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On the comaximal ideal graph of a commutative ring

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Abstract: Let R be a commutative ring with identity. We use $\Gamma(R)$ to denote the comaximal ideal graph. The vertices of $\Gamma(R)$ are proper ideals of R that are not contained in the Jacobson radical of R, and two vertices I and J are adjacent if and only if $I + J = R$. In this paper we show some properties of this graph together with the planarity and perfection of $\Gamma(R)$.

Key words: Chromatic number, clique number, planar graph, perfect graph

1. Introduction

For the sake of completeness, we explain some definitions and points used throughout this paper. A graph with vertex set V is said to be a graph on V. The vertex set of a graph G is referred to as $V(G)$ and its edge set as $E(G)$. Let v be a vertex of G. The *neighbourhood* of v is the set $N_G(v) = \{u \in G | vu \in G\}$. For a graph G, the *degree* of a vertex v in G, $deg(v)$, is the number of edges of G incident with v. A graph G is said to be *connected* if there is at least one path between every pair of vertices in G and the *distance* between two vertices v and w, $d(v,w)$, is the length of the shortest path connecting them. The *diameter* of a connected graph is the maximum of the distances between vertices. A *loop* of G is an edge that joins a vertex to itself. *Multiple edges* are two or more edges connecting the same two vertices within a *multigraph*. A *simple graph* is an unweighted, undirected graph containing no *loops* or *multiple edges*. A connected acyclic graph is called a *tree*. Acyclic graphs are usually called *forests*. A graph in which each pair of distinct vertices is joined by an edge is called a *complete graph*. We denote by K_n a complete graph with n vertices. A *complete bipartite graph* is a *bipartite graph* (i.e. a set of graph vertices decomposed into two disjoint sets X and Y such that no two graph vertices within the same set are adjacent) such that all pairs of graph vertices in the two sets are adjacent. We denote by $K_{n,m}$ a complete bipartite graph with $|X| = n$ and $|Y| = m$. We define a *coloring* of G to be an assignment of colors to the vertices of G, one color to each vertex, so that adjacent vertices are assigned distinct colors. If n colors are used, then the coloring is referred to as $n-coloring$. If there exists $n-coloring$ of G, then G is called $n-colorable$. The minimum n for which G is $n-colorable$ is called the *chromatic number* of G, and is denoted by $\chi(G)$. A subset S of the set of vertices of G is said to be a *clique* in G if every pair of distinct elements x and y of S is adjacent in G. The *clique number* of G is the maximum of the cardinality of all cliques in G and is denoted by $clique(G)$. The *complement* of

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G, $\overline{G}$, is the graph with the same vertex-set as G, where two distinct vertices are adjacent whenever they are nonadjacent in G. A graph is said to be *planar* if it can be drawn in the plane so that its edges intersect only at their ends. A *subdivision* of a graph is any graph that can be obtained from the original graph by replacing edges by paths. Kuratowski's theorem says that a graph is planar if and only if it contains no subdivision of K_5 or $K_{3,3}$ [1, Theorem 4.4.6]. A *subgraph* of G is a graph H such that $V(H) \subseteq V(G)$, $E(H) \subseteq E(G)$. The subgraph of G induced by a subset S of vertices of G is the subgraph whose vertex set is S and whose edges are all the edges of G with both ends in S [5]. Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two graphs with disjoint vertices set V_i and edges set E_i. The union of G_1 and G_2 is denoted by $G = G_1 \cup G_2$ with vertices set $V_1 \cup V_2$ and edges set $E_1 \cup E_2$. The join of G_1 and G_2 is denoted by $G = G_1 \vee G_2$ with vertices set $V_1 \cup V_2$ and the set of edges is $E_1 \cup E_2 \cup \{xy | x \in V_1 \textit{ and } y \in V_2\}$.

From now on let R be a commutative ring with identity. In [4], Sharma and Bhatwadekar defined a graph on R, with vertices as elements of R, where two distinct vertices a and b are adjacent if and only if $Ra + Rb = R$.

