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Almost co-Kähler manifolds satisfying some symmetry conditions

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Abstract: Let M^{2n+1} be an almost co-Kähler manifold of dimension > 3 with Kählerian leaves. In this paper, we first prove that if M^{2n+1} is locally symmetric, then either it is a co-Kähler manifold with locally symmetric Kählerian leaves, or the Reeb vector field ξ is harmonic and in this case M^{2n+1} is non-co-Kähler. We also prove that any almost co-Kähler manifold of dimension 3 is ϕ-symmetric if and only if it is locally isometric to either a flat Euclidean space $\mathbb{R}^3$ or a Riemannian product $\mathbb{R} \times N^2(c)$, where $N^2(c)$ denotes a Kähler surface of constant curvature $c \neq 0$.

Key words: Locally symmetric, ϕ-symmetric, almost co-Kähler manifold, Kählerian leaves

1. Introduction

In 1969, Tanno in [23] proved the well-known theorem that almost contact metric manifolds whose automorphism groups have maximum dimensions can be classified into three classes, which were later characterized as Sasakian, Kenmotsu, and co-Kähler manifolds, respectively. Notice that the notion of co-Kähler manifolds in this paper is just the terminology of cosymplectic manifolds earlier used by Blair [1, 2] and Goldberg and Yano [12]. Here we adopt new terminology due to the fact that the co-Kähler manifolds can be regarded as an odd-dimensional version of Kähler manifolds from certain topological viewpoints. For more details we refer the reader to Li [14], a recent survey by Cappelletti-Montano et al. [5], and the many references therein regarding geometric and topological results on such manifolds.

Among others, the study of locally symmetric almost contact metric manifolds has been an interesting problem in contact geometry for a long time. With regard to the complete classification of locally symmetric contact metric manifolds, we refer the reader to Blair [2, pp. 132–133] and references therein. In particular, Ghosh and Sharma in [11] proved that a locally symmetric contact strongly pseudoconvex integrable CR manifold of dimension $2n+1$, $n > 1$ and $n \neq 3$, is locally isometric to either the unit sphere $\mathbb{S}^{2n+1}$ or the Riemannian product $\mathbb{S}^n(4) \times \mathbb{R}^{n+1}$. Very recently, the present author and Liu in [24] proved that any CR-integrable almost Kenmotsu manifold of dimension greater than 3 is locally symmetric if and only if it is locally isometric to either the hyperbolic space $\mathbb{H}^{2n+1}(-1)$ or the product $\mathbb{H}^{n+1}(-4) \times \mathbb{R}^n$. We also observe that Perrone in [20] proved that any 3-dimensional locally symmetric almost co-Kähler manifold is locally isometric to either the Euclidean space $\mathbb{R}^3$ or a product space $\mathbb{R} \times N^2(c)$, where $N^2(c)$ denotes a Kähler surface of constant curvature $c \neq 0$.

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Motivated by these results mentioned above, one object of this paper is to study locally symmetric almost co-Kähler manifolds. We mainly prove that if an almost co-Kähler manifold M^{2n+1} of dimension greater than 3 with Kählerian leaves is locally symmetric, then either M^{2n+1} is a co-Kähler manifold with locally symmetric Kählerian leaves of dimension $2n$, or the Reeb vector field ξ is an eigenvector field of the Ricci operator with nonzero eigenvalue and hence ξ is harmonic and in this case M^{2n+1} is non-co-Kähler. Some corollaries of our main results are also provided.

On the other hand, by generalizing the notion of local symmetry, Takahashi in [22] introduced the notion of (weakly) *local ϕ-symmetry* in the context of Sasakian geometry (that is, the Riemannian curvature tensor is η-parallel), which is also an analogous notion of Hermitian symmetry in complex geometry. Many kinds of symmetries in the framework of contact geometry were introduced and studied after Takahashi's paper was published. For example, Boeckx et al. in [3] and [4] studied strongly local ϕ-symmetric contact metric manifolds and some $\mathcal{D}$-homothetic transformations of ϕ-symmetric spaces, respectively. Moreover, ϕ-recurrent and ϕ-symmetric Kenmotsu manifolds were studied by De et al. in [9] and [10]. Recently, the present author and Liu in [25] studied ϕ-recurrent and ϕ-symmetric almost Kenmotsu manifolds with the Reeb vector field satisfying some nullity conditions.

As far as we know, the studies of (locally) ϕ-symmetric co-Kähler manifolds are limited. The other object of the present paper is to study ϕ-symmetric almost co-Kähler manifolds of dimension three. Applying some results shown by Perrone [20, 21], we obtain a complete local classification theorem of such manifolds. Namely, any almost co-Kähler manifold of dimension 3 is ϕ-symmetric if and only if it is locally isometric to either a flat Euclidean space $\mathbb{R}^3$ or a Riemannian product $\mathbb{R}\times N^2(c)$, where $N^2(c)$ is a Kähler surface of constant curvature $c\neq 0$. This obviously extends the corresponding results of Perrone [20].

