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On algebraic properties of Veronese bi-type ideals arising from graphs

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Abstract: Some algebraic properties of the ideals of Veronese bi-type arising from graphs with loops are studied. More precisely, the property of these ideals to be bi-polymatroidal is discussed. Moreover, we are able to determine the structure of the ideals of vertex covers for such generalized graph ideals.

Key words: Veronese bi-type ideals, graph ideals, ideals of vertex covers

1. Introduction

Let $R = K[X_1, \dots, X_n; Y_1, \dots, Y_m]$ be a polynomial ring in two sets of variables over a field K. In some recent papers [5, 9, 16], monomial ideals of R were introduced and their connection to bipartite complete graphs was studied. Here we consider a class of monomial ideals of R, the so-called Veronese bi-type ideals, which are an extension of the ideals of Veronese type in a polynomial ring with two sets of variables. More precisely, the ideals of Veronese bi-type are monomial ideals of R generated in the same degree: $L_{q,s} = \sum_{k+r=q} I_{k,s}J_{r,s}$, with $k, r \geqslant 1$, $s \leqslant q$, where $I_{k,s}$ is the Veronese type ideal generated on degree k by the set $\{X_1^{a_{i_1}} \cdots X_n^{a_{i_n}} \,|\, \sum_{j=1}^{n} a_{i_j} = k, \;\; 0 \leqslant a_{i_j} \leqslant s, \;\; s \in \{1, \dots, k\}\}$ and $J_{r,s}$ is the Veronese type ideal generated on degree r by the set $\{Y_1^{b_{i_1}} \cdots Y_m^{b_{i_m}} \,|\, \sum_{j=1}^{m} b_{i_j} = r, \;\; 0 \leqslant b_{i_j} \leqslant s, \;\; s \in \{1, \dots, r\}\}$ [10–13, 15]. When $s = 2$, the Veronese bi-type ideals arise from bipartite graphs with loops, the so-called strong quasi-bipartite graphs [10]. A graph $\mathcal{G}$ with loops is quasi-bipartite if its vertex set V can be partitioned into disjoint subsets V_1 and V_2, any edge joins a vertex of V_1 with a vertex of V_2, and some vertices in V have loops. A quasi-bipartite graph $\mathcal{G}$ is strong if it is a complete bipartite graph and all its vertices have loops. A strong quasi-bipartite graph on vertices $x_1, \dots, x_n, y_1, \dots, y_m$ is denoted by $\mathcal{K}'_{n,m}$.

In this paper some properties of the above class of monomial ideals are discussed. In particular the Veronese bi-type ideals give an example of generalization of polymatroidal ideals. The class of polymatroidal ideals is one of the rare classes of monomial ideals with the property that all powers of an ideal in this class have a linear resolution. In fact, the powers of a polymatroidal ideal are polymatroidal ideals and the polymatroidal ideals have linear quotients; hence, they are an important class of monomial ideals with linear resolution. In [12] the author introduced the generalized notion of discrete bi-polymatroid and the related notion of bi-polymatroidal ideal in R. Let I be a monomial ideal of R generated in a single degree and $G(I)$ be its

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unique set of minimal generators. I is called a bi-polymatroidal ideal if for any two monomials $u, v \in G(I)$ such that $deg_{X_i}(u) > deg_{X_i}(v)$ or $deg_{Y_k}(u) > deg_{Y_k}(v)$ there exist $j \in [n]$ with $deg_{X_j}(u) < deg_{X_j}(v)$ or $l \in [m]$ with $deg_{Y_l}(u) < deg_{Y_l}(v)$ such that $X_j(u/X_i) \in G(I)$ or $Y_l(u/Y_k) \in G(I)$. A special class of discrete bi-polymatroids, namely the discrete bi-polymatroid of Veronese type whose set of bases is $B_{q,2} = \{(a;b) \in \mathbb{Z}_+^{n+m} : \ |a| = k, |b| = r, \ 0 \leqslant a_i, b_j \leqslant 2\}$, was introduced. The corresponding bi-polymatroidal ideal of R is the ideal of Veronese bi-type $L_{q,2}$ associated to a strong quasi-bipartite graph $\mathcal{K}'_{n,m}$. We denote $L_{q,2}$ by $I_q(\mathcal{K}'_{n,m})$. In [1] and [4], algebraic properties of polymatroidal ideals were examined, and in particular it was proved that a monomial localization of a polymatroidal ideal is a polymatroidal ideal. In Section 2 we generalize such a result to a bi-polymatroidal ideal $I_q(\mathcal{K}'_{n,m})$; namely, we verify that a monomial localization of $I_q(\mathcal{K}'_{n,m})$ is again a bi-polymatroidal ideal.

In addition, we are interested in some problems linked to the minimal vertex covers introduced in [6–8], or, more precisely, algebraic aspects concerning a generalization of the notion of minimal vertex covers that holds for complete bipartite graphs $\mathcal{K}_{n,m}$ [9]. Let $I \subset R = K[X_1, \dots, X_n; Y_1, \dots, Y_m]$ be a monomial ideal. The ideal of (minimal) covers of I, denoted by I_c, is generated by all monomials $X_{i_1} \cdots X_{i_k} Y_{j_1} \cdots Y_{j_h}$ such that $(X_{i_1}, \dots, X_{i_k}, Y_{j_1}, \dots, Y_{j_h})$ is an associated (minimal) prime ideal of I. For a strong quasi-bipartite graph $\mathcal{K}'_{n,m}$, the ideal of vertex covers of a generalized graph ideal $I_q(\mathcal{K}'{}_{n,m})$ is denoted by $(I_q)_c(\mathcal{K}'{}_{n,m})$. In Section 3 the structure of $(I_q)_c(\mathcal{K}'_{n,m})$ is entirely described.

