) y^{(k-1)}(0) + \omega_{j,k}^{(1)}(\lambda) y^{(k-1)}(a) \Big)_{j=1}^{4} = 0. \tag{2.8}$$

It follows from (2.3) that the representations of $\omega_{j,k}^{(0)}$ and $\omega_{j,k}^{(1)}$, $k = 1, 2, 3, 4$, $j = 1, 2, 3, 4$ are

$$\begin{cases} \omega_{j,k}^{(0)}(\lambda) = \alpha_{j,k} + i\lambda\beta_{j,k} \text{ if } j = 1, 2, \\ \omega_{j,k}^{(0)}(\lambda) = 0 \text{ if } j = 3, 4, \end{cases} \tag{2.9}$$

while

$$\begin{cases} \omega_{j,k}^{(1)}(\lambda) = 0 \text{ if } j = 1, 2, \\ \omega_{j,k}^{(1)}(\lambda) = \alpha_{j,k} + i\lambda\beta_{j,k} \text{ if } j = 3, 4. \end{cases} \tag{2.10}$$

3. Birkhoff regular problems

The characteristic function of (2.1) as defined in [5, (7.1.4)] is $\pi(\rho) = \rho^4 - 1$, and its zeros are i^{k-1}, $k = 1, \ldots, 4$. We can choose

$$C(x, \mu) = \operatorname{diag}(1, \mu, \mu^2, \mu^3)(i^{(j-1)(k-1)})_{j,k=1}^4$$

according to [5, Theorem 7.2.4 A], where $\lambda = \mu^2$. Then it follows that the boundary matrices defined in [5, (7.3.1)] of the problems (2.1), (2.2) are given by

$$W^{(u)}(\mu) := \left(\omega_{j,k}^{(u)}(\mu^2)\right)_{j,k=1}^4 C(a_u, \mu), \quad u = 0, 1, \tag{3.1}$$

where $a_u = 0$ for $u = 0$, while $a_u = a$ for $u = 1$.

It follows from (2.9), (2.10), and (3.1) that

$$W^{(0)}(\mu) = \begin{pmatrix} \gamma_{1,k} \\ \gamma_{2,k} \\ 0 \\ 0 \end{pmatrix}_{k=1}^4, \quad W^{(1)}(\mu) = \begin{pmatrix} 0 \\ 0 \\ \gamma_{3,k} \\ \gamma_{4,k} \end{pmatrix}_{k=1}^4, \tag{3.2}$$

where $\gamma_{j,k} = \sum_{s=0}^{3} \left(\alpha_{j,s} + i\mu^2\beta_{j,s}\right) i^{s(k-1)}\mu^s$, $k = 1, 2, 3, 4$ and $j = 1, 2, 3, 4$.

Recall that p_j and q_j are chosen according to Assumption 2.1, $\beta_{j,q_j} \neq 0$ if $q_j \in \{0, 1, 2, 3\}$ and $\alpha_{j,p_j} = 1$ if $j \in \Lambda$, see (2.4).

Let

$$\nu_j = \max\{p_j, q_j + 2\}. \tag{3.3}$$

Choosing $C_2(\mu) = \operatorname{diag}(\mu^{\nu_1}, \mu^{\nu_2}, \mu^{\nu_3}, \mu^{\nu_4})$ according to [5, Definition 7.3.1 and Theorem 7.3.2], it follows that $C_2(\mu)^{-1}W^{(u)} = W_0^{(u)} + O(\mu^{-1})$, $u = 0, 1$, where

$$W_0^{(0)} = \begin{pmatrix} \omega_{1,k} \\ \omega_{2,k} \\ 0 \\ 0 \end{pmatrix}_{k=1}^4, \quad W_0^{(1)} = \begin{pmatrix} 0 \\ 0 \\ \omega_{3,k} \\ \omega_{4,k} \end{pmatrix}_{k=1}^4, \tag{3.4}$$

and $\omega_{j,k}$ are the coefficients of the terms with the highest degrees of the polynomials $\gamma_{j,k}$ in μ, $k = 1, 2, 3, 4$, $j = 1, 2, 3, 4$.

