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An improved singular Trudinger-Moser inequality in dimension two

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Abstract: Let $\Omega \subset \mathbb{R}^2$ be a smooth bounded domain and $W_0^{1,2}(\Omega)$ be the usual Sobolev space. Let β, $0 \leq \beta < 2$, be fixed. Define for any real number $p > 1$,

$$\lambda_{p,\beta}(\Omega) = \inf_{u \in W_0^{1,2}(\Omega),\, u \not\equiv 0} \|\nabla u\|_2^2 / \|u\|_{p,\beta}^2,$$

where $\|\cdot\|_2$ denotes the standard L^2-norm in Ω and $\|u\|_{p,\beta} = (\int_\Omega |x|^{-\beta}|u|^p dx)^{1/p}$. Suppose that γ satisfies $\frac{\gamma}{4\pi} + \frac{\beta}{2} = 1$. Using a rearrangement argument, the author proves that

$$\sup_{u \in W_0^{1,2}(\Omega), \|\nabla u\|_2 \leq 1} \int_\Omega |x|^{-\beta} e^{\gamma u^2 \left(1 + \alpha \|u\|_{p,\beta}^2\right)} dx$$

is finite for any α, $0 \leq \alpha < \lambda_{p,\beta}(\mathbb{B}_R)$, where $\mathbb{B}_R$ stands for the disc centered at the origin with radius R verifying that πR^2 is equal to the area of Ω. Moreover, when $\Omega = \mathbb{B}_R$, the above supremum is infinity if $\alpha \geq \lambda_{p,\beta}(\mathbb{B}_R)$. This extends earlier results of Adimurthi and Druet, Y. Yang, Adimurthi and Sandeep, Adimurthi and Yang, Lu and Yang, and J. Zhu in dimension two.

Key words: Trudinger–Moser inequality, singular Trudinger–Moser inequality

1. Introduction

Let Ω be a bounded smooth domain in $\mathbb{R}^2$, and $W_0^{1,2}(\Omega)$ be the completion of $C_0^\infty(\Omega)$ under the norm $\|u\|_{W_0^{1,2}(\Omega)} = (\int_\Omega |\nabla u|^2 dx)^{1/2}$. The Trudinger–Moser inequality [9–11, 13, 19] says

$$\sup_{u \in W_0^{1,2}(\Omega), \|\nabla u\|_2 = 1} \int_\Omega e^{\gamma u^2} dx < +\infty \tag{1.1}$$

for any $\gamma \leq 4\pi$. Here and in the sequel, $\|\cdot\|_2$ denotes the standard L^2-norm. Moreover, for any $\gamma > 4\pi$, the supremum in (1.1) is infinity.

The inequality (1.1) was improved in many ways. It was proved by Adimurthi and Druet [1] that for any α, $0 \leq \alpha < \lambda_1(\Omega)$, the first eigenvalue of the Laplace operator with respect to the Dirichlet boundary

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condition, there holds

$$\sup_{u\in W_0^{1,2}(\Omega),\,\|\nabla u\|_2\le 1}\int_\Omega e^{4\pi u^2(1+\alpha\|u\|_2^2)}dx<+\infty; \tag{1.2}$$

While if $\alpha\ge\lambda_1(\Omega)$, then the supremum in (1.2) is infinity. It was then extended by Yang [14–16] to the general dimensional case and Riemannian surface cases, and by de Souza and do Ó [4] to $\mathbb{R}^2$. Moreover, Lu and Yang [8] extended L^2-norm to L^p-norm in (1.2). Precisely, letting $p>1$ and $0\le\alpha<\lambda_p(\Omega)$ be fixed, there holds

$$\sup_{u\in W_0^{1,2}(\Omega),\|\nabla u\|_2\le 1}\int_\Omega e^{4\pi u^2(1+\alpha\|u\|_p^2)}dx<+\infty; \tag{1.3}$$