Later, Maimani et al. [2] studied the graph structure defined by Sharma and Bhatwadekar and named such graph structure "*Comaximal Graphs*". They considered the subgraph of Sharma's graph, $\Gamma_2(R)$, which consists of nonunit elements.

In [6], Ye and Wu defined comaximal ideal graph, $\Gamma(R)$, with vertices as proper ideals of R that are not contained in the Jacobson radical of R, and two vertices I and J are adjacent if and only if $I + J = R$.

Some results of this paper for the graph $\Gamma(R)$ are similar to the results in [3] for the graph $\Gamma_2(R) \backslash J(R)$.

In this paper, we consider some properties of $\Gamma(R)$ and we investigate the planarity and perfection of this graph.

2. Properties of $\Gamma(R)$

Let $J(R)$ be Jacobson radical of R. R is said to be local if it has a unique maximal ideal. Let $Max(R)$ be the set of maximal ideals of R and $|Max(R)|$ denote the number of maximal ideals of R. For any maximal ideal M of R, $\mathcal{M}$ denotes the set of nonzero ideals contained in M and $|\mathcal{M}|$ denotes the number of ideals contained in M.

In [6], Ye and Wu showed that $\Gamma(R)$ has distance of at most 3. In what follows, first we characterize the cases in which two vertices have distance 1, 2, or 3. For any ideal I of R, let

$$M(I) = \{M \in Max(R) : I \subseteq M\}.$$

Lemma 2.1 *The elements I and J are adjacent in $\Gamma(R)$ if and only if there does not exist a maximal ideal M that contains both of them, that is,*

$$\{I, J\} \in E(\Gamma(R)) \iff M(I) \cap M(J) = \emptyset.$$

Proof Assume $I, J \subseteq M$, where $M \in Max(R)$; then $I + J \subseteq M$ and so I and J cannot be adjacent. Conversely, if I and J are not adjacent, then $I + J$ is a proper ideal of R; hence there exists a maximal ideal M such that $I + J \subseteq M$, and therefore $M(I) \cap M(J) \neq \emptyset$. □

Theorem 2.2 *([6], Theorem 2.4) For a ring R, $\Gamma(R)$ is a simple, connected graph with diameter less than or equal to three.*

Proposition 2.3 *Let $G = \Gamma(R)$ and $I, J, K \in G$ be distinct elements. Then the following are equivalent:*

(a) $K \in N_G(I) \cap N_G(J)$;

(b) $K \in N_G(IJ)$;

(c) $K \in N_G(I \cap J)$.

Proof (a)$\Rightarrow$(b): Suppose $K \in N_G(I) \cap N_G(J)$. Then $K + I = R = K + J$. Thus $k_1 + i = 1$ and $k_2 + j = 1$ for some $k_1, k_2 \in K, i \in I$, and $j \in J$. Therefore, $1 = ij + ik_2 + jk_1 + k_1k_2$, which implies that $IJ + K = R$. Hence $K \in N_G(IJ)$. (b)$\Rightarrow$(c): Assume $K + IJ = R$. As $IJ \subseteq I \cap J$, and so $K + (I \cap J) = R$ and $K \in N_G(I \cap J)$. (c)$\Rightarrow$(a): If $K + (I \cap J) = R$, then $K + I = R$ and $K + J = R$, which means that $K \in N_G(I)$ and $K \in N_G(J)$. Thus $K \in N_G(I) \cap N_G(J)$. □

Theorem 2.4 *Let $G = \Gamma(R)$ and $I, J \in G$ be distinct elements. Then the following hold.*

(a) $d(I, J) = 1$ if and only if $M(I) \cap M(J) = \emptyset$.

(b) $d(I, J) = 2$ if and only if $M(I) \cap M(J) \neq \emptyset$ and $IJ \nsubseteq J(R)$.