2. Preliminary notions

Let $R = K[X_1, \dots, X_n; Y_1, \dots, Y_m]$ be the polynomial ring over a field K in two sets of variables with $\deg(X_i) = \deg(Y_j) = 1$, for all $i = 1, \dots, n, \ j = 1, \dots, m$.

Let q, s, k, r be nonnegative integers with $s \leqslant q$ and $q = k + r, \ r, k \geqslant 1$. The *ideals of Veronese bi-type* of degree q are the monomial ideals of R

$$L_{q,s} = \sum_{k+r=q} I_{k,s} J_{r,s} \ ,$$

where $I_{k,s}$ is the ideal of Veronese type of degree k in the variables $X_1, \dots, X_n$ and $J_{r,s}$ is the ideal of Veronese type of degree r in the variables $Y_1, \dots, Y_m$.

$L_{q,s}$ is not trivial for $2 \leqslant q \leqslant s(n+m) - 1$.

Remark 2.1 In general $I_{k,s} \subseteq I_k$, where I_k is the *Veronese ideal* of degree k generated by all the monomials in the variables $X_1, \dots, X_n$ of degree k [17, 18].

One has $I_{k,s} = I_k$ for any $k \leqslant s$. If $s = 1$, $I_{k,1}$ is the square-free Veronese ideal of degree k generated by all the square-free monomials in the variables $X_1, \dots, X_n$ of degree k. Similar considerations hold for $J_{r,s} \subset K[Y_1, \dots, Y_m]$.

Example 2.2 Let $R = K[X_1, X_2; Y_1, Y_2]$ be a polynomial ring.

1) $L_{2,2} = I_{1,2}J_{1,2} = I_1J_1 = (X_1Y_1, X_1Y_2, X_2Y_1, X_2Y_2)$.

2) $L_{4,2} = I_{3,2}J_{1,2} + I_{1,2}J_{3,2} + I_{2,2}J_{2,2} = I_{3,2}J_1 + I_1J_{3,2} + I_2J_2 = (X_1^2X_2Y_1, X_1^2X_2Y_2, X_1X_2^2Y_1,$
$X_1X_2^2Y_2, X_1Y_1^2Y_2, X_2Y_1^2Y_2, X_1Y_1Y_2^2, X_2Y_1Y_2^2, X_1^2Y_1^2, X_1^2Y_1Y_2, X_1^2Y_2^2, X_2^2Y_1^2, X_2^2Y_2^2, X_2^2Y_1Y_2,$
$X_1X_2Y_1^2, X_1X_2Y_2^2, X_1X_2Y_1Y_2)$.

For $s=2$ and $q \geqslant 3$, the ideals $L_{q,s}$ are those associated to the walks of length $q-1$ of the strong quasi-bipartite graphs. Recall the following notions introduced in [10, 14].

Definition 2.3 A graph $\mathcal{G}$ with loops is said to be *quasi-bipartite* if its vertex set $V=\{x_1,\ldots,x_n,y_1,\ldots,y_m\}$ can be partitioned into the subsets $V_1=\{x_1,\ldots,x_n\}$ and $V_2=\{y_1,\ldots,y_m\}$, every edge joins a vertex of V_1 with a vertex of V_2, and some vertices in V have loops.

Definition 2.4 A quasi-bipartite graph $\mathcal{G}$ is called *strong* if it is a complete bipartite graph and all its vertices have loops.

A strong quasi-bipartite graph on vertices $x_1,\ldots,x_n,y_1,\ldots,y_m$ will be denoted by $\mathcal{K}'_{n,m}$.

Definition 2.5 Let $\mathcal{G}$ be a graph with loops in each of its n vertices. A *walk of length* q in $\mathcal{G}$ is an alternating sequence $w=\{v_{i_0},l_{i_1},v_{i_1},l_{i_2},\ldots,v_{i_{q-1}},l_{i_q},v_{i_q}\}$, where v_{i_0} or v_{i_g} is a vertex of $\mathcal{G}$ and $l_{i_g}=\{v_{i_{g-1}},v_{i_g}\}$, $g=1,\ldots,q$, is either the edge joining $v_{i_{g-1}}$ and v_{i_g} or a loop if $v_{i_{g-1}}=v_{i_g}$, $1\leqslant i_0\leqslant i_1\leqslant\ldots\leqslant i_q\leqslant n$.

Example 2.6 Let $\mathcal{K}'_{n,m}$ be a strong quasi-bipartite graph on vertices $x_1,\ldots,x_n,y_1,\ldots,y_m$. A walk of length 2 in $\mathcal{K}'_{n,m}$ is $\{x_i,l_i,x_i,l_{ij},y_j\}$ or $\{x_i,l_{ij},y_j,l_j,y_j\}$ where $l_i=\{x_i,x_i\}$, $l_j=\{y_j,y_j\}$ are loops, and l_{ij} is the edge joining x_i and y_j. Because $\mathcal{K}'_{n,m}$ is bipartite, any walk in it have not the edges $\{x_{i_h},x_{i_k}\}$, $i_h\neq i_k$, and $\{y_{j_h},y_{j_k}\}$, $j_h\neq j_k$.