The Birkhoff matrices are defined as

$$W_0^{(0)}\Delta_k + W_0^{(1)}(I - \Delta_k), \tag{3.5}$$

where Δ_k, $k = 1, 2, 3, 4$ are the 4×4 matrices with 2 consecutive ones and two consecutive zeros in the diagonal in a cyclic arrangement; see [5, Definition 7.3.1 and Proposition 4.1.7]. For definiteness Δ_1, Δ_2,

Δ_3, Δ_4 are respectively the matrices with 2 consecutive ones and two consecutive zeros in the diagonal with a cyclic arrangement starting respectively from the first, the second, the third, and the fourth columns. It is easy to see that after a permutation of columns, the matrices (3.5) are block diagonal matrices taken from two consecutive columns (in the sense of cyclic arrangement) of the first two rows of $W_0^{(0)}$ and the last two rows of $W_0^{(1)}$ respectively.

Let $\Gamma_{0,k}$ and $\Gamma_{2,k+2}$, $k = 1,2,3,4$ be the matrices respectively obtained from the first two rows and two consecutive columns of $W_0^{(0)}$ and the last two rows and two consecutive columns of $W_0^{(1)}$, defined by

$$\Gamma_{2u,k+2u} = \begin{pmatrix} \omega_{1+2u,k+2u} & \omega_{1+2u,k+1+2u} \\ \omega_{2+2u,k+2u} & \omega_{2+2u,k+1+2u} \end{pmatrix}, \; k = 1,2,3,4; u = 0,1. \tag{3.6}$$

The indices of the entries of the above matrix are such that $k+2u \equiv k+2u-4 \mod 4$ and $k+1+2u \equiv k+1+2u-4 \mod 4$, where $u = 0,1$. The determinants of the Birkhoff matrices are

$$\det\left[W_0^{(0)}\Delta_k + W_0^{(1)}(I-\Delta_k)\right] = \pm\det\Gamma_{0,k} \times \det\Gamma_{2,k+2}. \tag{3.7}$$

The eigenvalue problems under consideration can be classified by the powers p_j and q_j of the derivatives in the boundary conditions (2.8). Those classifications are given by:

$$\begin{cases} p_{j+2u} > q_{j+2u} + 2, \\ p_{j+2u} < q_{j+2u} + 2, \\ p_{j+2u} = q_{j+2u} + 2. \end{cases} \tag{3.8}$$

Hence, we have three different cases for each boundary condition. Since we have two boundary conditions per endpoint, then we have in total nine different cases for each endpoint of which three pairs are redundant. The three redundant pair cases are:

1) $p_{1+2u} > q_{1+2u} + 2$, $q_{2+2u} + 2 > p_{2+2u}$ and $q_{1+2u} + 2 > p_{1+2u}$, $p_{2+2u} > q_{2+2u} + 2$,
2) $p_{1+2u} > q_{1+2u} + 2$, $p_{2+2u} = q_{2+2u} + 2$ and $p_{1+2u} = q_{1+2u} + 2$, $p_{2+2u} > q_{2+2u} + 2$,
3) $q_{1+2u} + 2 > p_{1+2u}$, $p_{2+2u} = q_{2+2u} + 2$ and $p_{1+2u} = q_{1+2u} + 2$, $q_{2+2u} + 2 > p_{2+2u}$.

