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Global regularity for unsteady flow of third grade fluid in an annular region

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Abstract: This article develops global regularity criteria for unsteady and magnetohydrodynamic flow of third grade fluid in terms of bounded mean oscillations. Uniqueness of the solution is also verified.

Key words: Nonlinear problem, global regularity, third grade fluid, annular pipe, magnetohydrodynamic flow

1. Introduction

Non-Newtonian materials are now well recognized by scientists and engineers due to their industrial and technological applications. Several biological liquids also exhibit the rheological characteristics of non-Newtonian materials. Such materials having a magnetohydrodynamic character play a pivotal role in polymer processing, treatment of hyperthermia, cancer therapy, and many other fields. It is, however, well known that the flow of non-Newtonian fluids cannot be addressed by using the classical Navier–Stokes equations. This is because of their viscoelastic features in addition to the viscosity. Different non-Newtonian fluids have distinct rheological properties. Hence, several constitutive equations have been recommended for the flow analysis of non-Newtonian materials. The non-Newtonian fluids in general are classified into differential, rate, and integral categories. Several investigators in the field have chosen the simplest subclass of second grade fluid. The information about the flows of second grade fluid at present is quite sizeable. A few representative recent contributions in this direction can be seen in [1, 2, 4, 16, 17, and several references therein]. Note that second grade fluid cannot predict the stress relaxation and retardation time effects. Hence, the Maxwell fluid model is employed for stress relaxation time while an Oldroyd-B fluid captures both the stress retardation times (see [3, 9, 11–15, 20, 22, 23] for details).

Although the second grade fluid model is able to predict the normal stress effects, it does not capture the shear thinning and shear thickening properties that many materials show. Having such limitations in mind, researchers now prefer the third grade model. The third grade model predicts the shear thinning and shear thickening features. The equation of motion for even unidirectional flow of the third grade model is nonlinear, which is not the case in second grade fluid. Despite such complexities, recent workers in the field studied the flows of third grade fluid for different configurations and aspects (see [1, 5–7, 19]).

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2. Mathematical model and analysis

We consider the flow of an incompressible third grade fluid in an annular region. The fluid is conducting in the presence of an applied magnetic field B_0. The magnetic Reynolds number is taken to be small and thus the induced magnetic field is neglected. Moreover, the effect of the electric field is not considered. The present fundamental equations that govern the flow are:

$$div\bar{V} = 0, \tag{2.1}$$

$$\rho\frac{d\bar{V}}{dt} = div\bar{T} + \bar{J} \times \bar{B}_0, \tag{2.2}$$

in which $\bar{V}$ is the velocity field, ρ the density of fluid, and $\frac{d}{dt}$ material time differentiation. The taken Cauchy stress tensor $(\bar{T})$ for third grade fluid is [1, 5–7, 19]:

$$\bar{T} = -p\bar{I} + (\mu + \beta_3 tr(\bar{A}_1^2))\bar{A}_1 + \alpha_1\bar{A}_2 + \alpha_2\bar{A}_1^2. \tag{2.3}$$

Fosdick and Rajagopal [10] presented the thermodynamic analysis for the constitutive equation of third grade fluid. They found that all the fluid motions compatible with thermodynamics satisfy the Clausius–Dhem inequality provided that the material constants satisfy the following conditions:

$$\mu \geq 0, \beta_3 \geq 0, \beta_1 = \beta_2 = 0 \text{ and } |\alpha_1 + \alpha_2| \leq \sqrt{24\mu\beta_3}.$$

Here the specific Helmholtz free energy is minimum when the fluid is locally at rest. Note that the above conditions are necessary to determine the stability properties of flow. The first two Rivlin–Ericksen tensors are

$$\bar{A}_1 = (grad\bar{V}) + (grad\bar{V})^T, \tag{2.4}$$

$$\bar{A}_2 = \frac{d\bar{A}_1}{dt} + \bar{A}_1(grad\bar{V}) + (grad\bar{V})^T\bar{A}_1. \tag{2.5}$$

Here p is the pressure, $\bar{I}$ the identity tensor, T the matrix transpose, and $\bar{J}$ the current density. Now

$$\bar{J} \times \bar{B} \quad = \quad -\sigma B_0^2\bar{V}, \tag{2.6}$$

in which σ is electrical conductivity of the fluid. The velocity field for the present flow is

$$\bar{V} = w(r,t)e_z. \tag{2.7}$$

The above equation satisfies the continuity equation (2.1). By Eqs. (2.2), (2.6), and (2.7), we have in the absence of a modified pressure gradient:

$$\begin{aligned} \rho\frac{\partial w}{\partial t} \quad &= \quad \frac{\mu\partial^2 w}{\partial r^2} + \frac{\mu}{r}\frac{\partial w}{\partial r} + 6\beta_3(\frac{\partial w}{\partial r})^2\frac{\partial^2 w}{\partial r^2} + \frac{2\beta_3}{r}\left(\frac{\partial w}{\partial r}\right)^3 \\ &+ \quad \frac{\alpha_1}{r}\frac{\partial^2 w}{\partial r\partial t} + \frac{\alpha_1\partial^3 w}{\partial r^2\partial t} - \sigma B_0^2 w. \end{aligned} \tag{2.8}$$

The relevant conditions are prescribed as follows:

$$\begin{aligned} w(r,t) &= 0 \quad \text{at} \quad r = R_0 \\ w(r,t) &= W_1, \quad \text{at} \quad r = R_1 \\ w(r,0) &= w_0(r), \end{aligned} \tag{2.9}$$

in which $w_0(r)$ is the initial velocity field, W_1 is constant velocity, and R_0 and R_1 are the radii of inner and outer cylinder, respectively.