If $\alpha\ge\lambda_p(\Omega)$, then the supremum in (1.3) is infinity. Here

$$\lambda_p(\Omega)=\inf_{u\in W_0^{1,2}(\Omega),\,u\not\equiv 0}\frac{\int_\Omega|\nabla u|^2dx}{\left(\int_\Omega|u|^pdx\right)^{2/p}}.$$

Another interesting improvement of (1.1) is due to Adimurthi and Sandeep [2], who derived a singular Trudinger–Moser inequality. Namely, if $0\le\beta<n$ and $\frac{\gamma}{\alpha_n}+\frac{\beta}{n}=1$, then there holds

$$\sup_{u\in W_0^{1,n}(\Omega),\,\|\nabla u\|_n\le 1}\int_\Omega|x|^{-\beta}e^{\gamma|u|^{\frac{n}{n-1}}}dx<+\infty; \tag{1.4}$$

If $\gamma>\alpha_n(1-\beta/n)$, the supremum in (1.4) is infinity. Here $\alpha_n=n\omega_{n-1}^{1/(n-1)}$, ω_{n-1} is the area of the unit sphere in $\mathbb{R}^n$. This inequality was extended by Adimurthi and Yang [3] to $\mathbb{R}^n$, by de Souza and do Ó [5] to $\mathbb{R}^2$, and by Yang [17, 18] to $\mathbb{R}^4$ and Riemannian manifold.

In this note, we combine (1.3) and (1.4) in the case $n=2$. Let Ω be a smooth bounded domain in $\mathbb{R}^2$. Here and throughout this note we assume $0\in\Omega$. Let $p>1$ and $0\le\beta<2$ be fixed. We define

$$\lambda_{p,\beta}(\Omega)=\inf_{u\in W_0^{1,2}(\Omega),\,u\not\equiv 0}\frac{\int_\Omega|\nabla u|^2dx}{\left(\int_\Omega|x|^{-\beta}|u|^pdx\right)^{2/p}}. \tag{1.5}$$

In the sequel, we write for simplicity

$$\|u\|_{p,\beta}=\left(\int_\Omega|x|^{-\beta}|u|^pdx\right)^{1/p}. \tag{1.6}$$

Our main result is the following:

Theorem 1.1. *Let Ω be a smooth bounded domain in $\mathbb{R}^2$, and $\mathbb{B}_R\subset\mathbb{R}^2$ be the disc centered at the origin with radius R verifying that πR^2 is equal to the area of Ω. Let $p>1$ and $0\le\beta<2$ be fixed and $\lambda_{p,\beta}$ be defined as in (1.5). Then we have*

(i) for any α, $0\le\alpha<\lambda_{p,\beta}(\mathbb{B}_R)$, there holds

$$\sup_{u\in W_0^{1,2}(\Omega),\|\nabla u\|_2\le 1}\int_\Omega|x|^{-\beta}e^{\gamma u^2\left(1+\alpha\|u\|_{p,\beta}^2\right)}dx<+\infty; \tag{1.7}$$

(ii) when $\Omega = \mathbb{B}_R$, *for any* $\alpha \geq \lambda_{p,\beta}(\mathbb{B}_R)$,

$$\sup_{u \in W_0^{1,2}(\Omega), \|\nabla u\|_2 \leq 1} \int_\Omega |x|^{-\beta} e^{\gamma u^2 \left(1+\alpha \|u\|_{p,\beta}^2\right)} dx = +\infty.$$

Clearly Theorem 1.1 generalizes results of Adimurthi and Sandeep [2], Yang [14], and Lu and Yang [8] in dimension two. The proof of Theorem 1.1 is based on a rearrangement argument and test function computation.

The remaining part of this note is organized as follows. In Section 2, using a variational direct method and rearrangement argument, we prove three lemmas on eigenvalues. Theorem 1.1 is proved in Section 3.

2. Preliminary results

In this section, we study the properties of eigenvalues defined as in (1.5). The proof is based on a variational direct method, symmetrization, and change of variables.