This paper is arranged as follows. After providing some necessary preliminaries regarding almost co-Kähler manifolds in Section 2, we give some properties of such manifolds under a condition of local symmetry in Section 3. In the last section, we shall present the complete classification result of 3-dimensional almost co-Kähler manifolds under a condition of ϕ-symmetry.

2. Preliminaries

In this paper, by an *almost contact structure*, which is denoted by the triplet (ϕ,ξ,η), we mean that on a $(2n+1)$-dimensional smooth manifold M^{2n+1} there exist a $(1,1)$-type tensor field ϕ, a global vector field ξ, and a 1-form η such that

$$\phi^2=-\mathrm{id}+\eta\otimes\xi,\quad \eta(\xi)=1, \tag{2.1}$$

where id denotes the identity endomorphism, and ξ is called the characteristic or the Reeb vector field. It follows from (2.1) that $\phi(\xi)=0$, $\eta\circ\phi=0$, and $\mathrm{rank}(\phi)=2n$. We shall denote by (M^{2n+1},ϕ,ξ,η) a smooth manifold M^{2n+1} endowed with an almost contact structure, which is called an *almost contact manifold*.

The *fundamental 2-form* Φ on an almost contact metric manifold M^{2n+1} is defined by $\Phi(X,Y)=g(X,\phi Y)$ for any vector fields X and Y. We may define an almost complex structure J on product manifold $M^{2n+1}\times\mathbb{R}$ by

$$J\left(X,f\frac{d}{dt}\right)=\left(\phi X-f\xi,\eta(X)\frac{d}{dt}\right), \tag{2.2}$$

where X denotes the vector field tangent to M^{2n+1}, t is the coordinate of $\mathbb{R}$, and f is a smooth function defined on the product.

An almost contact structure is said to be *normal* if the above almost complex structure J is integrable, i.e. J is a complex structure. According to Blair [2], the *normality* of an almost contact structure is expressed by $[\phi,\phi]=-2d\eta\otimes\xi$, where $[\phi,\phi]$ denotes the Nijenhuis tensor of ϕ defined by

$$[\phi,\phi](X,Y)=\phi^2[X,Y]+[\phi X,\phi Y]-\phi[\phi X,Y]-\phi[X,\phi Y]$$

for any vector fields X,Y on M^{2n+1}.

If on an almost contact manifold there exists a Riemannian metric g satisfying

$$g(\phi X,\phi Y)=g(X,Y)-\eta(X)\eta(Y) \tag{2.3}$$

for any vector fields X,Y, then g is said to be *compatible* with the associated almost contact structure. In general, an almost contact manifold endowed with a compatible Riemannian metric is said to be an *almost contact metric manifold* and is denoted by $(M^{2n+1},\phi,\xi,\eta,g)$.

In this paper, by an *almost co-Kähler manifold*, we mean an almost contact metric manifold such that both the 1-form η and 2-form Φ are closed (see [5]). In particular, an almost co-Kähler manifold is said to be a *co-Kähler manifold* if the associated almost contact structure on it is normal, which is also equivalent to $\nabla\phi=0$, or equivalently $\nabla\Phi=0$. Notice that (almost) co-Kähler manifolds are just the (almost) cosymplectic manifolds studied in [1, 2, 12, 17, 18]. The simplest example of (almost) co-Kähler manifolds is the Riemannian product of a real line or a circle and a (almost) Kähler manifold. However, there do exist some examples of (almost) co-Kähler manifolds that are not globally the product of a (almost) Kähler manifold and a real line or a circle (see, for example, Dacko [17, Section 3]).

We now present some properties of almost co-Kähler manifolds. On an almost co-Kähler manifold $(M^{2n+1},\phi,\xi,\eta,g)$, we set $h=\frac{1}{2}\mathcal{L}_\xi\phi$ and $h'=h\circ\phi$ (notice that both h and h' are symmetric operators with respect to the metric g). Then the following formulas can be found in Olszak [17, 18] and Perrone [20]:

$$h\xi=0,\ \ h\phi+\phi h=0,\ \ \mathrm{tr}(h)=\mathrm{tr}(h')=0, \tag{2.4}$$

$$\nabla_\xi\phi=0,\ \ \nabla\xi=h',\ \ \mathrm{div}\xi=0, \tag{2.5}$$

$$\nabla_\xi h=-h^2\phi-\phi l, \tag{2.6}$$

$$\phi l\phi-l=2h^2, \tag{2.7}$$

where $l:=R(\cdot,\xi)\xi$ is the Jacobi operator along the Reeb vector field and the Riemannian curvature tensor R is defined by

$$R(X,Y)Z=\nabla_X\nabla_YZ-\nabla_Y\nabla_XZ-\nabla_{[X,Y]}Z,$$

and tr and div denote the trace and divergence operators, respectively.

3. Locally symmetric almost co-Kähler manifolds of dimension >3

In this section, we shall provide some classification results of locally symmetric almost co-Kähler manifolds of dimension greater than 3. First, following Olszak [17, Section 3] and O'Neill [16, p. 221], we see easily that any Riemannian product of a real line or a circle and a locally symmetric (almost) Kähler manifolds is a locally symmetric (almost) co-Kähler manifold. Next we present a useful lemma shown by Perrone [20].