We are going to adopt the following convention to eliminate the redundancies: if exactly one of the left endpoint boundary conditions depends on λ, we will enumerate it as the second boundary condition, and if exactly one of the right endpoint boundary conditions depends on λ, we will enumerate it as the fourth boundary condition. In the case where both boundary conditions at the same endpoint depend on λ then the one with the highest q_j power will be enumerated as the second boundary condition if the two boundary conditions are left endpoint boundary conditions, while it will be enumerated as the fourth boundary condition if the two boundary conditions are right endpoint boundary conditions. Hence, we are left with six different cases for each endpoint that we will denote by $\text{Case}^{(u)} r$, $u = 0,1$ and $r = 1,2,3,4,5,6$. The six different cases are:

$\text{Case}^{(\mathrm{u})}$ 1: $p_{1+2u} > q_{1+2u} + 2$, $p_{2+2u} > q_{2+2u} + 2$;

$\text{Case}^{(\mathrm{u})}$ 2: $p_{1+2u} > q_{1+2u} + 2$, $q_{2+2u} + 2 > p_{2+2u}$;

$\text{Case}^{(\mathrm{u})}$ 3: $p_{1+2u} > q_{1+2u} + 2$, $p_{2+2u} = q_{2+2u} + 2$;

$\text{Case}^{(\mathrm{u})}$ 4: $q_{1+2u} + 2 > p_{1+2u}$, $q_{2+2u} + 2 > p_{2+2u}$;

Case$^{(\mathrm{u})}$ 5: $q_{1+2u}+2>p_{1+2u}$, $p_{2+2u}=q_{2+2u}+2$;

Case$^{(\mathrm{u})}$ 6: $p_{1+2u}=q_{1+2u}+2$, $p_{2+2u}=q_{2+2u}+2$.

We are now going to evaluate $\det\Gamma_{2u,k+2u}$, where $u=0,1$ and $k=1,2,3,4$. For the above six cases we obtain:

Case$^{(\mathrm{u})}$ 1: $p_{1+2u}>q_{1+2u}+2$ and $p_{2+2u}>q_{2+2u}+2$.

$$\Gamma_{2u,k+2u}=\begin{pmatrix} i^{(k-1+2u)p_{1+2u}} & i^{(k+2u)p_{1+2u}} \\ i^{(k-1+2u)p_{2+2u}} & i^{(k+2u)p_{2+2u}} \end{pmatrix}$$

and

$$\det\Gamma_{2u,k+2u}=i^{(k+2u)(p_{1+2u}+p_{2+2u})}\left(i^{-p_{1+2u}}-i^{-p_{2+2u}}\right)\neq 0, \tag{3.9}$$

since $p_{1+2u}\neq p_{2+2u}$, $u=0,1$, according to Assumption 2.1.

Case$^{(\mathrm{u})}$ 2: $p_{1+2u}>q_{1+2u}+2$ and $q_{2+2u}+2>p_{2+2u}$.

$$\Gamma_{2u,k+2u}=\begin{pmatrix} i^{(k-1+2u)p_{1+2u}} & i^{(k+2u)p_{1+2u}} \\ i^{(k-1+2u)q_{2+2u}+1}\beta_{2+2u,q_{2+2u}} & i^{(k+2u)q_{2+2u}+1}\beta_{2+2u,q_{2+2u}} \end{pmatrix}$$

and

$$\det\Gamma_{2u,k+2u}=i^{(k+2u)(p_{1+2u}+q_{2+2u})+1}\beta_{2+2u,q_{2+2u}}\left(i^{-p_{1+2u}}-i^{-q_{2+2u}}\right)\neq 0, \tag{3.10}$$

as $p_{1+2u}\neq q_{2+2u}$, $u=0,1$, according to Assumption 2.1.