Lemma 2.1. *For any real number* $p > 1$ *and any* β, $0 \leq \beta < 2$, *we have* $\lambda_{p,\beta}(\Omega) > 0$. *Moreover,* $\lambda_{p,\beta}(\Omega)$ *can be attained by a function* $\phi_0 \in W_0^{1,2}(\Omega)$ *satisfying*

$$\begin{cases} -\Delta \phi_0 = \lambda_{p,\beta}(\Omega)|x|^{-\beta}\|\phi_0\|_{p,\beta}^{2-p}{\phi_0}^{p-1} \quad \text{in} \quad \Omega \\ \|\nabla \phi_0\|_2 = 1, \quad \phi_0 \geq 0 \quad \text{in} \quad \Omega, \end{cases} \tag{2.1}$$

where $\|\cdot\|_{p,\beta}$ *is defined as in (1.6).*

Proof. Choose a sequence of functions $u_k \in W_0^{1,2}(\Omega)$ such that $\|u_k\|_{p,\beta} = 1$ and $\|\nabla u_k\|_2^2 \to \lambda_{p,\beta}(\Omega)$. It follows that u_k is bounded in $W_0^{1,2}(\Omega)$. Without loss of generality, we assume

$$u_k \rightharpoonup u_0 \quad \text{weakly in} \quad W_0^{1,2}(\Omega), \tag{2.2}$$

$$u_k \to u_0 \quad \text{strongly in} \quad L^q(\Omega), \quad \forall q \geq 1. \tag{2.3}$$

In view of (2.3), the Hölder inequality leads to $\|u_0\|_{p,\beta} = 1$, while (2.2) implies that

$$\lim_{k \to +\infty} \int_\Omega \nabla u_k \nabla u_0 dx = \int_\Omega |\nabla u_0|^2 dx,$$

which leads to

$$\int_\Omega |\nabla u_0|^2 dx \leq \limsup_{k \to +\infty} \int_\Omega |\nabla u_k|^2 dx = \lambda_{p,\beta}(\Omega).$$

Hence u_0 attains $\lambda_{p,\beta}(\Omega)$ and in particular, $\lambda_{p,\beta}(\Omega) > 0$. Obviously, $|u_0|$ is also a minimizer and thus we can assume $u_0 \geq 0$. Set

$$\phi_0 = u_0/\|\nabla u_0\|_2.$$

Then ϕ_0 attains $\lambda_{p,\beta}(\Omega)$ and satisfies the Euler–Lagrange equation (2.1). By the elliptic regularity theory (see [6], Chapter 9), $\phi_0 \in C^1(\Omega \setminus \{0\}) \cap C^\nu(\Omega)$ for some $0 < \nu < 1$. □

If Ω is replaced by the disc $\mathbb{B}_R$ in Lemma 2.1, we have the following:

Lemma 2.2. *$\lambda_{p,\beta}(\mathbb{B}_R) > 0$, and $\lambda_{p,\beta}(\mathbb{B}_R)$ can be attained by some radially symmetric decreasing function $\phi_0 \in W_0^{1,2}(\mathbb{B}_R)$ satisfying (2.1) with Ω replaced by $\mathbb{B}_R$.*

Proof. For any $u \in C_0^\infty(\mathbb{B}_R)$, let u^* be the nonnegative decreasing rearrangement of $|u|$. Using the Hardy–Littlewood inequality (see for examples [7, 12]), we have

$$\int_{\mathbb{B}_R} |\nabla u^*|^2 dx \le \int_{\mathbb{B}_R} |\nabla u|^2 dx$$

and

$$\int_{\mathbb{B}_R} |x|^{-\beta}|u|^p dx \le \int_{\mathbb{B}_R} |x|^{-\beta} u^{*p} dx. \tag{2.4}$$

This together with the definition of $\lambda_{p,\beta}(\mathbb{B}_R)$ implies that

$$\lambda_{p,\beta}(\mathbb{B}_R) = \inf \frac{\|\nabla u\|_2^2}{\|u\|_{p,\beta}},$$

where the infimum takes over all nonnegative radially symmetric decreasing functions in $W_0^{1,2}(\mathbb{B}_R)$. Then by the same procedure as in the proof of Lemma 2.1 we can find the desired minimizer ϕ_0. □