Lemma 3.1 ([20]). *On any locally symmetric almost co-Kähler manifold we have* $\nabla_\xi h = 0$.

Proof Notice that the condition of local symmetry (i.e. $\nabla R = 0$) implies that $\nabla_\xi l = 0$. Then the proof follows from Eq. (3.3) of Perrone [20, Lemma 3.1].

Proposition 3.1. *Let* M^{2n+1} *be a locally symmetric almost co-Kähler manifold; then the multiplicity of the eigenvalue zero of* h *is at least three.*

Proof On any locally symmetric almost co-Kähler manifold M^{2n+1}, using $\nabla_\xi h = 0$ in equations (2.6) and (2.7) we obtain

$$l = -h^2. \tag{3.1}$$

Recall that the *rank* of a locally symmetric manifold is defined as the maximal dimension of a flat, totally geodesic submanifold of the manifold (see Helgason [13]). By this definition, we have $1 \leq \mathrm{rank}(M^{2n+1}) \leq 2n$. If the rank of M^{2n+1} equals one, M^{2n+1} must be of constant sectional curvature since the dimension is odd. On the other hand, according to Goldberg and Yano [12] and also Olszak [17, 18], in this context M^{2n+1} is a locally flat co-Kähler manifold and we obtain easily that $h = 0$. If M^{2n+1} does not have constant sectional curvature, its rank must be greater than one. This implies that for any point $p \in M^{2n+1}$ and each tangent vector at the point, there exists a flat, totally geodesic submanifold of dimension two through the point and tangent to the vector. In particular, there exists a nonzero vector X at each point p orthogonal to ξ such that $lX = 0$.

Let $\{\xi, E_i, \phi E_i\}$ be a local ϕ-frame of eigenvectors of h with the corresponding eigenvalues $\{0, \lambda_i, -\lambda_i\}$ respectively, where i ranges from 1 to n. Then we may write $X = \sum_{i=1}^n (X^i E_i + X^{*i}\phi E_i)$, and using this in equation (3.1) we obtain

$$\sum_{i=1}^n X^i(\lambda_i)^2 E_i + X^{*i}(\lambda_i)^2\phi E_i = 0.$$

Since X is a nonzero vector, by the previous relation, there exists $1 \leq j \leq n$ such that $X^j \neq 0$ (or $X^{*j} \neq 0$) and hence $\lambda_j = 0$. Obviously, ξ, E_j, and ϕE_j are three distinct nonzero eigenvectors of h with eigenvalue 0.

By the second term of relation (2.5), it is easy to check that $(\mathcal{L}_\xi g)(X,Y) = 2g(h'X,Y)$. Then ξ is a Killing vector field if and only if h is vanishing. The following corollary follows from [12, Proposition 3] and Proposition 3.1.

Corollary 3.1 ([20]). *Any three-dimensional locally symmetric almost co-Kähler manifold is co-Kähler.*

If on an almost co-Kähler manifold M^{2n+1} the Reeb vector field ξ satisfies

$$\begin{aligned} R(X,Y)\xi =& k(\eta(Y)X - \eta(X)Y) + \mu(\eta(Y)hX - \eta(X)hY) \\ &+ \nu(\eta(Y)h'X - \eta(X)h'Y) \end{aligned} \tag{3.2}$$

for certain smooth functions k, μ, and ν, we say that M^{2n+1} is an *almost co-Kähler* (k,μ,ν)*-manifold* (see Dacko and Olszak [8] and also [5]).

Proposition 3.2. *Any almost co-Kähler* (k,μ,ν)*-manifold is locally symmetric if and only if it is locally isometric to the product of a real line or a circle and a locally symmetric almost Kähler manifold.*

Proof Let M^{2n+1} be an almost co-Kähler (k,μ,ν)-manifold. Substituting $Y=\xi$ in (3.2) we get

$$l=-k\phi^2+\mu h+\nu h'.$$

Using this in equation (2.7) and in view of equation (2.6) we obtain

$$h^2=k\phi^2.$$

If $k\neq 0$, h has three distinct eigenvalues 0 and $\pm\sqrt{-k}$, contradicting Proposition 3.1. Having $k=0$ and hence $h=0$, according to Dacko [6, Theorem 2], we see that M^{2n+1} is locally isometric to the product of a real line or a circle and an almost Kähler manifold. The converse is trivial.

Proposition 3.3. *Let M^{2n+1} be a locally symmetric almost co-Kähler manifold; then the following relation holds:*

$$g((\nabla_X h')h'Y+(\nabla_Y h^2)X,Z)+g((\nabla_Z h')h'Y,X)=2g((\nabla_{h'Y}h')X,Z) \tag{3.3}$$

for any vector fields X,Y,Z.