Case$^{(\mathrm{u})}$ 3: $p_{1+2u}>q_{1+2u}+2$ and $p_{2+2u}=q_{2+2u}+2$.

$$\Gamma_{2u,k+2u}=\begin{pmatrix} i^{(k-1+2u)p_{1+2u}} & i^{(k+2u)p_{1+2u}} \\ i^{(k-1+2u)q_{2+2u}}\left(i\beta_{2+2u,q_{2+2u}}+(-1)^{k-1+2u}\right) & i^{(k+2u)q_{2+2u}}\left(i\beta_{2+2u,q_{2+2u}}+(-1)^{k+2u}\right) \end{pmatrix} \tag{3.11}$$

and

$$\begin{aligned}\det\Gamma_{2u,k+2u}=i^{(k+2u)(p_{1+2u}+q_{2+2u})}\Big[&i\beta_{2+2u,q_{2+2u}}\left(i^{-p_{1+2u}}-i^{-q_{2+2u}}\right)\\ &+(-1)^k\left(i^{-p_{1+2u}}+i^{-q_{2+2u}}\right)\Big].\end{aligned} \tag{3.12}$$

Since $p_{1+2u}\neq q_{2+2u}$, $u=0,1$, see Assumption 2.1, it follows from (3.12) that

$$\begin{aligned}\det\Gamma_{2u,k+2u}=0\Leftrightarrow\beta_{2+2u,q_{2+2u}}&=\frac{(-1)^{k+1}\left(i^{-p_{1+2u}}+i^{-q_{2+2u}}\right)}{i\left(i^{-p_{1+2u}}-i^{-q_{2+2u}}\right)}\\ &=\frac{(-1)^{k+1}\left(1+i^{p_{1+2u}-q_{2+2u}}\right)}{i\left(1-i^{p_{1+2u}-q_{2+2u}}\right)}.\end{aligned} \tag{3.13}$$

Recall that $p_{1+2u},q_{1+2u}\in\{0,1,2,3\}$. Since $p_{1+2u}>q_{2+2u}+2$, $u=0,1$, then $p_{1+2u}-q_{1+2u}=3$, $u=0,1$. Hence, it follows from (3.13) that

$$\det\Gamma_{2u,k+2u}=0\Leftrightarrow\beta_{2+2u,q_{2+2u}}=(-1)^k, \tag{3.14}$$

where $k = 1, 2$.

Case$^{(u)}$ 4: $q_{1+2u} + 2 > p_{1+2u}$ and $q_{2+2u} + 2 > p_{2+2u}$.

$$\Gamma_{2u,k+2u} = \begin{pmatrix} i^{(k-1+2u)q_{1+2u}+1}\beta_{1+2u,q_{1+2u}} & i^{(k+2u)q_{1+2u}+1}\beta_{1+2u,q_{1+2u}} \\ i^{(k-1+2u)q_{2+2u}+1}\beta_{2+2u,q_{2+2u}} & i^{(k+2u)q_{2+2u}+1}\beta_{2+2u,q_{2+2u}} \end{pmatrix}$$

and

$$\begin{aligned} \det \Gamma_{2u,k+2u} &= i^{(k+2u)(q_{1+2u}+q_{2+2u})+2}\beta_{1+2u,q_{1+2u}}\beta_{2+2u,q_{2+2u}}(i^{-q_{1+2u}} - i^{-q_{2+2u}}) \\ &\neq 0, \end{aligned} \tag{3.15}$$

as $q_{1+2u} \neq q_{2+2u}$, $u = 0, 1$, see Assumption 2.1.

Case$^{(u)}$ 5: $q_{1+2u} + 2 > p_{1+2u}$ and $p_{2+2u} = q_{2+2u} + 2$.

$$\Gamma_{2u,k+2u} = \begin{pmatrix} i^{(k-1+2u)q_{1+2u}+1}\beta_{1+2u,q_{1+2u}} & i^{(k+2u)q_{1+2u}+1}\beta_{1+2u,q_{1+2u}} \\ i^{(k-1+2u)q_{2+2u}}\left(i\beta_{2+2u,q_{2+2u}} + (-1)^{k-1+2u}\right) & i^{(k+2u)q_{2+2u}}\left(i\beta_{2+2u,q_{2+2u}} + (-1)^{k+2u}\right) \end{pmatrix} \tag{3.16}$$

and

$$\begin{aligned} \det \Gamma_{2u,k+2u} = i^{(k+2u)(q_{1+2u}+q_{2+2u})+1}\beta_{1+2u,q_{1+2u}}\Big[& i\beta_{2+2u,q_{2+2u}}(i^{-q_{1+2u}} - i^{-q_{2+2u}}) \\ &+ (-1)^k(i^{-q_{1+2u}} + i^{-q_{2+2u}})\Big]. \end{aligned} \tag{3.17}$$