For simplicity, we denote for any $q > 1$ and $r > 0$,

$$\|u\|_{q,\mathbb{B}_r} = \left(\int_{\mathbb{B}_r} |u|^q dx\right)^{1/q}, \quad \|u\|_{q,\beta,\mathbb{B}_r} = \left(\int_{\mathbb{B}_r} |x|^{-\beta}|u|^q dx\right)^{1/q}.$$

Lemma 2.3. *Let $p > 1$ and $0 \le \beta < 2$ be fixed. Then there holds*

$$\lambda_{p,\beta}(\mathbb{B}_R) = (1-\beta/2)^{1+2/p}\lambda_p(\mathbb{B}_{R^{1-\beta/2}}),$$

where $\lambda_p(\mathbb{B}_{R^{1-\beta/2}}) = \inf_{\|u\|_p=1} \|\nabla u\|_2^2$, and $\|\cdot\|_2$ denotes the $L^2(\mathbb{B}_{R^{1-\beta/2}})$-norm.

Proof. For simplicity, we write $a = 1-\beta/2$. On one hand, there exists some nonnegative radially symmetric function $v \in W_0^{1,2}(\mathbb{B}_{R^a})$ such that $\|v\|_{p,\mathbb{B}_{R^a}} = 1$ and

$$\|\nabla v\|_{2,\mathbb{B}_{R^a}}^2 = \lambda_p(\mathbb{B}_{R^a}). \tag{2.5}$$

We write $v(r) = v(x)$ with $r = |x|$. Define a new radially symmetric function

$$u(r) = a^{-1/2} v(r^a) \quad \text{for} \quad r \in [0, R].$$

Such a change of variable was also used by Adimurthi and Sandeep [2]. It follows that

$$\begin{aligned} \|u\|^2_{p,\beta,\mathbb{B}_R} &= \left(\int_{\mathbb{B}_R} |x|^{-\beta} u^p dx\right)^{2/p} \\ &= \left(\int_0^R 2\pi (u(r))^p r^{1-\beta} dr\right)^{2/p} \\ &= a^{-(1+2/p)} \left(\int_0^{R^a} 2\pi (v(t))^p t dt\right)^{2/p} \\ &= a^{-(1+2/p)} \|v\|^2_{p,\mathbb{B}_{R^a}} \\ &= a^{-(1+2/p)} \end{aligned} \tag{2.6}$$

and that

$$\begin{aligned} \|\nabla u\|^2_{2,\mathbb{B}_R} &= \int_0^R 2\pi r |u'(r)|^2 dr \\ &= a \int_0^R 2\pi r^{2a-1} |v'(r^a)|^2 dr \\ &= \int_0^{R^a} 2\pi t |v'(t)|^2 dt \\ &= \|\nabla v\|^2_{2,\mathbb{B}_{R^a}}. \end{aligned} \tag{2.7}$$

In view of (2.5) and the definition of $\lambda_{p,\beta}$, we conclude

$$a^{1+2/p} \lambda_p(\mathbb{B}_{R^a}) \geq \lambda_{p,\beta}(\mathbb{B}_R). \tag{2.8}$$

On the other hand, by Lemma 2.2, there exists some nonnegative radially symmetric function $u \in W_0^{1,2}(\mathbb{B}_R)$ such that $\|u\|^2_{p,\beta,\mathbb{B}_R} = 1$ and $\|\nabla u\|^2_{2,\mathbb{B}_R} = \lambda_{p,\beta}(\mathbb{B}_R)$. Set

$$v(r) = \sqrt{a} u(r^{1/a}) \quad \text{for} \quad r \in [0, R^a].$$

Repeating the above calculation, we have $\|\nabla v\|^2_{2,\mathbb{B}_{R^a}} = \|\nabla u\|^2_{2,\mathbb{B}_R}$ and

$$\|v\|^2_{p,\mathbb{B}_{R^a}} = a^{1+2/p} \left(\int_{\mathbb{B}_R} |x|^{-\beta} u^p dx\right)^{2/p} = a^{1+2/p} \|u\|^2_{p,\beta,\mathbb{B}_R} = a^{1+2/p}.$$

