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Overall approach to Mizoguchi–Takahashi type fixed point results

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Abstract: In this work, inspired by the recent technique of Jleli and Samet, we give a new generalization of the well-known Mizoguchi–Takahashi fixed point theorem, which is the closest answer to Reich's conjecture about the existence of fixed points of multivalued mappings on complete metric spaces. We also provide a nontrivial example showing that our result is a proper generalization of the Mizoguchi–Takahashi result.

Key words: Fixed point, multivalued mappings, Mizoguchi–Takahasi result, θ-contraction

1. Introduction and preliminaries

In 1922, Banach established the most famous fundamental fixed point theorem, called the Banach contraction principle, for metric fixed point theory. This principle is a very powerful test for the existence and uniqueness of the solution of considerable problems arising in mathematics and has played an important role in various fields of applied mathematical analysis. The Banach contraction principle asserts that if (X, d) is a complete metric space and $T : X \to X$ is a contraction mapping, that is, there exists $L \in [0, 1)$ such that

$$d(Tx, Ty) \leq Ld(x, y)$$

for all $x, y \in X$, then there exists a unique $x \in X$ such that $x = Tx$. This principle has been extended and generalized in many ways (see [3, 4, 11, 16, 25]). In 1969, Nadler [19] initiated the idea for multivalued contraction mapping and extended the Banach contraction principle to multivalued mappings and afterwards proved the following result:

Theorem 1 (Nadler [19]) *Let (X, d) be a complete metric space and $T : X \to \mathcal{CB}(X)$ be a multivalued mapping, where $\mathcal{CB}(X)$ is the family of all nonempty closed and bounded subsets of X. If T is a multivalued contraction, that is, if there exists $L \in [0, 1)$ such that*

$$H(Tx, Ty) \leq Ld(x, y)$$

for all $x, y \in X$, where H is the Pompeiu–Hausdorff metric on $\mathcal{CB}(X)$ defined by

$$H(A, B) = \max\left\{\sup_{x\in A} d(x, B), \sup_{y\in B} d(y, A)\right\},$$

and $d(x, B) = \inf\{d(x, y) : y \in B\}$, then there exists $z \in X$ such that $z \in Tz$.

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Inspired by his result, since then there has been continuous intense research activity for fixed point results concerning multivalued contractions, and by now, there are a number of results that extend this result in different ways (see [6, 7, 9, 14, 15]). Concerning these, Reich [20] proved the following result for multivalued nonlinear contractions.

Theorem 2 (Reich [**20**]) *Let (X,d) be a complete metric space and $T: X \to \mathcal{K}(X)$ be a multivalued mapping, where $\mathcal{K}(X)$ is the family of all nonempty compact subsets of X. If there exists a function $\alpha : (0,\infty) \to [0,1)$ such that*

$$\limsup_{t \to s^+} \alpha(t) < 1, \ \forall s \in (0,\infty) \tag{1.1}$$

satisfying

$$H(Tx,Ty) \leq \alpha(d(x,y))d(x,y),$$

for all $x, y \in X$ with $x \neq y$, then T has a fixed point.

In 1974, Reich [21] (see also [22]) asked if we can relax the compactness assumption on T to closed and bounded subsets of X in Theorem 2. This question is called Reich's conjecture in the literature. Although a lot of researchers studied this conjecture, it has not been completely solved. There are some partial positive answers to this conjecture and the nearest answer was given by Mizoguchi and Takahashi [18] in 1989 with the substitution $s \geq 0$ instead of $s > 0$ in assumption (1.1). They proved the following theorem:

Theorem 3 (Mizoguchi and Takahashi [**18**]) *Let (X,d) be a complete metric space and let $T: X \to \mathcal{CB}(X)$ be a multivalued mapping. If there exists a function $\alpha : (0,\infty) \to [0,1)$ such that*

$$\limsup_{t \to s^+} \alpha(t) < 1, \ \forall s \in [0,\infty) \tag{1.2}$$

satisfying

$$H(Tx,Ty) \leq \alpha(d(x,y))d(x,y), \tag{1.3}$$

for all $x, y \in X$ with $x \neq y$, then T has a fixed point.

We can find in [23] both a simple proof of Mizoguchi and Takahashi's result and an example showing that it is real generalization of Nadler's result. We can also find some general fixed point results in these directions in the literature (see [2, 5, 8, 17]).