Since $p_{1+2u} \neq q_{2+2u}$, $u = 0, 1$, see Assumption 2.1, it follows from (3.17) that

$$\begin{aligned} \det \Gamma_{2u,k+2u} = 0 \Leftrightarrow \beta_{2+2u,q_{2+2u}} &= \frac{(-1)^{k+1}(i^{-q_{1+2u}} + i^{-q_{2+2u}})}{i(i^{-q_{1+2u}} - i^{-q_{2+2u}})} \\ &= \frac{(-1)^{k+1}(1 + i^{q_{1+2u}-q_{2+2u}})}{i(1 - i^{q_{1+2u}-q_{2+2u}})}. \end{aligned} \tag{3.18}$$

Recall that the numbers p_{j+2u}, q_{j+2u}, $j = 1, 2$ and $u = 0, 1$ are mutually distinct; see Assumption 2.1. Recall as well that $q_{1+2u} \in \{0, 1, 2, 3\}$. Since $p_{2+2u} = q_{2+2u} + 2$, then if follows that the pair (p_{2+2u}, q_{2+2u}) only has two values, which are $(2, 0)$ and $(3, 1)$. If $q_{1+2u} = 0$, then it follows that $q_{2+2u} = 1$ and $p_{2+2u} = 3$, and hence $p_{1+2u} = 2$, which is not possible, since $q_{1+2u} + 2 > p_{1+2u}$. On the other hand if $q_{1+2u} = 1$, then $q_{2+2u} = 0$ and $p_{2+2u} = 2$, thus $p_{1+2u} = 3$ which is not possible since $q_{1+2u} + 2 > p_{1+2u}$. However, if $q_{1+2u} = 2$, then $q_{2+2u} = 1$ and $p_{2+2u} = 3$, and hence $p_{1+2u} = 0$. Finally, if $q_{1+2u} = 3$, then $q_{2+2u} = 0$ and $p_{2+2u} = 2$ and it follows that $p_{1+2u} = 1$. Hence, the possible values of the pair (q_{1+2u}, q_{2+2u}) are $(2, 1)$ and $(3, 0)$. Thus, it follows from (3.18) that

$$\det \Gamma_{2u,k+2u} = 0 \Leftrightarrow \beta_{2+2u,q_{2+2u}} = \begin{cases} (-1)^{k+1} & \text{if} \quad q_{1+2u} - q_{2+2u} = 1, \\ (-1)^k & \text{if} \quad q_{1+2u} - q_{2+2u} = 3, \end{cases} \tag{3.19}$$

where $k = 1, 2$.

Case$^{(u)}$ 6: $p_{1+2u} = q_{1+2u} + 2$ and $p_{2+2u} = q_{2+2u} + 2$.

Since for $p_j = q_j + 2$, $j = 1, 2, 3, 4$, we will have $p_j = 3$ and $q_j = 1$ or $p_j = 2$ and $q_j = 0$, $j = 1, 2, 3, 4$. Hence, $p_{1+2u} = 2$, $q_{1+2u} = 0$ while $p_{2+2u} = 3$, $q_{2+2u} = 1$, $u = 0, 1$ and it follows from the second row of the matrix (3.11) or (3.16) that

$$\Gamma_{2u,k+2u} = \begin{pmatrix} (-1)^{k-1} + i\beta_{1+2u,0} & (-1)^k + i\beta_{1+2u,0} \\ i^{k-1+2u}((-1)^{k-1+2u} + i\beta_{2+2u,1}) & i^{k+2u}((-1)^{k+2u} + i\beta_{2+2u,1}) \end{pmatrix}$$