The aim of our research is to prove the existence of a global classical solution of third grade fluid in terms of the bounded mean oscillation (BMO) norm. For this purpose we apply the procedure followed in the recent attempts [18, 25, 26] and obtain the bounded velocity and vorticity. BMO denotes the homogeneous space of BMOs associated with the norm

$$\|f\|_{\text{BMO}} \doteq \sup_{R^n, r>0} \frac{1}{|B_r(x)|} \int_{B_r(x)} \left| f(y) - \frac{1}{|B_r(y)|} \int_{B_r(y)} |f(z)| dz \right| dy.$$

For the details of BMO spaces, we refer readers to [23].

Let us recall the following Lemma (see [8]):

Lemma 1 *Let $1 < q < p < \infty$; then*

$$\|u\|_{L^p} \leq \tilde{C} \|u\|_{BMO}^{1-\frac{q}{p}} \|u\|_{L^q}^{\frac{q}{p}},$$

where $\tilde{C}$ is a constant. Our main results are:

Theorem 1 *Assuming $u_0 \in H^2(\Omega)$ and $\|w\|_{BMO}^2$, $\|g\|_{BMO}^2$, $\left\|\frac{\partial w}{\partial r}\right\|_{BMO}^2$, $\left\|\frac{\partial g}{\partial r}\right\|_{BMO}^2$, and $\left\|\frac{\partial^2 g}{\partial r^2}\right\|_{BMO}^2$ are sufficiently small then the system of (2.8) and (2.9) has a unique global classical solution $w(r,t)$ on $(0,T)$, where $g = \nabla \times \bar{V} = -\frac{\partial w}{\partial r}$ and $\Omega = [R_0, R_1]$.*

3. Proof of Theorem 1

For proving Theorem 1, we need the following propositions.

Proposition 1 *If w is the solution of (2.8) and (2.9) with $\|w\|_{BMO}^2$, $\|g\|_{BMO}^2$, $\left\|\frac{\partial w}{\partial r}\right\|_{BMO}^2$, and $\left\|\frac{\partial g}{\partial r}\right\|_{BMO}^2$ being sufficiently small, then velocity w and vorticity $g = \nabla \times \bar{V} = -\frac{\partial w}{\partial r}$ satisfy*

$$\begin{aligned} \sup_{0 \leq t \leq T} \Big(\rho \|w\|_{L^2}^2 + \alpha_1 \left\|\frac{\partial w}{\partial r}\right\|_{L^2}^2\Big) &\leq C_1\Big(\rho \|w_0\|_{L^2}^2 + \alpha_1 \left\|\frac{\partial w_0}{\partial r}\right\|_{L^2}^2\Big), \\ \sup_{0 \leq t \leq T} \Big(\rho \|g\|_{L^2}^2 + \alpha_1 \left\|\frac{\partial g}{\partial r}\right\|_{L^2}^2\Big) &\leq C\Big(\rho \|g_0\|_{L^2}^2 + \alpha_1 \left\|\frac{\partial g_0}{\partial r}\right\|_{L^2}^2\Big), \quad \text{on} \quad [0,T] \end{aligned}$$

in which C_1 and C depend on T.

Proof We assume that the solution of (2.8) is given by

$$w(r,t) = e^{-at}F(r). \tag{3.1}$$

Taking the inner product of (2.8) with w, and using integration by parts, we arrive at

$$\begin{aligned}
\frac{\rho}{2}\int_\Omega \frac{\partial}{\partial t}|w|^2 dr &= -\mu\int_\Omega \Big(\frac{\partial w}{\partial r}\Big)^2 dr + \mu\int_\Omega \frac{w}{r}\frac{\partial w}{\partial r} dr + 6\beta_3\int_\Omega w\frac{\partial^2 w}{\partial r^2}\Big(\frac{\partial w}{\partial r}\Big)^2 dr \\
&+ 2\beta_3\int_\Omega \frac{w}{r}\Big(\frac{\partial w}{\partial r}\Big)^3 dr + \alpha_1\int_\Omega \frac{w}{r}\frac{\partial^2 w}{\partial r\partial t} dr - \alpha_1\int_\Omega \frac{\partial w}{\partial r}\frac{\partial^2 w}{\partial r\partial t} dr \\
&- \sigma B_0^2\int_\Omega |w|^2 dr,
\end{aligned}$$

$$\begin{aligned}
\rho\frac{d}{dt}\|w\|_{L^2}^2 &\le -2\mu\left\|\frac{\partial w}{\partial r}\right\|_{L^2}^2 + \mu\int_\Omega \left|\frac{w}{r}\right|^2 dr + \mu\int_\Omega \left|\frac{\partial w}{\partial r}\right|^2 dr + I_1 \\
&+ 2\beta_3\int_\Omega \left|\frac{\partial w}{\partial r}\right|^2\Big(\left|\frac{w}{r}\right|^2 + \left|\frac{\partial w}{\partial r}\right|^2\Big) dr + 2\alpha_1\int_\Omega \left|\frac{w}{r}\right|\left|\frac{\partial^2 w}{\partial r\partial t}\right| dr \\
&- \alpha_1\int_\Omega \frac{\partial}{\partial t}\Big(\frac{\partial w}{\partial r}\Big)^2 dr - 2\sigma B_0^2\|w\|_{L^2}^2 \\
&\le -\mu\left\|\frac{\partial w}{\partial r}\right\|_{L^2}^2 + \mu\int_\Omega \left|\frac{w}{r}\right|^2 dr + I_1 + \beta_3\int_\Omega \left|\frac{w}{r}\right|^4 dr + 3\beta_3\left\|\frac{\partial w}{\partial r}\right\|_{L^4}^4 \\
&+ 2\alpha_1\int_\Omega \left|\frac{w}{r}\right|\left|\frac{\partial^2 w}{\partial r\partial t}\right| dr - \alpha_1\frac{d}{dt}\left\|\frac{\partial w}{\partial r}\right\|_{L^2}^2 - 2\sigma B_0^2\|w\|_{L^2}^2,
\end{aligned} \tag{3.2}$$