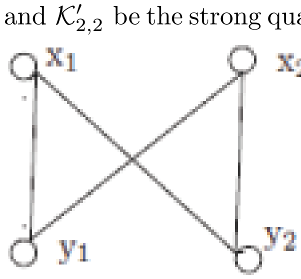
Let $\mathcal{G}$ be a graph with loops. The *generalized graph ideal* $I_q(\mathcal{G})$ associated to $\mathcal{G}$ is the ideal of the polynomial ring R generated by all the monomials of degree $q\geqslant 3$ corresponding to the walks of length $q-1$. Hence, the variables in each generator of $I_q(\mathcal{G})$ have at most degree 2.

For a strong quasi-bipartite graph $\mathcal{K}'_{n,m}$, the associated generalized graph ideals $I_q(\mathcal{K}'_{n,m})$ are therefore $L_{q,2}=\sum_{k+r=q} I_{k,2}J_{r,2}$, for $q\geqslant 3$ [10].

Remark 2.7 When $q=2$, the ideal $L_{q,2}$ does not describe the edge ideal $I(\mathcal{K}'_{n,m})=I_2(\mathcal{K}'_{n,m})$ of a strong quasi-bipartite graph. In fact, if we consider the strong quasi-bipartite graph $\mathcal{K}'_{2,2}$ on vertices x_1,x_2,y_1,y_2 then $I(\mathcal{K}'_{2,2})=(X_1Y_1,X_1Y_2,X_2Y_1,X_2Y_2,X_1^2,X_2^2,Y_1^2,Y_2^2)$, but $L_{2,2}=(X_1Y_1,X_1Y_2,X_2Y_1,X_2Y_2)$. Hence, $I(\mathcal{K}'_{2,2})\neq L_{2,2}$.

In the sequel, for strong quasi-bipartite graphs, we will often denote $L_{q,2}$, $q\geqslant 3$, by $I_q(\mathcal{K}'_{n,m})$.

Example 2.8 Let $R=K[X_1,X_2;Y_1,Y_2]$ and $\mathcal{K}'_{2,2}$ be the strong quasi-bipartite graph on vertices x_1,x_2,y_1,y_2:



$I_3(\mathcal{K}'_{2,2}) = I_1J_2 + I_2J_1 = (X_1Y_1Y_2, X_2Y_1Y_2, X_1Y_1^2, X_2Y_1^2, X_1Y_2^2, X_2Y_2^2, X_1X_2Y_1, X_1X_2Y_2, X_1^2Y_1, X_1^2Y_2, X_2^2Y_1, X_2^2Y_2)$.
$I_4(\mathcal{K}'_{2,2}) = I_{3,2}J_1 + I_1J_{3,2} + I_2J_2 = (X_1^2X_2Y_1, X_1^2X_2Y_2, X_1X_2^2Y_1, X_1X_2^2Y_2, X_1Y_1^2Y_2, X_2Y_1^2Y_2,$
$X_1Y_1Y_2^2, X_2Y_1Y_2^2, X_1^2Y_1^2, X_1^2Y_1Y_2, X_1^2Y_2^2, X_2^2Y_1^2, X_2^2Y_2^2, X_2^2Y_1Y_2, X_1X_2Y_1^2, X_1X_2Y_2^2, X_1X_2Y_1Y_2)$.

3. Monomial localization of bi-polymatroidal ideals

In [12] the ideals of Veronese bi-type $L_{q,s}$ are introduced as a class of bi-polymatroidal ideals. We recall some notions.
Let $n, m > 0$ be integers and $[n+m] = \{1, 2, \ldots, n+m\}$. Let $\mathbb{Z}_+$ be the set of nonnegative integers and $\mathbb{Z}_+^{n+m}$ be the set of the vectors $(a; b)$ with $a \in \mathbb{Z}_+^n$ and $b \in \mathbb{Z}_+^m$, i.e. $(a; b) = (a_1, \ldots, a_n; b_1, \ldots, b_m) \in \mathbb{Z}_+^{n+m}$ with each $a_i \geqslant 0,\ b_j \geqslant 0$.
The modulus of the vector $(a; b)$ is the number $|(a; b)| = |a| + |b| = \sum_{i=1}^n a_i + \sum_{j=1}^m b_j$. Let $(a; b)$ and $(c; d)$ be two vectors of $\mathbb{Z}_+^{n+m}$; one has $(a; b) \geqslant (c; d)$ if all components $(a_i - c_i; b_i - d_i)$ of the vector $(a - c; b - d)$ are nonnegative. If $(a; b) \geqslant (c; d)$ and $(a; b) \neq (c; d)$, one writes $(a; b) > (c; d)$. We say that $(c; d)$ is a subvector of $(a; b)$ if $(a; b) \geqslant (c; d)$. Moreover, we set $(a; b) \vee (c; d) = (max\{a_1, c_1\}, \ldots, max\{a_n, c_n\}; max\{b_1, d_1\}, \ldots, max\{b_m, d_m\})$. Hence, $(a; b) \leqslant (a; b) \vee (c; d)$ and $(c; d) \leqslant (a; b) \vee (c; d)$.