On the other hand, an attractive generalization of the Banach contraction principle given by Jleli and Samet [13] introduced a new type of contractive condition, which throughout this study we shall call θ-contraction. First we recall the basic definitions, relevant notions, and some related results concerning θ-contraction.

Let Θ be the set of all functions $\theta : (0,\infty) \to (1,\infty)$ satisfying the following conditions:

(θ_1) θ is nondecreasing;

(θ_2) For each sequence $\{t_n\} \subset (0,\infty)$, $\lim_{n\to\infty} \theta(t_n) = 1$ and $\lim_{n\to\infty} t_n = 0^+$ are equivalent;

(θ_3) There exist $r \in (0,1)$ and $l \in (0,\infty]$ such that $\lim_{t\to 0^+} \frac{\theta(t)-1}{t^r} = l$.

Let (X,d) be a metric space and $\theta \in \Theta$. A mapping $T: X \to X$ is said to be a θ-contraction if there exists a fixed constant $k \in [0,1)$ such that

$$\theta(d(Tx,Ty)) \leq [\theta(d(x,y))]^k \tag{1.4}$$

for all $x, y \in X$ with $d(Tx,Ty) > 0$.

An easy example of such mapping is the Banach contraction, which can be seen by taking $\theta(t) = e^{\sqrt{t}}$ in inequality (1.4). By choice of function $\theta(t) = e^{\sqrt{te^t}}$ in (1.4), we obtain a contraction type condition

$$\frac{d(Tx,Ty)}{d(x,y)} e^{d(Tx,Ty)-d(x,y)} \leq k^2, \tag{1.5}$$

for all $x, y \in X$ with $d(Tx,Ty) > 0$.

Now we give some of its important properties. Let $\theta_1, \theta_2 \in \Theta$. If $\theta = \frac{\theta_2}{\theta_1}$ is nondecreasing and $\theta_1(t) \leq \theta_2(t)$ for all $t \in (0,\infty)$, then it is easy to see that every θ_1-contraction is also a θ_2-contraction. Thus, if a mapping T is a Banach contraction, then it satisfies contraction type condition (1.5). In addition, it is clear that if T is a θ-contraction, then T is a contractive mapping, i.e. $d(Tx,Ty) < d(x,y)$ for all $x, y \in X$ with $x \neq y$. Hence, every θ-contraction on a metric space is continuous. Recently, Jleli and Samet [13] established a fixed point result for a type of such mappings on complete metric spaces:

Theorem 4 (Corollary 2.1 of [13]) *Let (X,d) be a complete metric space and $T : X \to X$ be a given mapping. If T is an θ-contraction, then T has a unique fixed point in X.*

In the theory of fixed point literature, we can find more papers dealing with θ-contraction mappings (see [1, 12]).

Naturally, the concept of θ-contraction was extended to multivalued mappings by Hançer et al. [10] (see also [24]) and they introduced the concept of multivalued θ-contraction: let (X,d) be a metric space, $T : X \to \mathcal{CB}(X)$ be a mapping, and $\theta \in \Theta$. Then T is said to be a multivalued θ-contraction if there exists a fixed constant $k \in [0,1)$ such that

$$\theta(H(Tx,Ty)) \leq [\theta(d(x,y))]^k \tag{1.6}$$

for all $x, y \in X$ with $H(Tx,Ty) > 0$.

Consequently, they established fixed point results for multivalued θ-contractions on complete metric spaces:

Theorem 5 ([10]) *Let (X,d) be a complete metric space and $T : X \to \mathcal{K}(X)$ be a multivalued θ-contraction. Then T has a fixed point.*

In the following example (Example 2.4 of [10]) we can see that $\mathcal{K}(X)$ cannot be replaced by $\mathcal{CB}(X)$ under the same conditions in Theorem 5.