Proof On any locally symmetric almost co-Kähler manifold, from equation (3.1) we have $lX=-h^2X$ for any $X\in\mathfrak{X}(M)$. Taking the covariant derivative of this relation, and using $\nabla R=0$ and relation (2.5), we obtain

$$R(\nabla_Y X,\xi)\xi+R(X,h'Y)\xi+R(X,\xi)h'Y=-\nabla_Y h^2X \tag{3.4}$$

for any $X,Y\in\mathfrak{X}(M)$. Using the second term of (2.5) we have

$$R(X,Y)\xi=(\nabla_X h')Y-(\nabla_Y h')X \tag{3.5}$$

for any $X,Y\in\mathfrak{X}(M)$. Applying (3.5) in (3.4) and using (3.1) again we get

$$(\nabla_X h')h'Y-(\nabla_{h'Y}h')X+R(X,\xi)h'Y+(\nabla_Y h^2)X=0$$

for any $X,Y\in\mathfrak{X}(M)$. By taking the inner product of the above equation with any vector field $Z\in\mathfrak{X}(M)$, and using the fact that $\nabla_X h'$ is a symmetric operator and using again (3.5), we obtain (3.3).

On an almost contact metric manifold, a symmetric $(1,1)$-type tensor field T is said to be *cyclic parallel* if it satisfies

$$g((\nabla_X T)Y,Z)+g((\nabla_Y T)Z,X)+g((\nabla_Z T)X,Y)=0$$

for any vector fields X,Y,Z. Obviously, if T is parallel then the above relation holds trivially. Under the condition of the cyclic parallelism of h', we obtain:

Corollary 3.2. *Let M^{2n+1} be an almost co-Kähler manifold such that the tensor field h' is cyclic parallel. If in addition M^{2n+1} is locally symmetric, then it is locally isometric to the product of a real line or a circle and a locally symmetric almost Kähler manifold.*

Proof From relation (3.3), by the condition of the cyclic parallelism of h' and the symmetry of h' we obtain

$$(\nabla_Y h^2)X-3(\nabla_{h'Y}h')X=0 \tag{3.6}$$

for any vector fields X,Y. Replacing X by ξ in (3.6) and making use of (2.5) we obtain $h'^3=0$. Since h is symmetric, we conclude that $h=0$ and hence the proof is similar to that of Proposition 3.2. This completes the proof.

On a Riemannian manifold, a $(1,1)$-type tensor field T is said to be of *Codazzi type* if it satisfies

$$(\nabla_X T)Y = (\nabla_Y T)X$$

for any vector fields X, Y. Using the above condition, we obtain:

Corollary 3.3. *Let M^{2n+1} be an almost co-Kähler manifold with h' being of Codazzi type. If in addition M^{2n+1} is locally symmetric, then it is locally isometric to the product of a real line or a circle and a locally symmetric almost Kähler manifold.*

Proof Suppose that h' is of Codazzi type; then from relation (3.5) we have

$$S(X, \xi) = 0$$

for any vector fields X, where S denotes the Ricci tensor defined by $S(X,Y) = \mathrm{tr}\{\cdot \to R(\cdot, X)Y\}$. Moreover, it follows from equation (3.1) that $S(\xi,\xi) = \mathrm{tr}(l) = -\mathrm{tr}(h^2)$. Comparing this with the above relation gives that $\mathrm{tr}(h^2) = 0$. Since h is symmetric, we conclude that $h = 0$. The rest of the proof is similar to that of Proposition 3.2. This completes the proof.

On an almost co-Kähler manifold M^{2n+1}, we shall denote by $\mathcal{D}$ the integrable distribution defined by $\mathcal{D} = \ker \eta$. Let M^{2n} be an integral manifold of $\mathcal{D}$, and then it is easy to check that M^{2n} and the restriction of ϕ on it admit an almost Kähler structure. If the associated almost Kähler structure is integrable, Olszak in [18] called M^{2n+1} an *almost co-Kähler manifold with Kählerian leaves*. Such a notion is analogous to that of contact strongly pseudoconvex integrable CR manifolds (see [11]) and CR-integrable almost Kenmotsu manifolds (see [24]). Moreover, Dacko and Olszak in [7] proved that any conformally flat almost co-Kähler manifold of dimension > 3 with Kählerian leaves is locally flat.

It is easy to check that any co-Kähler manifold has Kählerian leaves. Clearly, any almost co-Kähler manifold of dimension 3 has Kählerian leaves. From Dacko and Olszak [8], we know that any almost co-Kähler (k,μ,ν)-manifold with $k < 0$ has Kählerian leaves. For more strictly almost co-Kähler manifolds with Kählerian leaves, we refer the reader to Olszak [18, Section 3].

The following result characterizes the integrability of the associated almost Kähler structure.

Lemma 3.2 ([18, Proposition])**.** *An almost co-Kähler manifold has Kählerian leaves if and only if*

$$(\nabla_X \phi)(Y) = g(X, hY)\xi - \eta(Y)hX \tag{3.7}$$

for any vector fields X, Y.

Obviously, by (3.7), it follows that an almost co-Kähler manifold is co-Kähler if and only if it has Kählerian leaves and ξ is Killing.