This implies that

$$a^{1+2/p} \lambda_p(\mathbb{B}_{R^a}) \leq \lambda_{p,\beta}(\mathbb{B}_R). \tag{2.9}$$

Combining (2.8) and (2.9), we conclude the lemma. □

3. Proof of Theorem 1.1

In this section, we shall prove Theorem 1.1. For (i) of Theorem 1.1, we use a symmetrization argument and a change of variables, which was also used by Adimurthi and Sandeep [2]. For (ii) of Theorem 1.1, we employ the test function constructed by Yang [14] and Lu and Yang [8]. However, our calculation is more delicate; specifically the singular eigenvalue $\lambda_{p,\beta}(\mathbb{B}_R)$ is essentially involved.

Proof of (i) of Theorem 1.1. Let $p>1$ and $0\leq\beta<2$ be fixed. Suppose the area of Ω is equal to πR^2. For any $u\in W_0^{1,2}(\Omega)$, let u^* be the decreasing rearrangement of $|u|$. By the rearrangement argument, we have $u^*\in W_0^{1,2}(\mathbb{B}_R)$. Then we have the Polya–Szego inequality (see [3])

$$\int_{\mathbb{B}_R}|\nabla u^*|^2dx\leq\int_\Omega|\nabla u|^2dx$$

and by the Hardy–Littlewood inequality

$$\begin{aligned}\int_\Omega|x|^{-\beta}|u|^pdx &= \int_{\mathbb{B}_R}(|x|^{-\beta}|u|^p)^*dx\\ &\leq \int_{\mathbb{B}_R}(|x|^{-\beta})^*(|u|^p)^*dx\\ &\leq \int_{\mathbb{B}_R}|x|^{-\beta}u^{*p}dx.\end{aligned}$$

This leads to

$$\int_\Omega|x|^{-\beta}e^{\gamma u^2(1+\alpha\|u\|_{p,\beta}^2)}dx\leq\int_{\mathbb{B}_R}|x|^{-\beta}e^{\gamma {u^*}^2(1+\alpha\|u^*\|_{p,\beta,\mathbb{B}_R}^2)}dx.$$

Hence, to prove (1.7), it suffices to prove that for any α, $0\leq\alpha<\lambda_{p,\beta}(\mathbb{B}_R)$, and any nonnegative radially symmetric decreasing function $u\in W_0^{1,2}(\mathbb{B}_R)$ with $\|\nabla u\|_{2,\mathbb{B}_R}\leq 1$, there exists some constant C depending only on α, β, and R such that

$$\int_{\mathbb{B}_R}|x|^{-\beta}e^{\gamma u^2(1+\alpha\|u\|_{p,\beta,\mathbb{B}_R}^2)}dx\leq C, \tag{3.1}$$

where $\gamma=4\pi(1-\beta/2)$. For simplicity here we use u instead of u^*, but we need to understand that u is not the same as $u\in W_0^{1,2}(\Omega)$. Set $a=1-\beta/2$, and

$$v(r)=\sqrt{a}u(r^{1/a}).$$

By (2.6) and (2.7), we have

$$\|\nabla v\|_{2,\mathbb{B}_{R^a}}\leq 1 \tag{3.2}$$

and

$$\|u\|_{p,\beta,\mathbb{B}_R}^2=a^{-(1+2/p)}\|v\|_{p,\mathbb{B}_{R^a}}^2. \tag{3.3}$$