(c) $d(I, J) = 3$ if and only if $M(I) \cap M(J) \neq \emptyset$ and $IJ \subseteq J(R)$.

Proof (a): By Lemma 2.1. (b): Assume that $d(I, J) = 2$. Then $M(I) \cap M(J) \neq \emptyset$, by Lemma 2.1 and there is a K in $\Gamma(R)$ such that $K \in N_G(I)$ and $K \in N_G(J)$. Thus $K \in N_G(IJ)$, by Proposition 2.3, which implies that $deg(IJ) > 0$. Therefore $IJ \nsubseteq J(R)$, by [6,Proposition 2.1(2)]. Conversely, if $IJ \nsubseteq J(R)$, then $deg(IJ) > 0$ and there is a K in $\Gamma(R)$ such that $K + IJ = R$. Again according to Proposition 2.3, $K \in N_G(I) \cap N_G(J)$. Since $M(I) \cap M(J) \neq \emptyset$, $d(I, J) > 1$. Thus $d(I, J) = 2$.

(c): According to Theorem 2.2 and (b), $d(I, J) = 3$ if and only if $M(I) \cap M(J) \neq \emptyset$ and $IJ \subseteq J(R)$. □

In what follows, we investigate the condition that $\Gamma(R)$ is a planar graph.

Lemma 2.5 *If $\Gamma(R)$ is planar, then $|Max(R)| \leq 4$.*

Proof Assume to the contrary that $|Max(R)| \geq 5$. Let $M_1, ..., M_5$ be distinct maximal ideals of R. As every two maximal ideals are comaximal, K_5 is a subgraph of $\Gamma(R)$. Therefore $\Gamma(R)$ is not planar, by Kuratowski's theorem, which is a contradiction. Hence $|Max(R)| \leq 4$. □

If $|Max(R)| = 1$, then $\Gamma(R)$ is an empty graph, by [6, Proposition 2.1(1)] and it is planar. Suppose that $|Max(R)| = 2$. Then $\Gamma(R)$ is a complete bipartite graph, by [6,Lemma 4.1]. Thus $\Gamma(R)$ is planar if and only if $|\mathcal{M}_1 \backslash \mathcal{M}_2| \leq 2$ or $|\mathcal{M}_2 \backslash \mathcal{M}_1| \leq 2$. Otherwise it has $K_{3,3}$ as a subgraph and so it is not planar.

Assume that $|Max(R)| = 3$ and M_1, M_2, and M_3 are distinct maximal ideals of R. Set $V_i := \mathcal{M}_i \backslash \bigcup_{j \neq i} \mathcal{M}_j$, $V_{i_1 i_2} := (\mathcal{M}_{i_1} \bigcap \mathcal{M}_{i_2}) \backslash \mathcal{M}_j$ for $j \neq i_1, i_2$ and $1 \leq i_1 < i_2 \leq 3$. It is obvious that $|V_i| \geq 1$, since $M_i \in V_i$.

By the above notations, we have the following theorem.

Theorem 2.6 *Assume that $|Max(R)| = 3$. Then $\Gamma(R)$ is planar if and only if one of the following conditions hold.*

(a) For only one V_i, $|V_i| \geq 3$ and for $j \neq i$, $|V_j| = 1$. Moreover, $V_{jk} = \emptyset$ for distinct j, k, where $1 \leq j, k \leq 3$.

(b) $|V_i| = 2$ for all $1 \leq i \leq 3$, and $V_{ij} = \emptyset$ for all $1 \leq i < j \leq 3$.

(c) $|V_i| = |V_j| = 2$, $|V_k| = 1$ *with* $\{i, j, k\} = \{1, 2, 3\}$ *and* $V_{ki} = \emptyset$ *or* $V_{kj} = \emptyset$.

(d) There exists only one V_i *with* $|V_i| = 2$ *and* $|V_j| = 1$ *for all* $j \neq i$, *where* $1 \leq i, j \leq 3$.