Lemma 3.3 ([19, Proposition 10])**.** *Let M^{2n+1} be an almost co-Kähler manifold with Kählerian leaves, and then we have*

$$Q\phi - \phi Q = l\phi - \phi l + (\eta \circ Q\phi) \otimes \xi - \eta \otimes (\phi Q \xi), \tag{3.8}$$

where Q denotes the Ricci operator.

Together with the above two lemmas, we obtain the following main result.

Theorem 3.1. *Let M^{2n+1} be a locally symmetric almost co-Kähler manifold of dimension greater than 3 with Kählerian leaves. Then either M^{2n+1} is a co-Kähler manifold with locally symmetric Kählerian leaves, or ξ is an eigenvector field of the Ricci operator with negative constant eigenvalue.*

Proof On any locally symmetric almost co-Kähler manifold, from equation (3.1) we get $l\phi = \phi l$. Using this in equation (3.7) gives that

$$Q\phi - \phi Q = (\eta \circ Q\phi) \otimes \xi - \eta \otimes (\phi Q\xi). \tag{3.9}$$

Taking the covariant derivative of $Q\phi X - \phi QX = \eta(Q\phi X)\xi - \eta(X)\phi Q\xi$ along any vector field and using (2.5), we have

$$\begin{aligned}&(\nabla_Y Q)\phi X + Q(\nabla_Y \phi)X + (Q\phi - \phi Q)(\nabla_Y X) - (\nabla_Y \phi)QX - \phi(\nabla_Y Q)X\\ =&g(h'Y, Q\phi X)\xi + g(\xi, (\nabla_Y Q)\phi X + Q(\nabla_Y \phi)X + Q\phi(\nabla_Y X))\xi\\ &+ \eta(Q\phi X)h'Y - g(X, h'Y)\phi Q\xi - \eta(\nabla_Y X)\phi Q\xi\\ &- \eta(X)((\nabla_Y \phi)Q\xi + \phi(\nabla_Y Q)\xi + \phi Qh'Y)\end{aligned}$$

for any $X, Y \in \mathfrak{X}(M)$. The assumption of local symmetry implies that $\nabla Q = 0$. Using this and applying Lemma 3.2 to the previous equation we have

$$\begin{aligned}&g(X, hY)Q\xi - \eta(X)QhY + (Q\phi - \phi Q)(\nabla_Y X) - g(hQX, Y)\xi + \eta(QX)hY\\ =&g(h'Y, Q\phi X)\xi + g(Q\xi, g(hX, Y)\xi - \eta(X)hY + \phi(\nabla_Y X))\xi + \eta(Q\phi X)h'Y\\ &- g(h'X, Y)\phi Q\xi - \eta(\nabla_Y X)\phi Q\xi - \eta(X)(g(hQ\xi, Y)\xi - S(\xi, \xi)hY + \phi Qh'Y)\end{aligned}$$

for any $X, Y \in \mathfrak{X}(M)$. Applying equation (3.9) in the previous relation we obtain

$$\begin{aligned}&S(\xi, \xi)g(hX, Y)\xi + \eta(Q\phi X)h'Y - g(h'X, Y)\phi Q\xi + \eta(X)S(\xi, \xi)hY\\ =&g(hX, Y)Q\xi + \eta(QX)hY\end{aligned} \tag{3.10}$$

for any $X, Y \in \mathfrak{X}(M)$. Taking the inner product of (3.10) with X we get

$$S(X, \xi)g(hX, Y)\eta(X) - g(\phi Q\xi, X)g(h'X, Y) = g(Q\xi, X)g(hX, Y) \tag{3.11}$$

for any $X, Y \in \mathfrak{X}(M)$. Putting $X = \phi Q\xi$ into equation (3.11) we obtain

$$\|\phi Q\xi\|^2 hQ\xi = 0.$$

Clearly, by (2.1), $\phi Q\xi = 0$ implies that ξ is an eigenvector field of the Ricci operator Q. Moreover, by (3.1) we know that the eigenvalue is $-\mathrm{tr}(h^2)$. Otherwise, if $hQ\xi = 0$, putting $X = \phi Q\xi$ into equation (3.10) we obtain

$$\|\phi Q\xi\|^2 h' = 0.$$

It follows from the above relation that either $Q\xi = -\mathrm{tr}(h^2)\xi$ or h is vanishing. With regard to $h = 0$, applying Lemma 3.2 and Dacko [6, Theorem 2] we obtain that M^{2n+1} is a co-Kähler manifold with locally symmetric Kählerian leaves. Next, let us consider $h \neq 0$ and $Q\xi = -\mathrm{tr}(h^2)\xi$. Taking the covariant derivative of the latter equation and making use of the second term of relation (2.5) we have

$$Qh'X = -X(\mathrm{tr}(h^2))\xi - \mathrm{tr}(h^2)h'X \tag{3.12}$$

for any $X \in \mathfrak{X}(M)$. The inner product of the above relation with ξ implies that $\mathrm{tr}(h^2)$ is a positive constant and hence M^{2n+1} is a strictly almost co-Kähler manifold. This completes the proof.