We generalize the combinatorial concept of discrete polymatroid introduced in [3].

Definition 3.1 A *discrete bi-polymatroid* on the set $[n+m]$ is a nonempty finite subset $P \subset \mathbb{Z}_+^{n+m}$ satisfying the following conditions:

1) P contains with each $(a; b) \in P$ all its integral subvectors; that is, if $(a; b) \in P$ and $(c; d) \in \mathbb{Z}_+^{n+m}$ with $(c; d) \leqslant (a; b)$, then $(c; d) \in P$;

2) if for all $(a; b), (c; d) \in P$ with $|(a; b)| < |(c; d)|$, then there is a vector $(u; v) \in P$ such that $(a; b) < (u; v) < (a; b) \vee (c; d)$.

A *base* of a discrete bi-polymatroid P is a vector $(a; b) \in P$ such that $(a; b) < (c; d)$ for no $(c; d) \in P$. The set of bases of P is denoted by $B(P)$.

Remark 3.2 Each base of a discrete bi-polymatroid P has the same modulus that is said to be the *rank* of P. In fact, if $(a; b)$ and $(c; d)$ are bases of P with $|(a; b)| < |(c; d)|$, then by Definition 3.1 there exists $(u; v) \in P$ with $(a; b) < (u; v) < (a; b) \vee (c; d)$. This contradicts the maximality of $(a; b) \in B(P)$.

A characterization of discrete bi-polymatroids in terms of their set of bases is the following:

Theorem 3.3 *(Bi-exchange property, [12])*
P is a discrete bi-polymatroid $\Leftrightarrow$ if $(a; b), (c; d) \in B(P)$ with $a_i > c_i$ or $b_k > d_k$, then there exist $j \in \{1, \ldots, n\}$, $l \in \{1, \ldots, m\}$ with $a_j < c_j$ or $b_l < d_l$ such that $(a; b) - (e_i; 0) + (e_j; 0) \in B(P)$ or $(a; b) - (0; e'_k) + (0; e'_l) \in B(P)$, where e_i, e'_j denote the standard basis vectors of $\mathbb{Z}_+^n, \mathbb{Z}_+^m$ respectively.

For a monomial ideal $I \subset R$ we denote by $G(I)$ its unique set of minimal generators.
A monomial ideal generated by all the monomials corresponding to the set $B(P)$ of bases of a discrete bi-polymatroid is called a *bi-polymatroidal ideal* and it is generated by all the monomials $\underline{X}^a\underline{Y}^b$ with $(a, b) \in B(P)$,

where $\underline{X}^a\underline{Y}^b$ stands for $X_1^{a_1}\cdots X_n^{a_n}Y_1^{b_1}\cdots Y_m^{b_m}$. In [12] the following definition of bi-polymatroidal ideal is given as a consequence of Theorem 3.3.

Definition 3.4 A monomial ideal I of R generated in a single degree is called *bi-polymatroidal* if for all monomials $u = X_1^{a_1}\cdots X_n^{a_n}Y_1^{b_1}\cdots Y_m^{b_m}$, $v = X_1^{c_1}\cdots X_n^{c_n}Y_1^{d_1}\cdots Y_m^{d_m}$ in $G(I)$ and for each i with $a_i > c_i$ or k with $b_k > d_k$ one has $j \in [n]$ with $a_j < c_j$ or $l \in [m]$ with $b_l < d_l$ such that $X_j(u/X_i) \in G(I)$ or $Y_l(u/Y_k) \in G(I)$.

It follows that if for any two monomials $u, v \in G(I)$ such that $deg_{X_i}(u) > deg_{X_i}(v)$ or $deg_{Y_k}(u) > deg_{Y_k}(v)$ there exist $j \in [n]$ with $deg_{X_j}(u) < deg_{X_j}(v)$ or $l \in [m]$ with $deg_{Y_l}(u) < deg_{Y_l}(v)$ such that $X_j(u/X_i) \in G(I)$ or $Y_l(u/Y_k) \in G(I)$.

Theorem 3.5 *([12])*
The bi-polymatroidal ideals of $R = K[X_1,\dots,X_n;Y_1,\dots,Y_m]$ have linear quotients.

Now we consider the class of bi-polymatroidal ideals $L_{q,2} = I_q(\mathcal{K}'_{n,m})$.
Let q, r, k be nonnegative integers such that $k + r = q$; then

$$B_{q,2} = \{(a;b) \in \mathbb{Z}_+^{n+m} : \ |a| = k, |b| = r, \ 0 \leqslant a_i, b_j \leqslant 2\}$$

is the set of bases of a discrete bi-polymatroid of rank q. In fact, let $(a,b),(c,d) \in B_{q,2}$ with $a_i > c_i$ or $b_k > d_k$, Then for some j,l such that $a_j < c_j$ or $b_l < d_l$, one has $(a;b) - (e_i;0) + (e_j;0) = (a_1,\dots,a_i - 1,\dots,a_j+1,\dots,a_n;b_1,\dots,b_m)$, $(a;b) - (0;e'_k) + (0;e'_l) = (a_1,\dots,a_n;b_1,\dots,b_k-1,\dots,b_l+1,\dots,b_m)$. Hence, $(a;b) - (e_i;0) + (e_j;0) \in B_{q,2}$, $(a;b) - (0;e'_k) + (0;e'_l) \in B_{q,2}$, by definition.
This one is a *discrete bi-polymatroid of Veronese bi-type*.