where we used Young's inequality. Using integration by parts of I_1, we have

$$I_1 = -2\beta_3\left\|\frac{\partial w}{\partial r}\right\|_{L^4}^4.$$

From (3.1), we obtain

$$\begin{aligned}
2\alpha_1\int_\Omega \left|\frac{w}{r}\right|\left|\frac{\partial^2 w}{\partial r\partial t}\right| dr &= 2\alpha_1 a\int_\Omega \left|\frac{w}{r}\right|\left|\frac{\partial w}{\partial r}\right| dr \\
&= \alpha_1 a\int_\Omega \left|\frac{w}{r}\right|^2 dr + \alpha_1 a\left\|\frac{\partial w}{\partial r}\right\|_{L^2}^2,
\end{aligned}$$

where we used Young's inequality. Therefore, Eq. (3.2) becomes

$$\begin{aligned}\frac{d}{dt}\Big(\rho\left\|w\right\|_{L^2}^2+\alpha_1\left\|\frac{\partial w}{\partial r}\right\|_{L^2}^2\Big) &\le \Big(\alpha_1 a-\mu\Big)\left\|\frac{\partial w}{\partial r}\right\|_{L^2}^2+\Big(\mu+\alpha_1 a\Big)\int_\Omega\left|\frac{w}{r}\right|^2 dr\\ &+ \beta_3\int_\Omega\left|\frac{w}{r}\right|^4 dr+\beta_3\left\|\frac{\partial w}{\partial r}\right\|_{L^4}^4-2\sigma B_0^2\left\|w\right\|_{L^2}^2\\ &\le \Big(\alpha_1 a-\mu\Big)\left\|\frac{\partial w}{\partial r}\right\|_{L^2}^2+\Big(\mu+\alpha_1 a\Big)C_2\int_\Omega\left|\partial_i\left|\frac{w}{r}\right|^2\right| dr\\ &+ \beta_3 C_3\int_\Omega\left|\partial_i\left|\frac{w}{r}\right|^4\right| dr+\beta_3\left\|\frac{\partial w}{\partial r}\right\|_{L^4}^4-2\sigma B_0^2\left\|w\right\|_{L^2}^2 .\end{aligned}$$

Since $\partial_i w=\frac{x_i}{r}\frac{\partial w}{\partial r}$, and $\left|\partial_i\left|\frac{w}{r}\right|^2\right|\le 2\left|w\right|^2+2\left|w\right|\left|\frac{\partial w}{\partial r}\right|$ and $\left|\partial_i\left|\frac{w}{r}\right|^4\right|\le 8\left|w\right|^3+2\left|w\right|\left|\frac{\partial w}{\partial r}\right|$, after using Young's inequality we have:

$$\begin{aligned}\frac{d}{dt}\Big(\rho\left\|w\right\|_{L^2}^2+\alpha_1\left\|\frac{\partial w}{\partial r}\right\|_{L^2}^2\Big) &\le \Big(3\mu+3\alpha_1 a-2\sigma B_0^2\Big)\left\|w\right\|_{L^2}^2+2\alpha_1 a\left\|\frac{\partial w}{\partial r}\right\|_{L^2}^2\\ &+ 2\beta_3\left\|w\right\|_{L^4}^4+\beta_3\left\|\frac{\partial w}{\partial r}\right\|_{L^4}^4\\ &\le \Big(3\mu+3\alpha_1 a-2\sigma B_0^2\Big)\left\|w\right\|_{L^2}^2+2\alpha_1 a\left\|\frac{\partial w}{\partial r}\right\|_{L^2}^2\\ &+ 2\beta_3\tilde{C}\left\|w\right\|_{L^2}^2\left\|w\right\|_{BMO}^2+\beta_3\tilde{C}\left\|\frac{\partial w}{\partial r}\right\|_{L^2}^2\left\|\frac{\partial w}{\partial r}\right\|_{BMO}^2,\end{aligned}$$

where we used Lemma 1. Since $\left\|w\right\|_{BMO}^2$ and $\left\|\frac{\partial w}{\partial r}\right\|_{BMO}^2$ are sufficiently small, we can choose $2\beta_3\tilde{C}\left\|w\right\|_{BMO}^2\le C_2$ and $\beta_3\tilde{C}\left\|\frac{\partial w}{\partial r}\right\|_{BMO}^2\le C_3$, and therefore we obtain

$$\begin{aligned}\frac{d}{dt}\Big(\rho\left\|w\right\|_{L^2}^2+\alpha_1\left\|\frac{\partial w}{\partial r}\right\|_{L^2}^2\Big) &\le \Big(3\mu+3\alpha_1 a-2\sigma B_0^2+C_2\Big)\left\|w\right\|_{L^2}^2+\Big(2\alpha_1 a+C_3\Big)\left\|\frac{\partial w}{\partial r}\right\|_{L^2}^2\\ &\le \rho\left\|w\right\|_{L^2}^2+\alpha_1\left\|\frac{\partial w}{\partial r}\right\|_{L^2}^2\end{aligned}$$

where $\Big(3\mu+3\alpha_1 a-2\sigma B_0^2+C_2\Big)\le\rho$ and $\Big(2\alpha_1 a+C_3\Big)\le\alpha_1$. Using Gronwall's inequality, we have

$$\sup_{0\le t\le T}\Big(\rho\left\|w\right\|_{L^2}^2+\alpha_1\left\|\frac{\partial w}{\partial r}\right\|_{L^2}^2\Big)\le C_1\Big(\rho\left\|w_0\right\|_{L^2}^2+\alpha_1\left\|\frac{\partial w_0}{\partial r}\right\|_{L^2}^2\Big),$$