For a Riemannian manifold (M,g), we denote by (T^1M, g_S) the unit tangent sphere bundle admitting the well-known Sasakian metric. Then any unit vector field U defines a map $U:(M,g)\to(T^1M,g_S)$. Let M be closed and orientable; U is said to be *harmonic* if it is a critical point of the energy function defined by

$$E(U)=\frac{1}{2}\mathrm{vol}(M)+\frac{1}{2}\int_M \|U\|^2 dM.$$

Pak and Kim in [15] proved that the Reeb vector field of an almost co-Kähler manifold is harmonic if and only if it is an eigenvector field of the Ricci operator. Then we have the following corollary.

Corollary 3.4. *On a locally symmetric almost co-Kähler manifold with Kählerian leaves, the Reeb vector field ξ is harmonic.*

Proof If $h=0$, by equation (3.5) we obtain $R(X,Y)\xi=0$ and hence $Q\xi=0$. Otherwise, the proof follows from Theorem 3.1.

Corollary 3.5. *On a locally symmetric almost co-Kähler manifold with Kählerian leaves, the following relations hold:*

$$\mathrm{tr}(\nabla_X h^2)=(\mathrm{div}h')(h'X)=0$$

for any vector field X.

Proof First, let us consider a local orthonormal ϕ-frame $\{e_0=\xi, e_i, e_{n+i}=\phi e_i, 1\le i\le n\}$ on each point of M^{2n+1}. Since h' anticommutes with ϕ, then by using relation (3.7) we obtain

$$\begin{aligned}\mathrm{tr}(\nabla_X h') &= g((\nabla_X h')\xi,\xi)+\sum_{i=1}^n g((\nabla_X h')e_i,e_i)+\sum_{i=1}^n g((\nabla_X h')e_{n+i},e_{n+i})\\ &=\sum_{i=1}^n\Big(g((\nabla_X h')e_i,e_i)+g((\nabla_X h')\phi e_i,\phi e_i)\Big)=0\end{aligned}$$

for any vector field X. Replacing Y by e_j in (3.5) and taking the inner product of the resulting equation with e_j, and summing j over $0\le j\le 2n$, we obtain

$$S(X,\xi)=(\mathrm{div}h')(X)$$

for any vector field X. By Corollary 3.4 we know that ξ is an eigenvector field of Q. Then by (2.4) and the previous relation we obtain $(\mathrm{div}h')(h'X)=0$.

Using again $\mathrm{tr}(\nabla_X h')=0$ for any vector field X and replacing $X=Z=e_j$ in equation (3.3), and summing j over $0\le j\le 2n$, we obtain

$$\mathrm{tr}(\nabla_Y h^2)+2(\mathrm{div}h')(h'Y)=0$$

for any vector field Y. This completes the proof.

Corollary 3.6. *On a locally symmetric almost co-Kähler manifold with Kählerian leaves, the following relations hold:*

$$Q\phi=\phi Q,\ Qh'=h'Q,\ Qh=hQ. \tag{3.13}$$

Proof The first term follows from Corollary 3.4 and equation (3.9). The second and third terms hold naturally for the case $h = 0$. By Theorem 3.1, if $h \neq 0$ we get from (3.12) that

$$Qh' = -\mathrm{tr}(h^2)h'. \tag{3.14}$$

Obviously, ξ being an eigenvector of the Ricci operator implies that $Qh'\xi = h'Q\xi$ holds. Next, let X be an eigenvector field of h' orthogonal to ξ with a nonzero eigenvalue. Then we obtain directly from equation (3.14) that $QX = -\mathrm{tr}(h^2)X$ and hence $Qh'X = h'QX$. Finally, let $\{e_1, \phi e_1, \cdots, e_k, \phi e_k\}$ be all eigenvector fields of h' orthogonal to ξ with eigenvalues zero, where $1 \leq k \leq n-1$. Then, for any $1 \leq j \leq n-1$, from equation (3.14) we see that both Qe_j and $Q\phi e_j$ belong to $\mathrm{span}\{e_1, \phi e_1, \cdots, e_k, \phi e_k\}$. Then it follows that $Qh'e_j = h'Qe_j$ and $Qh'\phi e_j = h'Q\phi e_j$. This means that $Qh' = h'Q$. The third term of relation (3.13) follows directly from the previous two. This completes the proof.

Remark 3.1. *Goldberg and Yano [12] proved that an almost co-Kähler manifold is co-Kähler if and only if the curvature transformation R commutes with ϕ, i.e. $R(X,Y)\phi = \phi R(X,Y)$. This implies that $Q\phi = \phi Q$. From our main result, the previous relation holds even on a strictly almost co-Kähler manifold.*

4. ϕ-symmetric almost co-Kähler manifolds of dimension three

In this section, we investigate 3-dimensional almost co-Kähler manifolds under certain symmetry conditions. We first give the definition of ϕ*-symmetry* on an almost contact metric manifold (see also De et al. [9, 10] and [24]) as follows.