Example 3.6 Let $n = m = 2$. The set of bases of a discrete bi-polymatroid of rank 3 is
$B_{3,2} = \{(2,0;1,0),(2,0;0,1),(1,1;1,0),(1,1;0,1),(0,2;1,0),(0,2;0,1),(1,0;2,0),(1,0;0,2),$
$(1,0;1,1),(0,1;2,0),(0,1;0,2),(0,1;1,1)\}$.

The bi-polymatroidal ideal corresponding to the set $B_{q,2}$ is $L_{q,2} = I_q(\mathcal{K}'_{n,m})$, $q \geqslant 3$, where $\mathcal{K}'_{n,m}$ is a strong quasi-bipartite graph.
The monomial ideal $I_q(\mathcal{K}'_{n,m})$ satisfies Definition 3.4. In fact, let $u = X_1^{a_1}\cdots X_n^{a_n}Y_1^{b_1}\cdots Y_m^{b_m}$, $v = X_1^{c_1} \cdots X_n^{c_n}Y_1^{d_1}\cdots Y_m^{d_m} \in G(I_q(\mathcal{K}'_{n,m}))$, then $X_1^{a_1}\cdots X_n^{a_n} \in I_{k,2}$ and $Y_1^{b_1}\cdots Y_m^{b_m} \in J_{r,2}$ such that $\sum_{i=1}^n a_i + \sum_{j=1}^m b_j = q$, and $0 \leqslant a_i, b_j \leqslant 2$. Then it follows easily by the structure of $G(I_q(\mathcal{K}'_{n,m}))$ that for each i with $a_i > c_i$ or k with $b_k > d_k$ one has j with $a_j < c_j$ or l with $b_l < d_l$ such that $X_j(u/X_i) \in G(I_q(\mathcal{K}'_{n,m}))$ or $Y_l(u/Y_k) \in G(I_q(\mathcal{K}'_{n,m}))$.

Let us study some permanence properties of the bi-polymatroidal ideal $I_q(\mathcal{K}'_{n,m})$.

Theorem 3.7 *Let $I_q(\mathcal{K}'_{n,m}) \subset R = K[X_1,\dots,X_n;Y_1,\dots,Y_m]$ be the bi-polymatroidal ideal associated to strong quasi-bipartite graphs. For all monomials u of R one has:*

1. *$I_q(\mathcal{K}'_{n,m}) : u$ is a bi-polymatroidal ideal.*

2. $I_q(\mathcal{K}'_{n,m}) : u$ *has linear quotient.*

3. $I_q(\mathcal{K}'_{n,m}) : u$ *has a linear resolution.*

Proof Let $I_q(\mathcal{K}'_{n,m}) = \sum_{k+r=q} I_{k,2}J_{r,2},\ q \geqslant 3$.

1. It is enough to prove that for all variables X_i (resp. Y_k), $I_q(\mathcal{K}'_{n,m}) : X_i$ (resp. $I_q(\mathcal{K}'_{n,m}) : Y_k$) is a bi-polymatroidal ideal of R. Set $J = I_q(\mathcal{K}'_{n,m}) : X_i$. Let $u, v \in G(J)$, then $X_iu, X_iv \in X_iJ \subseteq I_q(\mathcal{K}'_{n,m})$. If $deg_{X_i}(u) = deg_{X_i}(v)$, then X_iu and X_iv satisfy the bi-exchange property of Definition 3.4 being $I_q(\mathcal{K}'_{n,m})$ bi-polymatroidal. Hence, this property is satisfied by u and v. If $deg_{X_i}(u) > deg_{X_i}(v)$, then X_i divides u. For a variable X_t with $deg_{X_t}(u) > deg_{X_t}(v)$, we prove that there exists a variable X_j with $deg_{X_j}(v) > deg_{X_j}(u)$, such that $X_j(u/X_t) \in G(J)$. Being $deg_{X_t}(X_iu) > deg_{X_t}(X_iv)$ and $I_q(\mathcal{K}'_{n,m})$ a bi-polymatroidal ideal, it follows that there exists a variable X_j with $deg_{X_j}(X_iu) < deg_{X_j}(X_iv)$ such that $X_j(X_iu/X_t) \in G(I_q(\mathcal{K}'_{n,m}))$. Then we have that $X_j(u/X_t) \in J$. The same result holds for $deg_{X_i}(u) < deg_{X_i}(v)$.
   In a similar way, if we set $J = I_q(\mathcal{K}'_{n,m}) : Y_k$ the thesis follows with the same argument. We conclude that $I_q(\mathcal{K}'_{n,m}) : u$ is bi-polymatroidal.