For simplicity we write $b=1+\alpha\|u\|_{p,\beta,\mathbb{B}_R}^2$. It follows from (3.3) that

$$b=1+\frac{\alpha}{a^{1+2/p}}\|v\|_{p,\mathbb{B}_{R^a}}^2.$$

By Lemma 2.2 and a straightforward calculation, we have

$$\begin{aligned} \int_{\mathbb{B}_R} |x|^{-\beta} e^{\gamma b u^2} dx &= \int_0^R 2\pi r^{2a-1} e^{\gamma b (u(r))^2} dr \\ &= \frac{1}{a} \int_0^{R^a} 2\pi t e^{4\pi b (v(t))^2} dt \\ &= \frac{1}{a} \int_{\mathbb{B}_{R^a}} e^{4\pi b v^2} dx \end{aligned} \tag{3.4}$$

and

$$\frac{\alpha}{a^{1+2/p}} < \frac{\lambda_{p,\beta}(\mathbb{B}_R)}{a^{1+2/p}} = \lambda_p(\mathbb{B}_{R^a}). \tag{3.5}$$

In view of (3.2) and (3.5), it follows from Theorem 1.1 in [8] that

$$\int_{\mathbb{B}_{R^a}} e^{4\pi b v^2} dx \leq \sup_{u \in W_0^{1,2}(\mathbb{B}_R),\, \|\nabla u\|_{2,\mathbb{B}_R} \leq 1} \int_{\mathbb{B}_R} e^{4\pi u^2 \left(1 + \frac{\alpha}{a^{1+2/p}} \|u\|_{p,\mathbb{B}_R}^2\right)} dx,$$

which together with (3.4) and (2.4) implies (3.1). □

Proof of (ii) of Theorem 1.1. Let $\Omega = \mathbb{B}_R$. We write $\lambda_{p,\beta} = \lambda_{p,\beta}(\mathbb{B}_R)$. By Lemma 2.2, $\lambda_{p,\beta}$ is attained by ϕ_0 verifying that ϕ_0 is a radially symmetric decreasing function, and $\phi_0 \in W_0^{1,2}(\mathbb{B}_R) \cap C^1(\mathbb{B}_R \setminus \{0\}) \cap C^\nu(\mathbb{B}_R)$ for some $0 < \nu < 1$. Clearly we have $\phi_0(0) = \max_{\mathbb{B}_R} \phi_0 > 0$. Denote $\phi_0(r) = \phi_0(x)$ for $0 \leq r = |x| \leq R$. Set

$$G(x) = -\frac{1}{2\pi} \log |x|, \quad |x| \leq R. \tag{3.6}$$

Following the lines of Yang [14] and Lu and Yang [8], we set

$$\phi_\epsilon(x) = \begin{cases} \sqrt{\frac{1}{2\pi} \log \frac{1}{\epsilon}}, & \text{when} \quad |x| < \epsilon \\ AG(x) + B, & \text{when} \quad \epsilon \leq |x| \leq \delta \\ t_\epsilon(\phi_0(\delta) + \eta(\phi_0 - \phi_0(\delta))), & \text{when} \quad \delta < |x| \leq R, \end{cases}$$

where

$$\begin{aligned} A &= \frac{\sqrt{\frac{1}{2\pi} \log \frac{1}{\epsilon}} - t_\epsilon \phi_0(\delta)}{\frac{1}{2\pi} \log \frac{1}{\epsilon} - \frac{1}{2\pi} \log \frac{1}{\delta}}, \\ B &= \frac{t_\epsilon \phi_0(\delta) \frac{1}{2\pi} \log \frac{1}{\epsilon} - \sqrt{\frac{1}{2\pi} \log \frac{1}{\epsilon}} \frac{1}{2\pi} \log \frac{1}{\delta}}{\frac{1}{2\pi} \log \frac{1}{\epsilon} - \frac{1}{2\pi} \log \frac{1}{\delta}}, \end{aligned}$$