Definition 4.1. *An almost contact metric manifold M is said to be ϕ-symmetric if its Riemannian curvature tensor satisfies*

$$\phi^2(\nabla_W R)(X,Y)Z = 0 \tag{4.1}$$

for any vector fields X, Y, Z, W on M.

Obviously, a locally symmetric almost contact metric manifold (i.e. $\nabla R = 0$) is ϕ-symmetric. Usually, the converse of the above assertion is not necessarily true. In this section, we aim to prove that on any almost co-Kähler manifold of dimension 3 the conditions of local symmetry and ϕ-symmetry are equivalent.

Let $(M^3, \phi, \xi, \eta, g)$ be an almost co-Kähler manifold of dimension 3. Following Perrone [21], let $\mathcal{U}_1$ be the open subset of M^3 on which $h \neq 0$ and $\mathcal{U}_2$ the open subset defined by $\mathcal{U}_2 = \{p \in M^3 : h = 0 \text{ in a neighborhood of } p\}$. Therefore, $\mathcal{U}_1 \cup \mathcal{U}_2$ is an open dense subset of M^3. For any point $p \in \mathcal{U}_1 \cup \mathcal{U}_2$, we may find a local orthonormal basis $\{\xi, e_1, e_2 = \phi e_1\}$ of three distinct unit eigenvector fields of h in a certain neighborhood of p. On $\mathcal{U}_1$ we may assume that $he_1 = \lambda e_1$ and hence $he_2 = -\lambda e_2$, where λ is a positive function.

Lemma 4.1 ([21, Lemma 2.1]). *On $\mathcal{U}_1$ we have*

$$\nabla_\xi e_1 = ae_2,\ \nabla_\xi e_2 = -ae_1,\ \nabla_{e_1}\xi = -\lambda e_2,\ \nabla_{e_2}\xi = -\lambda e_1,$$

$$\nabla_{e_1}e_1 = \frac{1}{2\lambda}(e_2(\lambda)+\sigma(e_1))e_2,\ \nabla_{e_2}e_2 = \frac{1}{2\lambda}(e_1(\lambda)+\sigma(e_2))e_1,$$

$$\nabla_{e_2}e_1 = \lambda\xi - \frac{1}{2\lambda}(e_1(\lambda)+\sigma(e_2))e_2,\ \nabla_{e_1}e_2 = \lambda\xi - \frac{1}{2\lambda}(e_2(\lambda)+\sigma(e_1))e_1,$$

$$le_1 = -\xi(\lambda)e_2 + (\lambda^2+2a\lambda)e_1,\ le_2 = -\xi(\lambda)e_1 + (\lambda^2-2a\lambda)e_2,$$

$$\nabla_\xi h = \left(\frac{1}{\lambda}\xi(\lambda)\mathrm{id} + 2a\phi\right)h,$$

where a is a smooth function and σ is the 1-form defined by $\sigma(\cdot) = S(\cdot\,,\xi)$.

Using the above lemma, one obtains the Ricci operator Q expressed as follows (see also Perrone [21]):

$$Q = \alpha\mathrm{id} + \beta\eta\otimes\xi + \phi\nabla_\xi h - \sigma(\phi^2)\otimes\xi + \sigma(e_1)\eta\otimes e_1 + \sigma(e_2)\eta\otimes e_2, \tag{4.2}$$

where $\alpha = \frac{1}{2}(r+\mathrm{tr}(h^2))$ and $\beta = -\frac{1}{2}(r+3\mathrm{tr}(h^2))$ and r is the scalar curvature.

Regarding the classification of 3-dimensional locally symmetric almost co-Kähler manifolds, we have:

Lemma 4.2 ([21, Proposition 3.1]). *A locally symmetric almost co-Kähler manifold of dimension 3 is locally either a flat Euclidean space $\mathbb{R}^3$ or a product space $\mathbb{R}\times N^2(c)$, where $N^2(c)$ denotes a Kähler surface of constant curvature $c\neq 0$.*

In what follows, we shall show that the conclusion of Lemma 3.1 holds even under a condition weaker than local symmetry, i.e. ϕ-symmetry.

Lemma 4.3. *Let M^{2n+1} be a ϕ-symmetric almost co-Kähler manifold of dimension ≥ 3, and then we have $\nabla_\xi h = 0$.*

Proof Clearly, by relation (2.1), the notion of ϕ-symmetry implies that

$$(\nabla_W R)(X,Y)Z = g((\nabla_W R)(X,Y)Z,\xi)\xi$$

for any vector fields X,Y,Z,W. Replacing both Y and Z by ξ in the above relation and using the second term of relation (2.4) we have

$$(\nabla_W l)X = R(X,h'W)\xi + R(X,\xi)h'W + g(lX,h'W)\xi + g((\nabla_W l)X,\xi)\xi$$

for any vector fields X,W. In view of $l\xi = 0$, replacing W by ξ in the previous relation we get

$$\nabla_\xi l = 0.$$

According to Perrone [20, Lemma 3.1], on an almost co-Kähler manifold $\nabla_\xi h = 0$ and $\nabla_\xi l = 0$ are equivalent. This completes the proof.