where C_1 depends on T. □

From (2.8) and (2.9), the vorticity g satisfies the following equations:

$$\begin{aligned}\rho\frac{\partial g}{\partial t} &= \frac{\mu\partial^2 g}{\partial r^2}-\frac{\mu g}{r^2}+\frac{\mu}{r}\frac{\partial g}{\partial r}+12\beta_3 g\Big(\frac{\partial g}{\partial r}\Big)^2+6\beta_3(g)^2\frac{\partial^2 g}{\partial r^2}+\frac{6\beta_3 g^2}{r}\frac{\partial g}{\partial r}\\ &- \frac{2\beta_3 g^3}{r^2}-\frac{\alpha_1}{r^2}\frac{\partial g}{\partial t}+\frac{\alpha_1}{r}\frac{\partial^2 g}{\partial r\partial t}+\alpha_1\frac{\partial^3 g}{\partial r^2\partial t}-\sigma B_0^2 g,\end{aligned} \tag{3.3}$$

$$
\begin{aligned}
g(r,t) &= 0 \quad \text{at} \quad r = R_0\\
g(r,t) &= 0, \quad \text{at} \quad r = R_1\\
g(r,0) &= g_0(r).
\end{aligned} \tag{3.4}
$$

We assume that the solution of (3.3) is given by

$$
g(r,t) = e^{-at} g(r). \tag{3.5}
$$

By taking the inner product of (3.3) with g and integrating by parts, it follows that

$$
\begin{aligned}
\frac{\rho}{2}\frac{d}{dt}\|g\|_{L^2}^2 &= -\mu\left\|\frac{\partial g}{\partial r}\right\|_{L^2}^2 - \mu\int_\Omega\left|\frac{g}{r}\right|^2 dr + \mu\int_\Omega \frac{g}{r}\frac{\partial g}{\partial r}dr - 6\beta_3\int_\Omega (g)^2\Big(\frac{\partial g}{\partial r}\Big)^2 dr\\
&- \alpha_1\int_\Omega \frac{g}{r^2}\frac{\partial g}{\partial t}dr + \alpha_1\int_\Omega \frac{g}{r}\frac{\partial^2 g}{\partial r\partial t}dr - \frac{\alpha_1}{2}\frac{d}{dt}\left\|\frac{\partial g}{\partial r}\right\|_{L^2}^2 - \sigma B_0^2\|g\|_{L^2}^2\\
&\leq -\frac{\mu}{2}\left\|\frac{\partial g}{\partial r}\right\|_{L^2}^2 - \frac{\mu}{2}\int_\Omega\left|\frac{g}{r}\right|^2 dr + 3\beta_3\|g\|_{L^4}^4 + 3\beta_3\left\|\frac{\partial g}{\partial r}\right\|_{L^4}^4\\
&+ \alpha_1\int_\Omega\left|\frac{g}{r^2}\right|\left|\frac{\partial g}{\partial t}\right|dr + \alpha_1\int_\Omega\left|\frac{g}{r}\right|\left|\frac{\partial^2 g}{\partial r\partial t}\right|dr - \frac{\alpha_1}{2}\frac{d}{dt}\left\|\frac{\partial g}{\partial r}\right\|_{L^2}^2 - \sigma B_0^2\|g\|_{L^2}^2\\
&= \frac{\mu}{2}\left\|\frac{\partial g}{\partial r}\right\|_{L^2}^2 + \Big(\frac{\alpha_1 a}{2} - \frac{\mu}{2}\Big)\int_\Omega\left|\frac{g}{r}\right|^2 dr + 3\beta_3\|g\|_{L^4}^4 + 3\beta_3\left\|\frac{\partial g}{\partial r}\right\|_{L^4}^4\\
&+ \alpha_1 a\int_\Omega\left|\frac{g}{r}\right|\left|\frac{\partial g}{\partial r}\right|dr - \frac{\alpha_1}{2}\frac{d}{dt}\left\|\frac{\partial g}{\partial r}\right\|_{L^2}^2 - \sigma B_0^2\|g\|_{L^2}^2\\
&\leq \Big(\frac{\alpha a}{2} + \frac{\mu}{2}\Big)\left\|\frac{\partial g}{\partial r}\right\|_{L^2}^2 + \Big(\alpha_1 a - \frac{\mu}{2}\Big)\int_\Omega\left|\frac{g}{r}\right|^2 dr + 3\beta_3\|g\|_{L^4}^4\\
&+ 3\beta_3\left\|\frac{\partial g}{\partial r}\right\|_{L^4}^4 - \frac{\alpha_1}{2}\frac{d}{dt}\left\|\frac{\partial g}{\partial r}\right\|_{L^2}^2 - \sigma B_0^2\|g\|_{L^2}^2\\
&\leq \Big(\frac{\alpha a}{2} + \frac{\mu}{2}\Big)\left\|\frac{\partial g}{\partial r}\right\|_{L^2}^2 + \Big(\alpha_1 a - \frac{\mu}{2}\Big)\int_\Omega\left|\partial_i\left|\frac{g}{r}\right|^2\right| dr + 3\beta_3\|g\|_{L^4}^4\\
&+ 3\beta_3\left\|\frac{\partial g}{\partial r}\right\|_{L^4}^4 - \frac{\alpha_1}{2}\frac{d}{dt}\left\|\frac{\partial g}{\partial r}\right\|_{L^2}^2 - \sigma B_0^2\|g\|_{L^2}^2,
\end{aligned}
$$

where we used Young's inequality, (3.5), and Poincare's inequality. Since $\partial_i g = \frac{x_i}{r}\frac{\partial g}{\partial r}$ and $\left|\partial_i\left|\frac{g}{r}\right|^2\right| \leq 2|g|^2 + 2|g|\left|\frac{\partial g}{\partial r}\right|$, after using Young's inequality we have