(e) $|V_i| = 1$ *for all* $1 \leq i \leq 3$.

Proof ($\Leftarrow$): Recall that each ideal in V_i is adjacent to all ideals of V_j, $j \neq i$, and all ideals in V_{jk}, $j \neq i$ and $k \neq i$, by the definition of $\Gamma(R)$. There is no edge between ideals of V_i and V_{ik}. There is also no edge between an ideal of V_{ik} and an ideal of V_{jk}. According to the given conditions, in all cases, graphs can be drawn in the plane. See Figures 1–5.

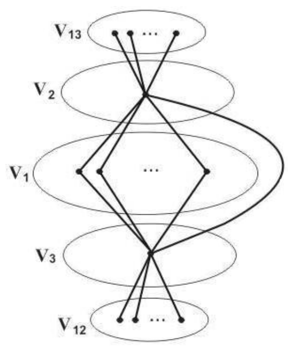


Figure 1.

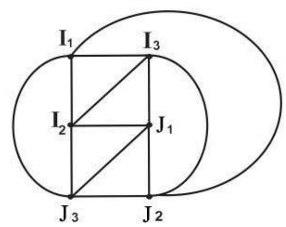


Figure 2.

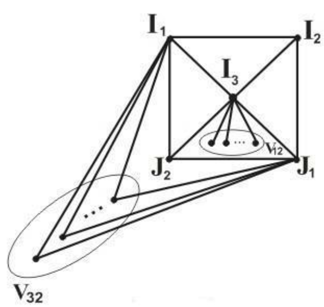


Figure 3.

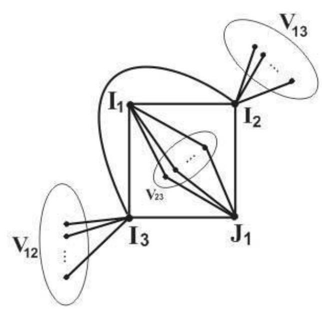


Figure 4.

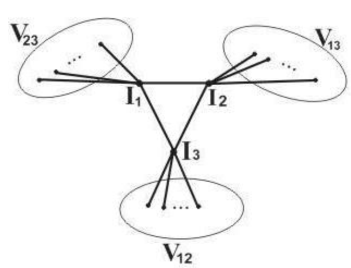


Figure 5.

Figure 1. (a). $|V_1| \geq 3, |V_2| = |V_3| = 1$, and $V_{23} = \emptyset$.

Figure 2. (b). $V_1 = \{I_1, J_1\}, V_2 = \{I_2, J_2\}, V_3 = \{I_3, J_3\}$, and $V_{12}, V_{13}, V_{23} = \emptyset$.

Figure 3. (c). $V_1 = \{I_1, J_1\}, V_2 = \{I_2, J_2\}, V_3 = \{I_3\}$, and $V_{13} = \emptyset$.

Figure 4. (d). $V_1 = \{I_1, J_1\}, V_2 = \{I_2\}, V_3 = \{I_3\}$.

Figure 5. (e). $V_1 = \{I_1\}, V_2 = \{I_2\}, V_3 = \{I_3\}$, and $V_{12}, V_{23}, V_{13} \neq \emptyset$.

$(\Rightarrow)$: we consider the following cases:

Case 1. If for distinct i and j with $1 \leq i, j \leq 3$, $|V_i|, |V_j| \geq 3$, then we have $K_{3,3}$ in $\Gamma(R)$ and so it is not planar.

Case 2. Let there exist only one V_i such that $|V_i| \geq 3$. Without loss of generality, we assume that $|V_1| \geq 3$. If $|V_2 \cup V_3| \geq 3$ or $|V_2 \cup V_3 \cup V_{23}| \geq 3$, then $K_{3,3}$ is the subgraph of $\Gamma(R)$ and so it is not planar. Therefore, in this case $\Gamma(R)$ is planar if $|V_2| = |V_3| = 1$ and $V_{23} = \emptyset$.