$\eta \in C^1(\overline{\mathbb{B}_R})$ satisfies $0 \leq \eta \leq 1$, $\eta \equiv 0$ when $|x| < \delta$, $\eta \equiv 1$ when $|x| \geq 2\delta$ and $|\nabla \eta| \leq 2/\delta$ for sufficiently small $\delta > 0$. One can see that $\phi_\epsilon \in W_0^{1,2}(\mathbb{B}_R)$. We choose t_ϵ such that $t_\epsilon \to 0$, $t_\epsilon^2 \log \frac{1}{\epsilon} \to +\infty$, and $t_\epsilon^3 \log \frac{1}{\epsilon} \to 0$. A straightforward calculation shows

$$\int_{\epsilon \leq |x| \leq \delta} |\nabla G|^2 dx = \frac{1}{2\pi} \log \frac{1}{\epsilon} - \frac{1}{2\pi} \log \frac{1}{\delta},$$

which gives

$$\begin{aligned}\int_{\epsilon\le|x|\le\delta}|\nabla\phi_\epsilon|^2dx &= A^2\int_{\epsilon\le|x|\le\delta}|\nabla G|^2dx\\ &= 1-\frac{2t_\epsilon\phi_0(\delta)}{\sqrt{\frac{1}{2\pi}\log\frac{1}{\epsilon}}}(1+o_\epsilon(1)),\end{aligned}$$

where $o_\epsilon(1)\to 0$ as $\epsilon\to 0$. Note that ϕ_0 is a distributional solution to

$$\begin{cases} -\Delta\phi_0=\lambda_{p,\beta}|x|^{-\beta}\|\phi_0\|_{p,\beta}^{2-p}{\phi_0}^{p-1} \quad \text{in} \quad \mathbb{B}_R\\ \|\nabla\phi_0\|_2=1, \quad \phi_0\ge 0 \quad \text{in} \quad \mathbb{B}_R.\end{cases} \tag{3.7}$$

Testing the above equation by $(\phi_0-\phi_0(2\delta))^+$, we have

$$\begin{aligned}\int_{\mathbb{B}_{2\delta}}|\nabla\phi_0|^2dx &= \int_{\mathbb{B}_R}\lambda_{p,\beta}|x|^{-\beta}\|\phi_0\|_{p,\beta}^{2-p}{\phi_0}^{p-1}(\phi_0-\phi_0(2\delta))^+dx\\ &\le \lambda_{p,\beta}\|\phi_0\|_{p,\beta}^{2-p}\int_{\mathbb{B}_{2\delta}}|x|^{-\beta}{\phi_0}^pdx\\ &\le \lambda_{p,\beta}\|\phi_0\|_{p,\beta}^{2-p}(\phi_0(0))^p\int_{\mathbb{B}_{2\delta}}|x|^{-\beta}dx\\ &= O(\delta^{2-\beta}).\end{aligned}$$

Since $\phi_0\in C^\nu(\mathbb{B}_R)$, it follows that $\int_{\delta\le|x|\le 2\delta}|\nabla\phi_\epsilon|^2dx=t_\epsilon^2O(\delta^\theta)$, where

$$\theta=\min\{2-\beta,2\nu\}.$$

Moreover, we can estimate the energy of ϕ_ϵ in domain $\mathbb{B}_R\setminus\mathbb{B}_{2\delta}$ as follows:

$$\begin{aligned}\int_{|x|>2\delta}|\nabla\phi_\epsilon|^2dx &= t_\epsilon^2\int_{|x|>2\delta}|\nabla\phi_0|^2dx\\ &= t_\epsilon^2(1-\int_{\mathbb{B}_{2\delta}}|\nabla\phi_0|^2dx)\\ &= t_\epsilon^2(1+O(\delta^{2-\beta})).\end{aligned}$$

Combining the above three estimates, we obtain

$$\int_{\mathbb{B}_R}|\nabla\phi_\epsilon|^2dx=1-\frac{2t_\epsilon\phi_0(\delta)}{\sqrt{\frac{1}{2\pi}\log\frac{1}{\epsilon}}}(1+o_\epsilon(1))+t_\epsilon^2(1+O(\delta^\theta)). \tag{3.8}$$