2. Any bi-polymatroidal ideal has linear quotients (Theorem 3.5), so for $I_q(\mathcal{K}'_{n,m}) : u$.

3. It follows from the general fact that ideals generated in a single degree with linear quotients have a linear resolution ([2], Lemma 4.1). □

Example 3.8 Let $R = K[X_1, X_2; Y_1, Y_2]$ and $L_{3,2} = (X_1Y_1Y_2, X_2Y_1Y_2, X_1Y_1^2, X_2Y_1^2, X_1Y_2^2,\ X_2Y_2^2, X_1X_2Y_1, X_1X_2Y_2, X_1^2Y_1, X_1^2Y_2, X_2^2Y_1, X_2^2Y_2)$ such that $L_{3,2} = I_3(\mathcal{K}'_{2,2})$, where $\mathcal{K}'_{2,2}$ is the strong quasi-bipartite graph on vertices x_1, x_2, y_1, y_2. Set $u = X_1$. One has:

$$I_3(\mathcal{K}'_{2,2}) : u = (Y_1Y_2, X_2Y_1Y_2, Y_1^2, X_2Y_1^2, Y_2^2, X_2Y_2^2, X_2Y_1, X_2Y_2, X_1Y_1, X_1Y_2, X_2^2Y_1, X_2^2Y_2).$$

Hence:

$$I_3(\mathcal{K}'_{2,2}) : u = (Y_1Y_2, Y_1^2, Y_2^2, X_2Y_1, X_2Y_2, X_1Y_1, X_1Y_2),$$

i.e. a bi-polymatroidal ideal. In fact, $I_3(\mathcal{K}'_{2,2}) : u$ easily satisfies the bi-exchange property.

We denote the set of monomial prime ideals of R by $\mathcal{P}(R)$. Let $\wp \in \mathcal{P}(R)$ be a monomial prime ideal. The *monomial localization* of a monomial ideal I with respect to $\wp$ is the monomial ideal $I(\wp)$, which is obtained from I by substituting the variables that do not belong to $\wp$ by 1.
More precisely, setting $\wp = (X_{i_1}, \dots, X_{i_r}, Y_{j_1}, \dots, Y_{j_t})$, we denote by $I(\wp)$ the monomial ideal in the polynomial ring $K[X_{i_1}, \dots, X_{i_r}, Y_{j_1}, \dots, Y_{j_t}]$ where $I(\wp) = I_C$ with $C = [n+m] \setminus \{i_1, \dots, i_r, j_1, \dots, j_t\}$. If I is a square-free monomial ideal, then $I(\wp) = I : X_C$ where $X_C = \prod_{i \in C} X_i$.

Example 3.9 Let $R = K[X_1, X_2; Y_1, Y_2]$. Set $\wp = (X_2, Y_1, Y_2)$. Compute the monomial localization of $I_3(\mathcal{K}'_{2,2})$ with respect to $\wp$, i.e. $(I_3(\mathcal{K}'_{2,2}))(\wp) = (I_3(\mathcal{K}'_{2,2}))_C$ where $C = \{1\}$:

$$(I_3(\mathcal{K}'_{2,2}))(\wp) = (Y_1Y_2, X_2Y_1Y_2, Y_1^2, X_2Y_1^2, Y_2^2, X_2Y_2^2, X_2Y_1, X_2Y_2, Y_1, Y_2, X_2^2Y_1, X_2^2Y_2) = (Y_1, Y_2),$$

obtained from $I(\mathcal{K}'_{2,2})$ by substituting the variable X_1 that does not belong to $\wp$ by 1.

Remark 3.10 For square-free bi-polymatroidal ideals the monomial localization is a bi-polymatroidal ideal by Theorem 3.7 (being $I(\wp) = I : u$, where $u = \prod_{i \in C} X_i$).

For the non-square-free bi-polymatroidal ideal $I_q(\mathcal{K}'_{n,m})$ we give the following:

Theorem 3.11 *Let $I_q(\mathcal{K}'_{n,m}) \subset R$ be a bi-polymatroidal ideal. For all $\wp \in \mathcal{P}(R)$ one has:*

1. *$(I_q(\mathcal{K}'_{n,m}))(\wp)$ is a bi-polymatroidal ideal.*
2. *$(I_q(\mathcal{K}'_{n,m}))(\wp)$ has linear quotients.*
3. *$(I_q(\mathcal{K}'_{n,m}))(\wp)$ has a linear resolution.*