Case 3. Assume that $|V_i| \leq 2$ for all $1 \leq i \leq 3$. First suppose that $|V_i| = 2$ for all $1 \leq i \leq 3$. Let there exist V_{ij}, say V_{12}, such that $V_{12} \neq \emptyset$. Let $V_1 = \{I_1, J_1\}, V_2 = \{I_2, J_2\}, V_3 = \{I_3, J_3\}$, and $K \in V_{12}$. As each ideal in V_i is adjacent to all ideals of V_j for $i \neq j$, and all ideals in V_{jk} for $j, k \neq i$, $\Gamma(R)$ has a subdivision of $K_{3,3}$ (Figure 6) and it is not planar. Therefore in this case $|V_i| = 2$ and $V_{ij} = \emptyset$ for all distinct i and j.

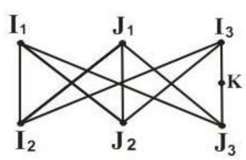


Figure 6.

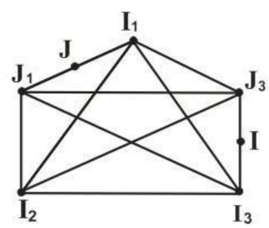


Figure 7.

Now let without loss of generality, $|V_1| = |V_3| = 2$ and $|V_2| = 1$. Let $V_{12}, V_{23} \neq \emptyset$, $V_1 = \{I_1, J_1\}, V_2 = \{I_2\}, V_3 = \{I_3, J_3\}$, $I \in V_{12}$, and $J \in V_{23}$. Then the subgraph generated by $\{I_1, J_1, I_2, I_3, J_3, I, J\}$ is a subdivision of K_5 as shown in Figure 7. Therefore $\Gamma(R)$ is not planar.

At the end, let for a unique V_i, $|V_i| = 2$ and $|V_j| = 1$ for $1 \leq j \neq i \leq 3$ or $|V_i| = 1$ for all $1 \leq i \leq 3$. It is obvious that in these cases $\Gamma(R)$ is planar. □

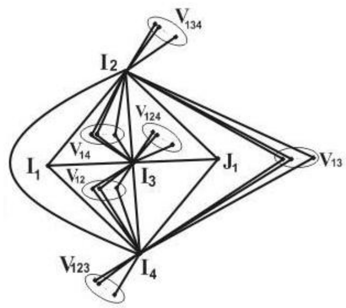
Now suppose that $|Max(R)| = 4$. Set $V_i := \mathcal{M}_i \setminus \bigcup_{j \neq i} \mathcal{M}_j$, $V_{i_1 i_2} := (\mathcal{M}_{i_1} \bigcap \mathcal{M}_{i_2}) \setminus \bigcup_{j \neq i_1, i_2} \mathcal{M}_j$, $V_{i_1 i_2 i_3}$ $:= (\mathcal{M}_{i_1} \bigcap \mathcal{M}_{i_2} \bigcap \mathcal{M}_{i_3}) \setminus \mathcal{M}_j$ for $j \neq i_1, i_2, i_3$, $1 \leq i, j \leq 4$, where $1 \leq i_1 < i_2 < i_3 \leq 4$.

Theorem 2.7 *Assume that $|Max(R)| = 4$. Then $\Gamma(R)$ is planar if and only if one of the following conditions hold.*

(a) There exists only one V_i with $|V_i| = 2$. Also $V_{jk} = \emptyset$ and $V_{jkl} = \emptyset$ for distinct $1 \leq i, j, k, l \leq 4$.

(b) $|V_i| = 1$ for all $1 \leq i \leq 4$.

Proof $(\Leftarrow)$: Note that each ideal in V_i is adjacent to all ideals of V_j for $1 \leq i \neq j \leq 4$, all ideals in V_{jk} for $1 \leq j, k \neq i \leq 4$, and all ideals in V_{jkl} for $1 \leq j, k, l \neq i \leq 4$, according to the definition of $\Gamma(R)$. Similar to Theorem 2.6, in the conditions (a) and (b) of the theorem, we can draw the figure of $\Gamma(R)$ as Figure 8 and Figure 9. Hence $\Gamma(R)$ is planar.