Example 1 ([10]) *Consider the complete metric space (X,d), where $X = [0,2]$, and $d(x,y) = 0$ if $x = y$ and $d(x,y) = 1 + |x-y|$ if $x \neq y$. Define a mapping $T : X \to \mathcal{CB}(X)$, by $Tx = \mathbb{Q}$ if $x \in X \backslash \mathbb{Q}$ and $Tx = X \backslash \mathbb{Q}$ if $x \in \mathbb{Q}$, where $\mathbb{Q}$ is the set of all rational numbers in X. Then T is a multivalued θ-contraction with $\theta \in \Theta$ defined by $\theta(t) = e^{\sqrt{t}}$ if $t \leq 1$ and $\theta(t) = 9$ if $t > 1$, but T has no fixed points.*

However, by taking into account the following condition, which is not strong, on θ, this replacement can be made:

$$(\theta_4)\, \theta(\inf A) = \inf \theta(A) \text{ for all } A \subset (0, \infty) \text{ with } \inf A > 0.$$

Note that if θ is right continuous and satisfies (θ_1), then (θ_4) holds. Conversely, if (θ_4) holds, then θ is right continuous. Let Ξ be the family of all functions θ satisfying (θ_1)–(θ_4).

Theorem 6 ([10]) *Let (X, d) be a complete metric space and $T : X \to \mathcal{CB}(X)$ be a multivalued θ-contraction with $\theta \in \Xi$. Then T has a fixed point.*

In the present paper, we give a new generalization of Mizoguchi and Takahashi's result using this new approach for multivalued mappings. We will consider the contractive constant k as a function of $d(x, y)$ in (1.6) and therefore we will introduce a new concept called multivalued nonlinear θ-contraction. Later, we give some fixed point results for mappings of this type on complete metric spaces. In a special case, we obtain the Mizoguchi–Takahashi result. We also give an example showing that our result is a real generalization of the Mizoguchi–Takahashi result.

2. The results

Let (X, d) be a metric space, $T : X \to \mathcal{CB}(X)$, and $\theta \in \Theta$. Then we say that T is a multivalued nonlinear θ-contraction if there exists a function $k : (0, \infty) \to [0, 1)$ such that

$$\theta(H(Tx, Ty)) \leq [\theta(d(x, y))]^{k(d(x,y))}, \tag{2.1}$$

for all $x, y \in X$ with $H(Tx, Ty) > 0$.

If $k \in [0, 1)$ is a constant, then T is a multivalued θ-contraction, and also, if $\theta(t) = e^{\sqrt{t}}$, then T is a multivalued contraction.

Our first result is connected to mapping $T : X \to \mathcal{K}(X)$. For this, we will use the following lemma:

Lemma 1 *Let (X, d) be a metric space and A be compact subset of X. Then, for $x \in X$, there exists $a \in A$ such that $d(x, a) = d(x, A)$.*

Theorem 7 *Let (X, d) be a complete metric space and $T : X \to \mathcal{K}(X)$ be a multivalued nonlinear θ-contraction. Then T has a fixed point provided that*

$$\limsup_{t \to s^+} k(t) < 1, \ \forall s \in [0, \infty) \tag{2.2}$$

holds.

Proof Suppose that T has no fixed point, i.e. $d(x, Tx) > 0$ for all $x \in X$. Let $x_0 \in X$ and $x_1 \in Tx_0$. Since $0 < d(x_1, Tx_1) \leq H(Tx_0, Tx_1)$, then from (θ_1) and using (2.1), we get

$$\theta(d(x_1, Tx_1)) \leq \theta(H(Tx_0, Tx_1)) \leq [\theta(d(x_0, x_1))]^{k(d(x_0,x_1))}. \tag{2.3}$$

Since Tx_1 is compact, then from Lemma 1 there exists $x_2 \in Tx_1$ such that $d(x_1, x_2) = d(x_1, Tx_1)$. From (2.3),

$$\theta(d(x_1, x_2)) \leq \theta(H(Tx_0, Tx_1)) \leq [\theta(d(x_0, x_1))]^{k(d(x_0,x_1))}.$$