and

$$\begin{aligned} \det \Gamma_{2u,k+2u} &= i^{k+2u}\left((-1)^{k-1} + i\beta_{1+2u,0}\right)\left((-1)^k + i\beta_{2+2u,1}\right) \\ &\quad - i^{k-1+2u}\left((-1)^k + i\beta_{1+2u,0}\right)\left((-1)^{k-1} + i\beta_{2+2u,1}\right) \\ &= i^{k-1+2u}\left[\left((-1)^{k-1}i - \beta_{1+2u,0}\right)\left((-1)^k + i\beta_{2+2u,1}\right)\right. \\ &\quad \left. - \left((-1)^k + i\beta_{1+2u,0}\right)\left((-1)^{k-1} + i\beta_{2+2u,1}\right)\right] \\ &= i^{k-1+2u}\left[-i + (-1)^k\beta_{2+2u,1} - (-1)^k\beta_{1+2u,0}\right. \\ &\quad -\beta_{1+2u,0}\beta_{2+2u,1}i + 1 - (-1)^k\beta_{2+2u,1}i + (-1)^k\beta_{1+2u,0}i \\ &\quad \left. +\beta_{1+2u,0}\beta_{2+2u,1}\right] \\ &= (1-i)i^{k-1+2u}\left[1 - (-1)^k\beta_{1+2u,0} + \beta_{2+2u,1}((-1)^k + \beta_{1+2u,0})\right]. \end{aligned} \tag{3.20}$$

It follows from (3.20) that if

$$\beta_{1+2u,0} = (-1)^{k-1} \quad \text{then} \quad \det \Gamma_{2u,k+2u} = 2(1-i)i^{k-1+2u} \neq 0. \tag{3.21}$$

However, if $\beta_{1+2u,0} \neq (-1)^{k-1}$ then

$$\det \Gamma_{2u,k+2u} = 0 \Leftrightarrow \beta_{2+2u,1} = \frac{-1 + (-1)^k\beta_{1+2u,0}}{(-1)^k + \beta_{1+2u,0}}. \tag{3.22}$$

It follows from (3.9), (3.10), (3.15), and (3.21) and from $\text{Case}^{(\text{u})}$ 1, $\text{Case}^{(\text{u})}$ 2, $\text{Case}^{(\text{u})}$ 4, and $\text{Case}^{(\text{u})}$ 6 with $\beta_{1+2u,0} = (-1)^{k-1}$ that:

Proposition 3.1 *Let $p_j, q_j \in \{0, 1, 2, 3\}$, where p_j, q_j are as defined in Assumption 2.1, $j = 1, 2, 3, 4$. Let u such that $u = 0$ if $j = 1, 2$ and $u = 1$ if $j = 3, 4$. Let $k = 1, 2, 3, 4$. Then*

$$\det \Gamma_{2u,k+2u} \neq 0$$

for the following conditions:

1) $p_{1+2u} > q_{1+2u} + 2$ *and* $p_{2+2u} > q_{2+2u} + 2$,
2) $p_{1+2u} > q_{1+2u} + 2$ *and* $q_{2+2u} + 2 > p_{2+2u}$,
3) $q_{1+2u} + 2 > p_{1+2u}$ *and* $q_{2+2u} + 2 > p_{2+2u}$,
4) $p_{1+2u} = q_{1+2u} + 2$, $p_{2+2u} = q_{2+2u} + 2$ *and* $\beta_{1+2u,0} = (-1)^{k-1}$.

Remark 3.2 For the remaining following three cases, $\text{Case}^{(\text{u})}$ 3, $\text{Case}^{(\text{u})}$ 5, and $\text{Case}^{(\text{u})}$ 6 with $\beta_{1+2u,0} \neq (-1)^{k-1}$, $k = 1, 2$, some additional conditions are needed for $\det \Gamma_{2u,k+2u} \neq 0$. These conditions are given in Proposition 3.3.