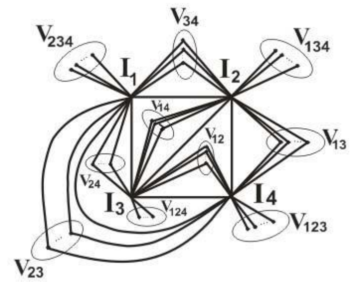


Figure 8. **Figure 9.**

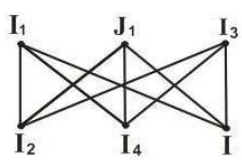
Figure 8. (a). $V_1 = \{I_1, J_1\}$, $V_2 = \{I_2\}$, $V_3 = \{I_3\}$, $V_4 = \{I_4\}$, V_{12}, V_{13}, $V_{14}, V_{123}, V_{124}, V_{134} \neq \emptyset$, and $V_{23}, V_{24}, V_{34}, V_{234} = \emptyset$.

Figure 9. (b). $V_1 = \{I_1\}$, $V_2 = \{I_2\}$, $V_3 = \{I_3\}$, $V_4 = \{I_4\}$, and $V_{12}, V_{13}, V_{14}, V_{23}$, V_{24}, V_{34}, V_{123}, V_{124}, V_{134}, $V_{234} \neq \emptyset$.

($\Rightarrow$): Assume that for some i with $1 \leq i \leq 4$, $|V_i| \geq 2$. Let $|V_1| \geq 2$. We have the following cases:

Case 1. Let for some j with $2 \leq j \leq 4$, $|V_j| \geq 2$. Without loss of generality, let $|V_2| \geq 2$. Then we have the subdivision of $K_{3,3}$ in $\Gamma(R)$, where $V_1 = \{I_1, J_1\}$, $V_2 = \{I_2, J_2\}$, $K \in V_3$, and $L \in V_4$. Hence $\Gamma(R)$ is not planar.

Case 2. Assume that for only one V_i, $|V_i| = 2$, $|V_j| = 1$, for all $1 \leq j \neq i \leq 4$ and for some $1 < j < k$, $V_{jk} \neq \emptyset$. Let $i = 1$ and $V_{24} \neq \emptyset$ or $V_{234} \neq \emptyset$. If $V_{24} \neq \emptyset$, then $\Gamma(R)$ has the subdivision of $K_{3,3}$, where $V_1 = \{I_1, J_1\}$, $V_2 = \{I_2\}$, $V_3 = \{I_3\}$, $V_4 = \{I_4\}$, and $I \in V_{24}$ (Figure 10). Hence $\Gamma(R)$ is not planar. Now let $V_{234} \neq \emptyset, V_1 = \{I_1, J_1\}, V_2 = \{I_2\}, V_3 = \{I_3\}, V_4 = \{I_4\}$, and $I \in V_{234}$. Then $\Gamma(R)$ has the subdivision of K_5 (Figure 11). Thus $\Gamma(R)$ is not planar. □



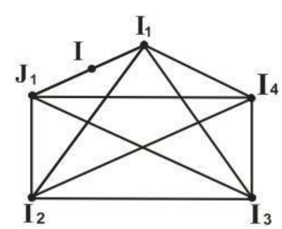


Figure 10. **Figure 11.**

3. Perfect comaximal ideal graph of a commutative ring

In this section, we investigate the perfection of $\Gamma(R)$. Firstly we recall some definitions and notations on perfect graphs.