Proof

1. Let $I_q(\mathcal{K}'_{n,m}) \subset R = K[X_1, \ldots, X_n; Y_1, \ldots, Y_m]$ be the bi-polymatroidal ideal generated in degree q. Set, for any $i \in [n+m]$, $(I_q(\mathcal{K}'_{n,m}))_{\{i\}}$ the monomial ideal obtained from $I_q(\mathcal{K}'_{n,m})$ by substituting the variable of index i by 1. Let $\underline{X}^a \underline{Y}^b \in I_q(\mathcal{K}'_{n,m})$, where $\underline{X}^a \underline{Y}^b = X_1^{a_1} \cdots X_n^{a_n} Y_1^{b_1} \cdots Y_m^{b_m}$ and $(a; b) = (a_1, ..., a_n; b_1, ..., b_m) \in \mathbb{Z}_+^{n+m}$. Then $I_q(\mathcal{K}'_{n,m}) = (\{\underline{X}^a \underline{Y}^b \mid (a; b) \in B_{q,2}\})$. Suppose $i \in [n]$; this means that $(I_q(\mathcal{K}'_{n,m}))_{\{i\}}$ is the monomial ideal obtained from $I_q(\mathcal{K}'_{n,m})$ by substituting the variable X_i by 1. It follows that $(I_q(\mathcal{K}'_{n,m}))_{\{i\}} = (\{\underline{X}^{a'} \underline{Y}^b \mid (a; b) \in B_{q,2}\})$, where for all $(a; b) \in B_{q,2}$ we put $\underline{X}^{a'} \underline{Y}^b = \underline{X}^a \underline{Y}^b / X_i^{a_i}$.
We prove that $(I_q(\mathcal{K}'_{n,m}))_{\{i\}}$ is a bi-polymatroidal ideal.
The first step is to show that $(I_q(\mathcal{K}'_{n,m}))_{\{i\}}$ is generated in a single degree. If $h_i = max\{a_i \mid (a; b) \in B_{q,2}\}$, then we prove that $G((I_q(\mathcal{K}'_{n,m}))_{\{i\}}) = \{\underline{X}^a \underline{Y}^b / X_i^{h_i} \mid (a; b) \in B_{q,2},\ a_i = h_i\}$. Let $(c; d) \in B_{q,2}$ then $c_i < h_i$. Now we show that there exists $(u, v) \in B_{q,2}$ with $u_i = h_i$ and such that $\underline{X}^{u'} \underline{Y}^v$ divides $\underline{X}^{c'} \underline{Y}^d$. We proceed by induction on $h_i - c_i$. If $h_i - c_i = 0$, it is clear. Suppose, now, that $h_i - c_i > 0$, i.e. $c_i < h_i$. Let $(a, b) \in B_{q,2}$ with $a_i = h_i$. Applying Theorem 3.3 there exists an integer $j \in [n]$ with $a_j < c_j$ such that $(a; b) - (e_i; 0) + (e_j; 0) \in B_{q,2}$, and by symmetry $(c, d) - (e_j; 0) + (e_i; 0) \in B_{q,2}$. Set $(z, t) = (c, d) - (e_j; 0) + (e_i; 0)$. Hence, one has that $\underline{X}^{z'} \underline{Y}^t$ divides $\underline{X}^{c'} \underline{Y}^d$. Being $h_i - z_i < h_i - c_i$, by induction hypothesis there exists $(u, v) \in B_{q,2}$ with $u_i = h_i$ and such that $\underline{X}^{u'} \underline{Y}^v$ divides $\underline{X}^{z'} \underline{Y}^t$. It follows that $\underline{X}^{u'} \underline{Y}^v$ divides $\underline{X}^{c'} \underline{Y}^d$; this yields the desired conclusion.
The second step is to prove that the set $B'_{q,2} = \{(a', b) | \underline{X}^{a'} \underline{Y}^b \in G((I_q(\mathcal{K}'_{n,m}))_{\{i\}})\}$ is the set of bases of a discrete bi-polymatroid P' of rank $q - h_i$ on the set $[n+m] \backslash \{i\}$.
For all $(a'; b) \in B'_{q,2}$ one has $|(a', b)| = q - h_i$. Now we verify the bi-exchange property: let $(a', b), (c', d) \in B'_{q,2}$ with $a'_k > c'_k$, then $k \neq i$. By hypothesis for $(a, b), (c, d) \in B_{q,2}$: let $a_k = a'_k > c'_k = c_k$, then there exists $l \in [n]$ such that $a_l < c_l$ and $(a; b) - (e_k; 0) + (e_l; 0) \in B_{q,2}$. Set $(x, y) = (a; b) - (e_k; 0) + (e_l; 0)$; since $a_i = c_i = h_i$, it follows that $l \neq i$ and $x_i = a_i$. Hence, it follows that $(x', y) \in B'_{q,2}$, i.e. $(a'; b) - (e_k; 0) + (e_l; 0) \in B'_{q,2}$ as required. Hence, $B'_{q,2}$ is a discrete bi-polymatroid on the set $[n+m] \backslash \{i\}$. It follows that $(I_q(\mathcal{K}'_{n,m}))_{\{i\}}$ is a bi-polymatroidal ideal.
If we suppose $i \in [m]$, the thesis follows by the same argument. In all cases one has that $(I_q(\mathcal{K}'_{n,m}))_{\{i\}}$ is bi-polymatroidal ideal. As a consequence, if $\wp = (X_{i_1}, \ldots, X_{i_r}, Y_{j_1}, \ldots,\ Y_{j_r})$ is a prime ideal of R,

the monomial localization $(I_q(\mathcal{K}'_{n,m}))(\wp) = I_q(\mathcal{K}'_{n,m})_C$, with $C = [n+m]\setminus\{i_1,\dots,i_r,j_1,\dots,j_r\}$, is a bi-polymatroidal ideal.

2. It follows from Theorem 3.5.

3. It follows from the general fact that ideals generated in the same degree with linear quotients have a linear resolution ([2], Lemma 4.1).

□

Example 3.12 Let $R = K[X_1,X_2;Y_1,Y_2]$. Consider the ideal $(I_3(\mathcal{K}'_{2,2})) = (X_1Y_1Y_2, X_2Y_1Y_2,\ X_1Y_1^2, X_2Y_1^2, X_1Y_2^2, X_2Y_2^2, X_1X_2Y_1, X_1X_2Y_2, X_1^2Y_1, X_1^2Y_2, X_2^2Y_1, X_2^2Y_2)$ and $\wp = (X_2,Y_1,Y_2)$, $C=\{1\}$. One has $(I_3(\mathcal{K}'_{2,2}))(\wp) = I_3(\mathcal{K}'_{2,2})_C = (Y_1,Y_2)$, i.e. a bi-polymatroidal ideal of R.