By induction, we can find a sequence $\{x_n\}$ in X such that $x_{n+1} \in Tx_n$ and

$$\theta(d(x_n, x_{n+1})) \leq [\theta(d(x_{n-1}, x_n))]^{k(d(x_{n-1},x_n))} < \theta(d(x_{n-1}, x_n)) \tag{2.4}$$

for all $n \in \mathbb{N}$. Thus, by taking into account (θ_1), the sequence $\{d(x_n, x_{n+1})\}$ is decreasing and hence convergent. From (2.2), there exists $b \in (0,1)$ and $n_0 \in \mathbb{N}$ such that $k(d(x_n, x_{n+1})) < b$ for all $n \geq n_0$. Thus, we obtain, for all $n \geq n_0$,

$$\begin{aligned}
1 &< \theta(d(x_n, x_{n+1})) \\
&\leq [\theta(d(x_{n-1}, x_n))]^{k(d(x_{n-1},x_n))} \\
&\leq [\theta(d(x_{n-2}, x_{n-1}))]^{k(d(x_{n-2},x_{n-1}))k(d(x_{n-1},x_n))} \\
&\;\;\vdots \\
&\leq [\theta(d(x_0, x_1))]^{k(d(x_0,x_1))\cdots k(d(x_{n-2},x_{n-1}))k(d(x_{n-1},x_n))} \\
&= [\theta(d(x_0, x_1))]^{k(d(x_0,x_1))\cdots k(d(x_{n_0-1},x_{n_0}))k(d(x_{n_0},x_{n_0+1}))\cdots k(d(x_{n-2},x_{n-1}))k(d(x_{n-1},x_n))} \\
&\leq [\theta(d(x_0, x_1))]^{k(d(x_{n_0},x_{n_0+1}))\cdots k(d(x_{n-2},x_{n-1}))k(d(x_{n-1},x_n))} \\
&\leq [\theta(d(x_0, x_1))]^{b^{(n-n_0)}}.
\end{aligned}$$

Thus, we obtain

$$1 < \theta(d(x_n, x_{n+1})) \leq [\theta(d(x_0, x_1))]^{b^{(n-n_0)}} \tag{2.5}$$

for all $n \geq n_0$. Letting $n \to \infty$ in (2.5), we obtain

$$\lim_{n\to\infty} \theta(d(x_n, x_{n+1})) = 1. \tag{2.6}$$

From (θ_2), $\lim_{n\to\infty} d(x_n, x_{n+1}) = 0^+$ and so from (θ_3) there exist $r \in (0,1)$ and $l \in (0,\infty]$ such that

$$\lim_{n\to\infty} \frac{\theta(d(x_n, x_{n+1})) - 1}{[d(x_n, x_{n+1})]^r} = l.$$

Suppose that $l < \infty$. In this case, let $B = \frac{l}{2} > 0$. From the definition of the limit, there exists $n_0 \in \mathbb{N}$ such that, for all $n \geq n_0$,

$$\left| \frac{\theta(d(x_n, x_{n+1})) - 1}{[d(x_n, x_{n+1})]^r} - l \right| \leq B.$$

This implies that, for all $n \geq n_0$,

$$\frac{\theta(d(x_n, x_{n+1})) - 1}{[d(x_n, x_{n+1})]^r} \geq l - B = B.$$

Then, for all $n \geq n_0$,

$$n\,[d(x_n, x_{n+1})]^r \leq An\,[\theta(d(x_n, x_{n+1})) - 1],$$

where $A = 1/B$.

Suppose now that $l = \infty$. Let $B > 0$ be an arbitrary positive number. From the definition of the limit, there exists $n_0 \in \mathbb{N}$ such that, for all $n \geq n_0$,

$$\frac{\theta(d(x_n, x_{n+1})) - 1}{[d(x_n, x_{n+1})]^r} \geq B.$$

This implies that, for all $n \geq n_0$,

$$n\,[d(x_n, x_{n+1})]^r \leq An\,[\theta(d(x_n, x_{n+1})) - 1]\,,$$

where $A = 1/B$.

Thus, in all cases, there exist $A > 0$ and $n_0 \in \mathbb{N}$ such that, for all $n \geq n_0$,

$$n\,[d(x_n, x_{n+1})]^r \leq An\,[\theta(d(x_n, x_{n+1})) - 1]\,.$$

Using (2.5), we obtain, for all $n \geq n_0$,

$$n\,[d(x_n, x_{n+1})]^r \leq An\left[[\theta(d(x_0, x_1))]^{b^{(n-n_0)}} - 1\right].$$