$$\begin{aligned}\frac{d}{dt}\Big(\rho\,\|g\|^2_{L^2} &+ \alpha_1\left\|\frac{\partial g}{\partial r}\right\|^2_{L^2}\Big) \le \Big(4\alpha a - 2\mu + \epsilon\Big(2\alpha a-\mu\Big) - \sigma B_0^2\Big)\|g\|^2_{L^2}\\ &+ \Big(\alpha a + \mu + \frac{1}{\epsilon}\Big(2\alpha a - \mu\Big)\Big)\left\|\frac{\partial g}{\partial r}\right\|^2_{L^2} + 3\beta_3\|g\|^4_{L^4} + 3\beta_3\left\|\frac{\partial g}{\partial r}\right\|^4_{L^4}\\ &\le \Big(4\alpha a - 2\mu + \epsilon\Big(2\alpha a-\mu\Big) - \sigma B_0^2\Big)\|g\|^2_{L^2} + \Big(\alpha a + \mu + \frac{1}{\epsilon}\Big(2\alpha a - \mu\Big)\Big)\left\|\frac{\partial g}{\partial r}\right\|^2_{L^2}\\ &+ 3\beta_3\tilde{C}\,\|g\|^2_{L^2}\,\|g\|^2_{BMO} + 3\beta_3\tilde{C}\left\|\frac{\partial g}{\partial r}\right\|^2_{L^2}\left\|\frac{\partial g}{\partial r}\right\|^2_{BMO},\end{aligned}$$

where we used Lemma 1. Since $\|g\|^2_{BMO}$ and $\left\|\frac{\partial g}{\partial r}\right\|^2_{BMO}$ are sufficiently small, we can choose $3\beta_3\tilde{C}\,\|g\|^2_{BMO} \le C_4$ and $3\beta_3\tilde{C}\left\|\frac{\partial g}{\partial r}\right\|^2_{BMO} \le C_5$ and therefore we get

$$\begin{aligned}\frac{d}{dt}\Big(\rho\,\|g\|^2_{L^2} + \alpha_1\left\|\frac{\partial g}{\partial r}\right\|^2_{L^2}\Big) &\le \Big(4\alpha a - 2\mu + \epsilon\Big(2\alpha a-\mu\Big) + C_4 - \sigma B_0^2\Big)\|g\|^2_{L^2}\\ &+ \Big(\alpha a + \mu + \frac{1}{\epsilon}\Big(2\alpha a - \mu\Big) + C_5\Big)\left\|\frac{\partial g}{\partial r}\right\|^2_{L^2}\\ &\le \rho\,\|g\|^2_{L^2} + \alpha_1\left\|\frac{\partial g}{\partial r}\right\|^2_{L^2}\end{aligned}$$

in which $\left|4\alpha a - 2\mu + \epsilon\Big(2\alpha a-\mu\Big) + C_4 - \sigma B_0^2\right| \le \rho$ and $\left|\alpha a + \mu + \frac{1}{\epsilon}\Big(2\alpha a - \mu\Big) + C_5\right| \le \alpha_1$. Now applying Gronwall's inequality, we obtain

$$\sup_{0\le t\le T}\Big(\rho\,\|g\|^2_{L^2} + \alpha_1\left\|\frac{\partial g}{\partial r}\right\|^2_{L^2}\Big) \le C\Big(\rho\,\|g_0\|^2_{L^2} + \alpha_1\left\|\frac{\partial g_0}{\partial r}\right\|^2_{L^2}\Big),$$

where C depends on T.

Proposition 2 *If g is the solution of (3.3) and (3.4) with $\|g\|^2_{BMO}$, $\left\|\frac{\partial g}{\partial r}\right\|^2_{BMO}$, and $\left\|\frac{\partial^2 g}{\partial r^2}\right\|^2_{BMO}$ being sufficiently small, then the vorticity $g = \nabla \times \bar{V}$ satisfies*

$$\sup_{0\le t\le T}\Big(\rho\left\|\frac{\partial g}{\partial r}\right\|^2_{L^2} + \alpha_1\left\|\frac{\partial^2 g}{\partial r^2}\right\|^2_{L^2}\Big) \le C\Big(\rho\left\|\frac{\partial g_0}{\partial r}\right\|^2_{L^2} + \alpha_1\left\|\frac{\partial^2 g_0}{\partial r^2}\right\|^2_{L^2}\Big) \tag{3.6}$$

on $[0,T]$ and C depends on g_0 and T.

Proof Taking the inner product of (3.3) with $-\frac{\partial^2 g}{\partial r^2}$ and integrating by parts, we get

$$
\begin{aligned}
\frac{\rho}{2}\frac{d}{dt}\left\|\frac{\partial g}{\partial r}\right\|_{L^2}^2 &= -\mu\left\|\frac{\partial^2 g}{\partial r^2}\right\|_{L^2}^2 + \mu\int_\Omega \frac{g}{r^2}\frac{\partial^2 g}{\partial r^2}dr - \mu\int_\Omega \frac{1}{r}\frac{\partial g}{\partial r}\frac{\partial^2 g}{\partial r^2}dr + 4\beta_3\int_\Omega\Big(\frac{\partial g}{\partial r}\Big)^4 dr \\
&+ 6\beta_3\int_\Omega |g|^2\left|\frac{\partial^2 g}{\partial r^2}\right|^2 dr - 9\beta_3\int_\Omega \frac{g^2}{r^2}\Big(\frac{\partial g}{\partial r}\Big)^2 dr + 6\beta_3\int_\Omega \frac{g}{r}\Big(\frac{\partial g}{\partial r}\Big)^3 dr \\
&+ 4\beta_3\int_\Omega \frac{g^3}{r^3}\Big(\frac{\partial g}{\partial r}\Big)dr + \alpha\int_\Omega \frac{1}{r^2}\frac{\partial g}{\partial t}\frac{\partial^2 g}{\partial r^2}dr - \alpha_1\int_\Omega \frac{1}{r}\frac{\partial^2 g}{\partial t\partial r}\frac{\partial^2 g}{\partial r^2}dr \\
&- \frac{\alpha_1}{2}\frac{d}{dt}\left\|\frac{\partial^2 g}{\partial r^2}\right\|_{L^2}^2 - \sigma B_0^2\left\|\frac{\partial g}{\partial r}\right\|_{L^2}^2
\end{aligned}
$$