By Definition 4.1, we easily obtain the following:

Lemma 4.4. *On a ϕ-symmetric almost co-Kähler manifold of dimension ≥ 3 we have*

$$(\nabla_W Q)X = g((\nabla_W Q)X, \xi)\xi \tag{4.3}$$

for any vector fields X, W.

Theorem 4.1. *Let M^3 be an almost co-Kähler manifold of dimension 3, and then M^3 is locally symmetric if and only if it is ϕ-symmetric.*

Proof If $h = 0$ holds identically, the proof follows from Lemma 4.2. Therefore, we now need only to consider the case that h is not identically zero. Without loss of generality we can assume that $\lambda > 0$. We remark that Lemma 4.1 is applicable in this case. If M^3 is ϕ-symmetric, by Lemma 4.3 we may use $\nabla_\xi h = 0$ in Lemma 4.1 and obtain

$$\xi(\lambda) = a = 0. \tag{4.4}$$

In view of the above relation and (2.1), it follows from equation (4.2) that

$$Q\xi = -\mathrm{tr}(h^2)\xi + \sigma(e_1)e_1 + \sigma(e_2)e_2,\ Qe_1 = \sigma(e_1)\xi + \alpha e_1,\ Qe_2 = \sigma(e_2)\xi + \alpha e_2.$$

Using relation (4.4) and applying Lemma 4.1, by a simple calculation we obtain

$$(\nabla_{e_1} Q)e_1 = \left(e_1(\sigma(e_1)) - \frac{1}{2\lambda}(e_2(\lambda) + \sigma(e_1))\sigma(e_2)\right)\xi + e_1(\alpha)e_1 - \lambda\sigma(e_1)e_2.$$

Comparing this relation with (4.3) and in view of the assumption $\lambda > 0$ we may get

$$e_1(\alpha) = \sigma(e_1) = 0. \tag{4.5}$$

Similarly, applying Lemma 4.1 and using equation (4.5) we obtain

$$(\nabla_{e_2} Q)e_2 = \left(e_2(\sigma(e_2)) - \frac{1}{2\lambda}(e_1(\lambda) + \sigma(e_2))\sigma(e_1)\right)\xi + e_2(\alpha)e_2 - \lambda\sigma(e_2)e_1.$$

Comparing this relation with relation (4.3) and in view of $\lambda > 0$ we get

$$e_2(\alpha) = \sigma(e_2) = 0. \tag{4.6}$$

In view of $a = 0$, using equations (4.5) and (4.6) and applying again Lemma 4.1 we have

$$(\nabla_\xi Q)e_1 = \xi(\alpha)e_1.$$

Comparing this with relation (4.3) we have that

$$\xi(\alpha) = 0. \tag{4.7}$$

Together with equations (4.5)–(4.7) we obtain that $2\alpha = r + \mathrm{tr}(h^2)$ is a constant and ξ is an eigenvector field of the Ricci operator Q. Applying again Lemma 4.1 and using $Q\xi = -\mathrm{tr}(h^2)\xi$ we get

$$(\nabla_{e_1} Q)\xi = -e_1(\mathrm{tr}(h^2))\xi + \frac{\lambda}{2}(r + 3\mathrm{tr}(h^2))e_2.$$

Comparing this with relation (4.3) and in view of $\lambda > 0$ we have

$$r + 3\mathrm{tr}(h^2) = 0. \tag{4.8}$$

Then we conclude that both the scalar curvature r and $\mathrm{tr}(h^2)$ are constants. In particular, applying Lemma 4.3 and using (4.4)–(4.8) in equation (4.2) gives that $Q = \frac{r}{3}\mathrm{id}$. It is well known that on any three-dimensional Riemannian manifold the curvature tensor R is given by

$$\begin{aligned} R(X,Y)Z =& g(Y,Z)QX - g(X,Z)QY + g(QY,Z)X \\ &- g(QX,Z)Y - \frac{r}{2}(g(Y,Z)X - g(X,Z)Y) \end{aligned}$$

for any vector fields X, Y, Z. Hence, an Einstein condition $Q = \frac{r}{3}\mathrm{id}$ on the above relation gives that M^3 is of constant sectional curvature. Moreover, following Olszak [18, Theorem 3] we know that any 3-dimensional almost co-Kähler manifold of constant sectional curvature is locally flat. Then using $l = 0$ in equation (2.7) gives that $h = 0$ and hence $\lambda = 0$, a contradiction. Then the proof follows from Lemma 4.2. The converse is trivial.

Applying Lemma 4.2 and Theorem 4.1 we have:

Theorem 4.2. *On any almost co-Kähler manifold M^3 of dimension 3, the following statements are equivalent:*
1) *M^3 is locally symmetric.*
2) *M^3 is ϕ-symmetric.*
3) *M^3 is locally isometric to a product $\mathbb{R} \times N^2(c)$, where $N^2(c)$ denotes a Kähler surface of constant curvature c.*

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