Letting $n \to \infty$ in the above inequality, we obtain

$$\lim_{n\to\infty} n\,[d(x_n, x_{n+1})]^r = 0.$$

Thus, there exists $n_1 \in \mathbb{N}$ such that $n\,[d(x_n, x_{n+1})]^r \leq 1$ for all $n \geq n_1$. Therefore, we have, for all $n \geq n_1$,

$$d(x_n, x_{n+1}) \leq \frac{1}{n^{1/r}}. \tag{2.7}$$

In order to show that $\{x_n\}$ is a Cauchy sequence, consider $m, n \in \mathbb{N}$ such that $m > n \geq n_1$. Using the triangular inequality for the metric and from (2.7), we have

$$\begin{aligned} d(x_n, x_m) &\leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \cdots + d(x_{m-1}, x_m) \\ &= \sum_{i=n}^{m-1} d(x_i, x_{i+1}) \leq \sum_{i=n}^{\infty} d(x_i, x_{i+1}) \leq \sum_{i=n}^{\infty} \frac{1}{i^{1/r}}. \end{aligned}$$

By the convergence of the series $\sum\limits_{i=1}^{\infty} \frac{1}{i^{1/r}}$, letting to limit $n \to \infty$, we get $d(x_n, x_m) \to 0$. This yields that $\{x_n\}$ is a Cauchy sequence in (X, d). Since (X, d) is a complete metric space, the sequence $\{x_n\}$ converges to some point $z \in X$, that is, $\lim_{n\to\infty} x_n = z$.

On the other hand, from (2.1), for all $x, y \in X$ with $H(Tx, Ty) > 0$, we get

$$H(Tx, Ty) < d(x, y)$$

and so

$$H(Tx, Ty) \leq d(x, y)$$

for all $x, y \in X$. Then we get

$$d(x_{n+1}, Tz) \leq H(Tx_n, Tz) \leq d(x_n, z).$$

Letting $n \to \infty$ in the above, we obtain $d(z, Tz) = 0$. This contradicts that T has no fixed point. Thereby, this completes the proof. □

As shown in Example 1, we cannot take $\mathcal{CB}(X)$ instead of $\mathcal{K}(X)$ in Theorem 7. However, by adding the condition (θ_4) on θ, we can give the following:

Theorem 8 *Let (X,d) be a complete metric space and $T : X \to \mathcal{CB}(X)$ be a multivalued nonlinear θ-contraction with $\theta \in \Xi$. Then T has a fixed point provided that the condition (2.2) holds.*

Proof Suppose that T has no fixed point, i.e. $d(x, Tx) > 0$ for all $x \in X$. Let $x_0 \in X$ and $x_1 \in Tx_0$. Since $0 < d(x_1, Tx_1) \leq H(Tx_0, Tx_1)$, then from (θ_1) and using (2.1), we get

$$\theta(d(x_1, Tx_1)) \leq \theta(H(Tx_0, Tx_1)) \leq [\theta(d(x_0, x_1))]^{k(d(x_0,x_1))}. \tag{2.8}$$

From (θ_4), we can write

$$\theta(d(x_1, Tx_1)) = \inf_{y \in Tx_1} \theta(d(x_1, y))$$

and so from (2.8) we have

$$\begin{aligned} \inf_{y \in Tx_1} \theta(d(x_1, y)) &\leq [\theta(d(x_0, x_1))]^{k(d(x_0,x_1))} \\ &< [\theta(d(x_0, x_1))]^{\sqrt{k(d(x_0,x_1))}}. \end{aligned} \tag{2.9}$$

Then, from (2.9), there exists $x_2 \in Tx_1$ such that

$$\theta(d(x_1, x_2)) \leq [\theta(d(x_0, x_1))]^{\sqrt{k(d(x_0,x_1))}}.$$

By induction, we find a sequence $\{x_n\}$ in X such that $x_{n+1} \in Tx_n$ and

$$\theta(d(x_n, x_{n+1})) \leq [\theta(d(x_n, x_{n-1}))]^{\sqrt{k(d(x_n,x_{n-1}))}}$$

for all $n \in \mathbb{N}$. The rest of the proof can be completed as in the proof of Theorem 7. □

By considering $\theta(t) = e^{\sqrt{t}}$ and $k(t) = \sqrt{\alpha(t)}$ in Theorem 8, we can obtain the following corollary, which is the famous Mizoguchi–Takahashi fixed point result for multivalued nonlinear contractions:

Corollary 1 **(**Mizoguchi–Takahashi) *Let (X,d) be a complete metric space. Suppose that $T : X \to \mathcal{CB}(X)$ satisfies*

$$H(Tx, Ty) \leq \alpha(d(x, y))d(x, y),$$

for all $x, y \in X$, $x \neq y$, where $\alpha : (0, \infty) \to [0, 1)$ satisfying $\limsup_{t \to s^+} \alpha(t) < 1$ for all $s \in [0, \infty)$. Then T has a fixed point.