Let $v_\epsilon = \phi_\epsilon/\|\nabla\phi_\epsilon\|_2$. Then $v_\epsilon \in W_0^{1,2}(\mathbb{B}_R)$ and $\|\nabla v_\epsilon\|_2 = 1$. Combining (2.1) with (3.8) and noting that $(\int_{\mathbb{B}_R}|x|^{-\beta}u_0^p dx)^{-2/p} = \lambda_{p,\beta}$, we have

$$\begin{aligned}
\lambda_{p,\beta}\|v_\epsilon\|_{p,\beta}^2 &\geq \frac{\lambda_{p,\beta}}{\|\nabla\phi_\epsilon\|_2^2}\left(\int_{|x|>2\delta}|x|^{-\beta}t_\epsilon^p\phi_0^p dx\right)^{2/p}\\
&= \frac{\lambda_{p,\beta}}{\|\nabla\phi_\epsilon\|_2^2}t_\epsilon^2\left(\int_{\mathbb{B}_R}|x|^{-\beta}\phi_0^p dx - \int_{\mathbb{B}_{2\delta}}|x|^{-\beta}\phi_0^p dx\right)^{2/p}\\
&= \frac{t_\epsilon^2}{\|\nabla\phi_\epsilon\|_2^2}(1+O(\delta^{2-\beta}))\\
&= t_\epsilon^2(1+O(\delta^{2-\beta})+O(t_\epsilon^2)).
\end{aligned}$$

Here we also used the estimate

$$\frac{1}{\|\nabla\phi_\epsilon\|_2^2} = 1 + \frac{2t_\epsilon\phi_0(\delta)}{\sqrt{\frac{1}{2\pi}\log\frac{1}{\epsilon}}}(1+o_\epsilon(1)) - t_\epsilon^2(1+O(\delta^\theta)).$$

Recall that $\gamma = 4\pi(1-\beta/2)$. A straightforward calculation shows on domain $\mathbb{B}_\epsilon$,

$$\begin{aligned}
&\gamma v_\epsilon^2\left(1+\lambda_{p,\beta}\|v_\epsilon\|_{p,\beta}^2\right) \qquad (3.9)\\
&\geq (2-\beta)\log\frac{1}{\epsilon} + (4-2\beta)\sqrt{2\pi}t_\epsilon\sqrt{\log\frac{1}{\epsilon}}\phi_0(\delta)(1+o_\epsilon(1))\\
&\quad +(2-\beta)t_\epsilon^2\log\frac{1}{\epsilon}\left(O(\delta^\theta)+O(t_\epsilon^2)\right).
\end{aligned}$$

Taking

$$\delta = \frac{1}{\left(t_\epsilon^2\log\frac{1}{\epsilon}\right)^{2/\theta}},$$

one gets $\epsilon/\delta = o_\epsilon(1)$ and $t_\epsilon^2\log\frac{1}{\epsilon}O(\delta^\theta) = o_\epsilon(1)$. Moreover, we have $t_\epsilon^4\log\frac{1}{\epsilon} = o_\epsilon(1)$ and $\phi_0(\delta) = \phi_0(0) + O(\delta^\nu)$. Since $2\nu/\theta \geq 1$, we have $t_\epsilon\sqrt{\log\frac{1}{\epsilon}}\,\delta^\nu = o_\epsilon(1)$. Therefore, it follows from (3.9) that for any $\alpha \geq \lambda_{p,\beta}$,

$$\begin{aligned}
\int_{\mathbb{B}_R}|x|^{-\beta}e^{\gamma v_\epsilon^2(1+\alpha\|v_\epsilon\|_{p,\beta}^2)}dx &\geq \int_{|x|\leq\epsilon}|x|^{-\beta}e^{\gamma v_\epsilon^2(1+\lambda_{p,\beta}\|v_\epsilon\|_{p,\beta}^2)}dx\\
&\geq \frac{2\pi}{2-\beta}e^{(4-2\beta)\sqrt{2\pi}\phi_0(0)t_\epsilon\sqrt{\log\frac{1}{\epsilon}}+o_\epsilon(1)}\\
&\to +\infty \quad \text{as} \quad \epsilon\to 0.
\end{aligned}$$

Here, in the second inequality, we have used the fact $\int_{|x|\leq\epsilon}|x|^{-\beta}dx = \frac{2\pi}{2-\beta}\epsilon^{2-\beta}$. Hence (ii) of Theorem 1.1 follows. □

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