$$
\begin{aligned}
\frac{d}{dt}\Big(\rho\left\|\frac{\partial g}{\partial r}\right\|_{L^2}^2 &+ \alpha_1\left\|\frac{\partial^2 g}{\partial r^2}\right\|_{L^2}^2\Big) \le \mu\int_\Omega\left|\frac{g}{r^2}\right|^2 dr + \mu\int_\Omega\left|\frac{1}{r}\frac{\partial g}{\partial r}\right|^2 dr + 28\beta_3\left\|\frac{\partial g}{\partial r}\right\|_{L^4}^4 \\
&+ 6\beta_3\|g\|_{L^4}^4 + 18\beta_3\int_\Omega\left|\frac{g}{r}\right|^4 dr + 6\beta_3\left\|\frac{\partial^2 g}{\partial r^2}\right\|_{L^4}^4 - 2\alpha a\int_\Omega \frac{g}{r^2}\frac{\partial^2 g}{\partial r^2}dr \\
&+ 2\alpha a\int_\Omega \frac{1}{r}\frac{\partial g}{\partial r}\frac{\partial^2 g}{\partial r^2}dr - 2\sigma B_0^2\left\|\frac{\partial g}{\partial r}\right\|_{L^2}^2 \\
&\le 2\alpha a\left\|\frac{\partial^2 g}{\partial r^2}\right\|_{L^2}^2 + \Big(\mu+\alpha a\Big)\int_\Omega\left|\frac{g}{r^2}\right|^2 dr + \Big(\mu+\alpha a\Big)\int_\Omega\left|\frac{1}{r}\frac{\partial g}{\partial r}\right|^2 dr \\
&+ 28\beta_3\left\|\frac{\partial g}{\partial r}\right\|_{L^4}^4 + 6\beta_3\|g\|_{L^4}^4 + 6\beta_3\left\|\frac{\partial^2 g}{\partial r^2}\right\|_{L^4}^4 + 18\beta_3\int_\Omega\left|\frac{g}{r}\right|^4 dr - 2\sigma B_0^2\left\|\frac{\partial g}{\partial r}\right\|_{L^2}^2 \\
&\le 2\alpha a\left\|\frac{\partial^2 g}{\partial r^2}\right\|_{L^2}^2 + \Big(\mu+\alpha a\Big)C_7\int_\Omega\left|\partial_i\left|\frac{g}{r^2}\right|^2\right|dr + \Big(\mu+\alpha a\Big)C_8\int_\Omega\left|\partial_i\left|\frac{1}{r}\frac{\partial g}{\partial r}\right|^2\right|dr \\
&+ 28\beta_3\left\|\frac{\partial g}{\partial r}\right\|_{L^4}^4 + 6\beta_3\|g\|_{L^4}^4 + 6\beta_3\left\|\frac{\partial^2 g}{\partial r^2}\right\|_{L^4}^4 + 18\beta_3\int_\Omega\left|\frac{g}{r}\right|^4 dr - 2\sigma B_0^2\left\|\frac{\partial g}{\partial r}\right\|_{L^2}^2,
\end{aligned}
$$

where we used Young's inequality, (3.5), and Poincare's inequality. Since $\partial_i g = \frac{x_i}{r}\frac{\partial g}{\partial r}$, and $\left|\partial_i\left|\frac{g}{r}\right|^4\right| \le 2|g|^2 + 2|g|\left|\frac{\partial g}{\partial r}\right|$, $\left|\partial_i\left|\frac{g}{r^2}\right|^2\right| \le 4|g|^2 + 2|g|\left|\frac{\partial g}{\partial r}\right|$, and $\left|\partial_i\left|\frac{1}{r}\frac{\partial g}{\partial r}\right|^2\right| \le 2|\frac{\partial g}{\partial r}| + 2\left|\frac{\partial^2 g}{\partial r^2}\right|\left|\frac{\partial g}{\partial r}\right|$, after using Young's inequality we have

$$
\begin{aligned}
\frac{d}{dt}\Big(\rho\left\|\frac{\partial g}{\partial r}\right\|_{L^2}^2 &+ \alpha_1\left\|\frac{\partial^2 g}{\partial r^2}\right\|_{L^2}^2\Big) \le \Big(5\mu + 5\alpha a + 48\beta_3 C_9\Big)\|g\|_{L^2}^2 \\
&+ \Big((\mu+\alpha a)C_7 + 3(\mu+\alpha a)C_8 + 18\beta_3 C_9 - 2\sigma B_0^2\Big)\left\|\frac{\partial g}{\partial r}\right\|_{L^2}^2 \\
&+ \Big(2\alpha a + (\mu+\alpha a)C_8\Big)\left\|\frac{\partial^2 g}{\partial r^2}\right\|_{L^2}^2 + 6\beta_3\|g\|_{L^4}^4 + 28\beta_3\left\|\frac{\partial g}{\partial r}\right\|_{L^4}^4 + 6\beta_3\left\|\frac{\partial^2 g}{\partial r^2}\right\|_{L^4}^4
\end{aligned}
$$