The following provided nontrivial example shows that the investigation of this paper is significant.

Example 2 *Consider the complete metric space (X,d), where $X = \{\frac{1}{n} : n \in \mathbb{N}\} \cup \{0\}$ and*

$$d(x, y) = \begin{cases} 0 & , \quad x = y \\ \max\{x, y\} & , \quad x \neq y \end{cases}.$$

Define a mapping $T : X \to \mathcal{CB}(X)$ *by*

$$Tx = \begin{cases} \{0\} & , \quad x = 0 \\ \left\{0, \frac{1}{n+1}, \frac{1}{n+2}, \cdots\right\} & , \quad x = \frac{1}{n}, \ n \in \mathbb{N} \end{cases}.$$

We claim that T *is multivalued nonlinear* θ*-contraction with* $\theta(t) = e^{\sqrt{te^t}}$ *and* $k : (0, \infty) \to [0, 1)$ *defined by*

$$k(t) = \begin{cases} \sqrt{e^{\frac{1}{n+1} - \frac{1}{n}}} & , \quad \text{if } t = \frac{1}{n} \text{ for } n \in \mathbb{N} \\ 0 & , \quad \text{otherwise} \end{cases}.$$

It is clear that $\limsup_{t \to s^+} k(t) = 0 < 1$ *for all* $s \in [0, \infty)$*. Observe that taking* $\theta(t) = e^{\sqrt{te^t}}$*, the contractive condition (2.1) turns to*

$$\frac{H(Tx, Ty)}{d(x, y)} e^{H(Tx,Ty) - d(x,y)} \leq [k(d(x, y))]^2.$$

In fact, if $x = \frac{1}{n}$ *and* $y = \frac{1}{m}$ *with* $m > n$*, then*

$$\begin{aligned} \frac{H(Tx, Ty)}{d(x, y)} e^{H(Tx,Ty) - d(x,y)} &\leq \frac{\frac{1}{n+1}}{\frac{1}{n}} e^{\frac{1}{n+1} - \frac{1}{n}} \\ &\leq e^{\frac{1}{n+1} - \frac{1}{n}} \\ &= k^2(\frac{1}{n}) \\ &= k^2(d(x, y)), \end{aligned}$$

and if $x = \frac{1}{n}$ *and* $y = 0$*, then*

$$\begin{aligned} \frac{H(Tx, Ty)}{d(x, y)} e^{H(Tx,Ty) - d(x,y)} &\leq \frac{\frac{1}{n+1}}{\frac{1}{n}} e^{\frac{1}{n+1} - \frac{1}{n}} \\ &\leq e^{\frac{1}{n+1} - \frac{1}{n}} \\ &= k^2(\frac{1}{n}) \\ &= k^2(d(x, y)). \end{aligned}$$

This shows that T *is a multivalued nonlinear* θ*-contraction, and therefore all conditions of Theorem 8 are satisfied and so* T *has a fixed point.*

Now we show that Mizoguchi and Takahashi's result cannot be applied to this example.

Suppose that there exists a function $\alpha : (0, \infty) \to [0, 1)$ *satisfying (1.2) and (1.3). Then, for* $x = 0$ *and* $y = \frac{1}{n}$*, we get*

$$H(Tx, Ty) = \frac{1}{n+1} \text{ and } d(x, y) = \frac{1}{n},$$

and so we obtain

$$H(Tx, Ty) \leq \alpha(d(x, y))d(x, y)$$

or

$$\frac{n}{n+1} \leq \alpha(\frac{1}{n}).$$

Taking the limit supremum as $n \to \infty$ in above, we have

$$1 \leq \limsup_{n\to\infty} \alpha(\frac{1}{n}) \leq \limsup_{t\to 0^+} \alpha(t) < 1,$$

which is a contradiction.

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