Definition 3.1 *([1]) A graph G is perfect if for every induced subgraph H of G, $\chi(H) = clique(H)$.*

Definition 3.2 *([1]) A graph is chordal (or triangulated) if each of its cycles of length at least 4 has a chord, i.e. if it contains no induced cycles other than triangles.*

([1]) Let G be a graph with induced subgraphs G_1 and G_2 such that $G = G_1 \cup G_2$. Let $S = G_1 \cap G_2$; we say that G arises from G_1 and G_2 by pasting these graphs together along S (Figure 12).

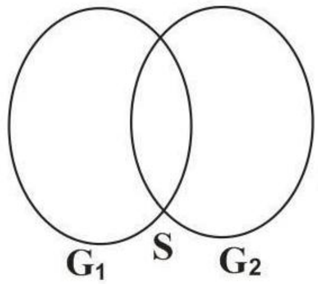


Figure 12.

Proposition 3.3 *([1], Proposition 5.5.1) A graph is chordal if and only if it can be constructed recursively by pasting along complete subgraphs, starting from complete graphs.*

Proposition 3.4 *([1], Proposition 5.5.2) Every chordal graph is perfect.*

([1]) Complete graphs, empty graphs, and complete k-partite graphs are perfect.

([1]) If G is obtained from two chordal graphs G_1 and G_2 by pasting them together along a complete subgraph S, then G is chordal.

Theorem 3.5 *([1], Berge 1966) A graph G is perfect if and only if neither G or $\overline{G}$ contains an odd cycle of length at least 5 as an induced subgraph.*

Theorem 3.6 *If $|Max(R)| \leq 4$, then $\Gamma(R)$ is a perfect graph.*

Proof Case 1. Let R be a local ring. Then $\Gamma(R)$ is an empty graph, by [6, *Proposition* 2.1(1)] and so it is perfect.

Case 2. Assume that R has only two maximal ideals. Then $\Gamma(R)$ is a complete bipartite graph and so it is perfect.

Case 3. Let $|Max(R)| = 3$. We show that $\overline{\Gamma(R)}$ is chordal. Let V_i and V_{ij} be defined as in Section 2. The connection between these sets is as Figure 13.

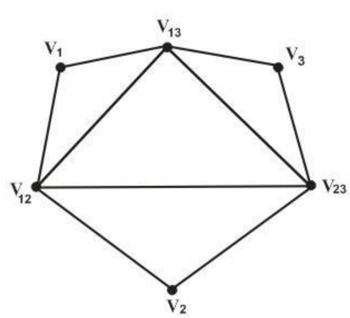


Figure 13. V_i 's are independent of each other and V_{jk} too, $j \neq i$ and $k \neq i$, each V_i is a complete graph, by the definition of $\overline{\Gamma(R)}$.

Consider $G_1 = \overline{\Gamma(R)}[V_1 \cup V_{12} \cup V_{13}]$ and $G_2 = \overline{\Gamma(R)}[V_3 \cup V_{13} \cup V_{23}]$. Both G_1 and G_2 are complete graphs. G' arises from G_1 and G_2 by pasting these graphs together along $S = \overline{\Gamma(R)}[V_{13}]$. Thus G' is a chordal graph, since $\overline{\Gamma(R)}[V_{13}]$ is complete. Now define $G'' = \overline{\Gamma(R)}[V_2 \cup V_{12} \cup V_{23}]$. Then G'' is a complete graph. $\overline{\Gamma(R)}$ arises from G' and G'' by pasting these graphs together along $S = \overline{\Gamma(R)}[V_{12} \cup V_{23}]$. Since $\overline{\Gamma(R)}[V_{12} \cup V_{23}]$ is complete, $\overline{\Gamma(R)}$ is a chordal graph and so it is perfect.

Case 4. Suppose that $|Max(R)| = 4$. We show that $\Gamma(R)$ is chordal. Consider V_i, V_{ij}, and V_{ijk} as in Section 2.

Let G denote the induced subgraph of $\Gamma(R)$ generated by $V_1 \cup V_2 \cup V_3 \cup V_4 \cup V_{12} \cup V_{13} \cup V_{14} \cup V_{23} \cup V_{24} \cup V_{34}$. G is denoted in Figure 14.