$$\leq \Big(5\mu + 5\alpha a + 48\beta_3 C_9\Big) \|g\|_{L^2}^2 + \Big(2\alpha a + (\mu+\alpha a)C_8\Big) \left\|\frac{\partial^2 g}{\partial r^2}\right\|_{L^2}^2$$
$$+ \Big((\mu+\alpha a)C_7 + 3(\mu+\alpha a)C_8 + 18\beta_3 C_9 - 2\sigma B_0^2\Big) \left\|\frac{\partial g}{\partial r}\right\|_{L^2}^2$$
$$+ 6\beta_3 \tilde{C} \|g\|_{L^2}^2 \|g\|_{BMO}^2 + 28\beta_3 \tilde{C} \left\|\frac{\partial g}{\partial r}\right\|_{L^2}^2 \left\|\frac{\partial g}{\partial r}\right\|_{BMO}^2 + 6\beta_3 \tilde{C} \left\|\frac{\partial^2 g}{\partial r^2}\right\|_{L^2}^2 \left\|\frac{\partial^2 g}{\partial r^2}\right\|_{BMO}^2,$$

where we used Lemma 1. Since $\|g\|_{BMO}^2$ and $\left\|\frac{\partial g}{\partial r}\right\|_{BMO}^2$ are sufficiently small, we can choose $6\beta_3 \tilde{C} \|g\|_{BMO}^2 \leq C_{10}$, $28\beta_3 \tilde{C} \left\|\frac{\partial g}{\partial r}\right\|_{BMO}^2 \leq C_{11}$, and $6\beta_3 \left\|\frac{\partial^2 g}{\partial r^2}\right\|_{BMO}^2 \leq C_{12}$ and therefore we get

$$\frac{d}{dt}\Big(\rho \left\|\frac{\partial g}{\partial r}\right\|_{L^2}^2 + \alpha_1 \left\|\frac{\partial^2 g}{\partial r^2}\right\|_{L^2}^2\Big) \leq \Big(5\mu + 5\alpha a + 48\beta_3 C_9 + C_{10}\Big) \|g\|_{L^2}^2 + \Big((\mu+\alpha a)C_7 + 3(\mu+\alpha a)C_8$$
$$+ C_{11} - 2\sigma B_0^2\Big) \left\|\frac{\partial g}{\partial r}\right\|_{L^2}^2 + \Big(2\alpha a + (\mu+\alpha a)C_8 + C_{12}\Big) \left\|\frac{\partial^2 g}{\partial r^2}\right\|_{L^2}^2$$
$$\leq \Big(5\mu + 5\alpha a + 48\beta_3 C_9 + C_{10}\Big) \|g\|_{L^2}^2 + \rho \left\|\frac{\partial g}{\partial r}\right\|_{L^2}^2 + \alpha \left\|\frac{\partial^2 g}{\partial r^2}\right\|_{L^2}^2,$$

where $(\mu+\alpha a)C_7 + 3(\mu+\alpha a)C_8 + C_{11} - 2\sigma B_0^2 \leq \rho$ and $2\alpha a + (\mu+\alpha a)C_8 + C_{12} \leq \alpha$. Now applying Proposition 1 and Gronwall's inequality, we get the required (3.6). □

For the uniqueness we assume that w_1 and w_2 are two solutions of (2.8) having the same initial condition. Therefore:

$$\rho\frac{\partial w_1}{\partial t} = \frac{\mu\partial^2 w_1}{\partial r^2} + \frac{\mu}{r}\frac{\partial w_1}{\partial r} + 6\beta_3(\frac{\partial w_1}{\partial r})^2\frac{\partial^2 w_1}{\partial r^2} + \frac{2\beta_3}{r}\Big(\frac{\partial w_1}{\partial r}\Big)^3$$
$$+ \frac{\alpha_1}{r}\frac{\partial^2 w_1}{\partial r\partial t} + \frac{\alpha_1\partial^3 w_1}{\partial r^2\partial t} - M^2 w_1 \tag{3.7}$$

$$\rho\frac{\partial w_2}{\partial t} = \frac{\mu\partial^2 w_2}{\partial r^2} + \frac{\mu}{r}\frac{\partial w_2}{\partial r} + 6\beta_3(\frac{\partial w_2}{\partial r})^2\frac{\partial^2 w_2}{\partial r^2} + \frac{2\beta_3}{r}\Big(\frac{\partial w_2}{\partial r}\Big)^3$$
$$+ \frac{\alpha_1}{r}\frac{\partial^2 w_2}{\partial r\partial t} + \frac{\alpha_1\partial^3 w_2}{\partial r^2\partial t} - M^2 w_2. \tag{3.8}$$

Subtracting (3.8) from (3.7) and letting $h = w_1 - w_2$, we have

$$\rho\frac{\partial h}{\partial t} = \frac{\mu\partial^2 h}{\partial r^2} + \frac{\mu}{r}\frac{\partial h}{\partial r} + 6\beta_3\left[\Big(\frac{\partial w_1}{\partial r}\Big)^2\frac{\partial^2 w_1}{\partial r^2} - \Big(\frac{\partial w_2}{\partial r}\Big)^2\frac{\partial^2 w_2}{\partial r^2}\right]$$
$$+ \frac{2\beta_3}{r}\left[\Big(\frac{\partial w_1}{\partial r}\Big)^3 - \Big(\frac{\partial w_2}{\partial r}\Big)^3\right] + \frac{\alpha_1\partial^3 h}{\partial r^2\partial t} - \sigma B_0^2 h. \tag{3.9}$$

$$h(r,t) = 0 \quad \text{at} \quad r = R_0$$
$$h(r,t) = 0, \quad \text{at} \quad r = R_1$$
$$h(y,0) = 0. \tag{3.10}$$