It follows from (3.14), (3.19), and (3.22) that:

Proposition 3.3 *Let* $p_j, q_j \in \{0,1,2,3\}$, *where* p_j, q_j *are as defined in Assumption 2.1,* $j = 1,2,3,4$. *Let* u *such that* $u = 0$ *if* $j = 1,2$ *and* $u = 1$ *if* $j = 3,4$. *Let* $k = 1,2,3,4$. *Then*

$$\det \Gamma_{2u,k+2u} \neq 0$$

for the following three conditions:
1) $p_{1+2u} > q_{1+2u} + 2$, $p_{2+2u} = q_{2+2u} + 2$ *and* $\beta_{2+2u,q_{2+2u}} \neq (-1)^k$,
2) $q_{1+2u} + 2 > p_{1+2u}$, $p_{2+2u} = q_{2+u} + 2$ *and*

$$\beta_{2+2u,q_{2+2u}} \neq \begin{cases} (-1)^{k+1} & \text{if} \quad q_{1+2u} - q_{2+2u} = 1, \\ (-1)^k & \text{if} \quad q_{1+2u} - q_{2+2u} = 3, \end{cases}$$

3) $p_{1+2u} = q_{1+2u} + 2$, $p_{2+2u} = q_{2+2u} + 2$, $\beta_{1+2u,0} \neq (-1)^{k-1}$ *and* $\beta_{2+2u,1} \neq \dfrac{-1 + (-1)^k \beta_{1+2u,0}}{(-1)^k + \beta_{1+2u,0}}$.

Let $C(r,u)$, $r = 1,2,3,4,5,6$, $u = 0,1$, be the following conditions:
$C(1,u)$: $p_{1+2u} > q_{1+2u} + 2$, $p_{2+2u} > q_{2+2u} + 2$;
$C(2,u)$: $p_{1+2u} > q_{1+2u} + 2$, $q_{2+2u} + 2 > p_{2+2u}$;
$C(3,u)$: $p_{1+2u} > q_{1+2u} + 2$, $p_{2+2u} = q_{2+2u} + 2$ and $\beta_{2+2u,q_{2+2u}} \neq (-1)^k$, where $k = 1,2$;
$C(4,u)$: $q_{1+2u} + 2 > p_{1+2u}$, $q_{2+2u} + 2 > p_{2+2u}$;
$C(5,u)$: $q_{1+2u} + 2 > p_{1+2u}$, $p_{2+2u} = q_{2+2u} + 2$,

$$\beta_{2+2u,q_{2+2u}} \neq \begin{cases} (-1)^{k+1} & \text{if} \quad q_{1+2u} - q_{2+2u} = 1, \\ (-1)^k & \text{if} \quad q_{1+2u} - q_{2+2u} = 3, \end{cases}$$

where $k = 1,2$;
$C(6,u)$: $p_{1+2u} = q_{1+2u} + 2$, $p_{2+2u} = q_{2+2u} + 2$ and $\beta_{1+2u,0} = (-1)^{k-1}$ or $p_{1+2u} = q_{1+2u} + 2$, $p_{2+2u} = q_{2+2u} + 2$, $\beta_{1+2u,0} \neq (-1)^{k-1}$ and $\beta_{2+2u,1} \neq \dfrac{-1 + (-1)^k \beta_{1+2u,0}}{(-1)^k + \beta_{1+2u,0}}$, where $k = 1,2$.

Then it follows from Proposition 3.1, Proposition 3.3, and [5, Definition 7.3.1 and Proposition 4.1.7] that:

Theorem 3.4 *The problems* (2.1), (2.2) *are Birkhoff regular if and only if there are* $r_0, r_1 \in \{1,2,3,4,5,6\}$ *such that the conditions* $C(r_0, 0)$ *and* $C(r_1, 1)$ *hold.*

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