We show that this graph has no odd cycle of length at least 5 as an induced subgraph. Assume to the contrary that there is an induced 5-cycle, C, in graph G. Let there exist two vertices I and J, one in V_i and the other in V_j in C. Without loss of generality, let $I \in V_1$, $J \in V_2$ and the neighbour of J be in V_{13} or in V_{14}. If $K \in V_{13}$ and $JK \in E(C)$, every neighbour of K is in V_4 or V_{24}, which are joined to I. Thus C has a chord, which is a contradiction. A similar case will occur if $K \in V_{14}$.

Now let C have only one vertex from $\bigcup_{i=1}^{4} V_i$. Without loss of generality, assume that $I \in V_1$, $I \in C$, $J \in V_{23}$ is adjacent vertex of I in C and $C : I - J - K - L - M - \cdots$. If $K \in V_{14}$, then $L \in V_2$ or V_3. Thus $IL \in E(C)$ is a chord in C, which is a contradiction. If $K \in V_4$, then $IK \in E(C)$ is a chord, a contradiction. Therefore C has no vertex in $\bigcup_{i=1}^{4} V_i$. However, the induced subgraph of $\bigcup_{1 \le i < j \le 4} V_{ij}$, G, is a forest and has no cycle and so G is perfect. Now $\Gamma(R)$ is the graph shown in Figure 15.

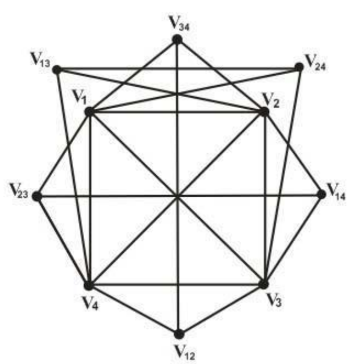


Figure 14.

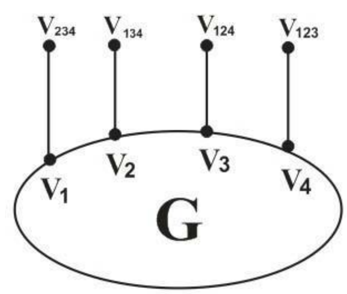


Figure 15.

It is obvious that $\Gamma(R)$ has no odd cycle of length at least 5 as an induced subgraph. Thus $\Gamma(R)$ is perfect. □

For $|Max(R)| \ge 5$, maybe $\overline{\Gamma(R)}$ is not perfect.

Example 3.7 Consider the ring $R = \mathbb{Z}_{2310}$ with $Max(R) = \{M_1 =< 2 >, M_2 =< 3 >, M_3 =< 5 >, M_4 =< 7 >, M_5 =< 11 >\}$ and $< 6 > \in V_{12}$, $< 15 > \in V_{23}$, $< 35 > \in V_{34}$, $< 77 > \in V_{45}$, $< 22 > \in V_{15}$.

Clearly, the above ideals are distinct and they are not contained in $J(R)$. It is easy to check that the subgraph G of $\overline{\Gamma(R)}$ induced on $\{< 6 >, < 15 >, < 35 >, < 77 >, < 22 >\}$ is a C_5. Therefore, $\overline{\Gamma(R)}$ is imperfect.

Now we give the main result on $\Gamma(R)$.

Corollary 3.8 *If* $\Gamma(R)$ *is planar, then* $\Gamma(R)$ *is also perfect.*

Proof By Lemma 2.5 and Theorem 3.6. □

The converse of the above corollary does not hold in general.

Example 3.9 Consider the ring $R = \mathbb{Z}_{36}$. Clearly $Max(R) = \{<2>, <3>\}$ with $|<2> \setminus <3>| \geq 3$ and $|<3> \setminus <2>| \geq 3$. Then $\Gamma(R)$ is perfect but it is not planar, since it has a subdivision of $K_{3,3}$.

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