For the case $\left(\frac{\partial w_1}{\partial r}\right)^2 \geq \left(\frac{\partial w_2}{\partial r}\right)^2$ and $\left(\frac{\partial w_1}{\partial r}\right)^3 \geq \left(\frac{\partial w_2}{\partial r}\right)^3$, we have

$$\rho\frac{\partial h}{\partial t} \leq \frac{\mu\partial^2 h}{\partial r^2} + \frac{\mu}{r}\frac{\partial h}{\partial r} + \frac{\alpha_1\partial^3 h}{\partial r^2\partial t} + 6\beta_3\Big(\frac{\partial w_1}{\partial r}\Big)^2\frac{\partial^2 h}{\partial r^2} + \frac{4\beta_3}{r}\Big(\frac{\partial w_1}{\partial r}\Big)^3 - M^2 h. \tag{3.11}$$

Multiplying (3.11) by h and using integration by parts, we have

$$\begin{aligned}\frac{\rho}{2}\frac{d}{dt}\|h\|_{L^2}^2 \quad &\leq \quad -\mu\left\|\frac{\partial h}{\partial r}\right\|_{L^2}^2 + \mu\int_\Omega \frac{h}{r}\frac{\partial h}{\partial r}dr - \frac{\alpha_1}{2}\int_\Omega \frac{\partial}{\partial t}\Big(\frac{\partial h}{\partial r}\Big)^2 dr \\ &+ \quad 6\beta_3\int_\Omega h\left|\frac{\partial w_1}{\partial r}\right|^2\frac{\partial^2 h}{\partial r^2}dr + \frac{4\beta_3}{r}\int_\Omega\left|\frac{\partial w_1}{\partial r}\right|^3 h dr - M^2\|h\|_{L^2}^2 .\end{aligned}$$

Since w_1 is the solution of (2.8) and from Proposition 1, we can choose $\left|\frac{\partial w_1}{\partial r}\right|^2$ and $\left|\frac{\partial w_1}{\partial r}\right|^3$ less than or equal to C, and integrating the fourth term on the right hand side, we get

$$\begin{aligned}\frac{\rho}{2}\frac{d}{dt}\|h\|_{L^2}^2 \quad &\leq \quad -\Big(\frac{\mu}{2} + 6\beta_3 C\Big)\left\|\frac{\partial h}{\partial r}\right\|_{L^2}^2 + \Big(\frac{\mu}{2} + 4\beta_3 CC_{11}\Big)\int_\Omega\left|\frac{h}{r}\right|^2 dr \\ &- \quad \frac{\alpha_1}{2}\int_\Omega\frac{\partial}{\partial t}\Big(\frac{\partial h}{\partial r}\Big)^2 dr - M^2\|h\|_{L^2}^2 \\ &\leq \quad -\Big(\frac{\mu}{2} + 6\beta_3 C\Big)\left\|\frac{\partial h}{\partial r}\right\|_{L^2}^2 + \Big(\frac{\mu}{2} + 4\beta_3 CC_{11}\Big)C_{12}\int_\Omega\left|\partial_i\left|\frac{h}{r}\right|^2\right| dr \\ &- \quad \frac{\alpha_1}{2}\int_\Omega\frac{\partial}{\partial t}\Big(\frac{\partial h}{\partial r}\Big)^2 dr - M^2\|h\|_{L^2}^2 ,\end{aligned}$$

where we used Young's inequality and Poincare's inequality. Since $\partial_i h = \frac{x_i}{r}\frac{\partial h}{\partial r}$ and $\left|\partial_i\left|\frac{h}{r}\right|^2\right| \leq 2|h|^2 + 2|h|\left|\frac{\partial h}{\partial r}\right|$, and after using Young's inequality, we have

$$\begin{aligned}\frac{d}{dt}\Big(\rho\|h\|_{L^2}^2 + \alpha\left\|\frac{\partial h}{\partial r}\right\|_{L^2}^2\Big) \quad &\leq \quad \Big(4\mu C_{12} + 24\beta_3 CC_{11}C_{12} - 2M^2\Big)\|h\|_{L^2}^2 \\ &+ \quad \Big(\mu C_{12} + 4\beta_3 CC_{11}C_{12} - \frac{\mu}{2} - 6\beta_3 C\Big)\left\|\frac{\partial h}{\partial r}\right\|_{L^2}^2 \\ &\leq \quad \left|4\mu C_{12} + 24\beta_3 CC_{11}C_{12} - 2M^2\right|\|h\|_{L^2}^2 \\ &+ \quad \left|\mu C_{12} + 4\beta_3 CC_{11}C_{12} - \frac{\mu}{2} - 6\beta_3 C\right|\left\|\frac{\partial h}{\partial r}\right\|_{L^2}^2 \\ &\leq \quad \rho\|h\|_{L^2}^2 + \alpha\left\|\frac{\partial h}{\partial r}\right\|_{L^2}^2 ,\end{aligned}$$

where $\left|4\mu C_{12} + 24\beta_3 CC_{11}C_{12} - 2M^2\right| \leq \rho$ and $\left|\mu C_{12} + 4\beta_3 CC_{11}C_{12} - \frac{\mu}{2} - 6\beta_3 C\right| \leq \alpha$. Applying Grownwall's inequality, we obtain

$$\rho\|h\|_{L^2}^2 + \alpha\left\|\frac{\partial h}{\partial r}\right\|_{L^2}^2 \leq 0,$$

which implies that $\|h\|_{L^2}^2 = 0$; that is, $h = 0$ and $w_1 = w_2$.

For the case $\left(\frac{\partial w_1}{\partial r}\right)^2 < \left(\frac{\partial w_2}{\partial r}\right)^2$, subtracting (3.7) from (3.8) and using the same procedure as for the case $\left(\frac{\partial w_1}{\partial r}\right)^2 \geq \left(\frac{\partial w_2}{\partial r}\right)^2$, we get $w_1 = w_2$.

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