4. Ideals of vertex covers for the generalized graph ideals of a strong quasi-bipartite graph

Definition 4.1 Let $\mathcal{G}$ be any graph with loops on vertex set $[n] = \{v_1,\dots,v_n\}$. A subset C of $[n]$ is said to be a *generalized vertex cover* of $\mathcal{G}$ if every walk of $\mathcal{G}$ is incident with one vertex in C. C is said *minimal* if no proper subset of C is a generalized vertex cover of $\mathcal{G}$.

Remark 4.2 There exists a one-to-one correspondence between generalized vertex covers of $\mathcal{G}$ and prime ideals of the generalized graph ideal $I_q(\mathcal{G})$ that preserves the minimality. In fact, $\wp$ is a minimal prime ideal of $I_q(\mathcal{G})$ if and only if $\wp = (C)$, for some minimal generalized vertex cover C of $\mathcal{G}$. Thus, $I_q(\mathcal{G})$ has primary decomposition $(C_1)\cap\dots\cap(C_r)$, where $C_1,\dots,C_r$ are the minimal generalized vertex covers of $\mathcal{G}$.

An algebraic aspect linked to the generalized vertex covers of $\mathcal{G}$ is the notion of the ideal of vertex covers for the generalized graph ideals associated to $\mathcal{G}$.

Definition 4.3 The *ideal of vertex covers* for the generalized graph ideal $I_q(\mathcal{G})$, denoted by $(I_q)_c(\mathcal{G})$, is the ideal of R generated by all monomials $X_{i_1}\cdots X_{i_r}$ such that $(X_{i_1},\dots,X_{i_r})$ is an associated prime ideal of $I_q(\mathcal{G})$.

Hence, $(I_q)_c(\mathcal{G}) = (\{X_{i_1}\cdots X_{i_r} \mid \{v_{i_1},\dots,v_{i_r}\}$ is a generalized vertex cover of $\mathcal{G}\})$, and the minimal generators of $(I_q)_c(\mathcal{G})$ correspond to the minimal generalized vertex covers.

The following generalizes the characterization of the ideals of vertex covers given in [18].

Property 4.1 $(I_q)_c(\mathcal{G}) = \Big(\bigcap_{\{v_{i_1},l_{i_2},\dots,v_{i_q}\}\text{ walk in }\mathcal{G}} (X_{i_1},\dots,X_{i_q})\Big)$, $\forall\, q \geqslant 2$.

From now on, let $R = K[X_1,\dots,X_n;Y_1,\dots,Y_m]$, $n \geqslant m$, be the polynomial ring over a field K in two sets of variables and $\mathcal{K}'_{n,m}$ be a strong quasi-bipartite graph on vertex set $[n+m] = \{x_1,\dots,x_n,y_1,\dots,y_m\}$.

Let $I_q(\mathcal{K}'_{n,m})$ denote the generalized graph ideals of $\mathcal{K}'_{n,m}$.

Recall that they are the ideals of Veronese bi-type of degree q, $\sum_{k+r=q} I_{k,2}J_{r,2}$, where $I_{k,2}$ is the ideal of Veronese type of degree k in $X_1,\dots,X_n$, and $J_{r,2}$ is the one of degree r in $Y_1,\dots,Y_m$.

For instance, $I_{3,2}$ has generators $X_1^2X_2,\dots,X_{n-1}X_n^2;X_1X_2X_3,\dots,X_{n-2}X_{n-1}X_n$,
$I_{4,2}=(X_1^2X_2^2,\dots,X_{n-1}^2X_n^2;X_1^2X_2X_3,\dots,X_{n-2}X_{n-1}X_n^2;X_1\cdots X_4,\dots,X_{n-3}\cdots X_n),\dots;$
$I_{2h,2}$ has $\binom{n}{h}+\binom{n}{h-1}\binom{n-h+1}{2}+\binom{n}{h-2}\binom{n-h+2}{4}+\cdots+\binom{n}{1}\binom{n-1}{2(h-1)}+\binom{n}{2h}$ generators of even degree $2h\leqslant 2n$, which are monomials with h, or $h+1,\dots$, or $2h$, of the n variables;
$I_{2h+1,2}$ has $\binom{n}{h}\binom{n-h}{1}+\binom{n}{h-1}\binom{n-h+1}{3}+\cdots+\binom{n}{1}\binom{n-1}{2h-1}+\binom{n}{2h+1}$ generators of odd degree $2h+1\leqslant 2n$, which are monomials with $h+1$, or $h+2,\dots$, or $2h+1$, of the n variables.

The generators of $I_q(\mathcal{K}'_{n,m})$ correspond to all the walks of length $q-1$ in $\mathcal{K}'_{n,m}$, where $3\leqslant q\leqslant 2(n+m)-1$. They are monomials of degree q in the variables X_i,X_i^2,Y_j,Y_j^2. For instance, the generalized graph ideals of $\mathcal{K}'_{5,3}$ are the following ones:

$I_3(\mathcal{K}'_{5,3})=I_1J_2+I_2J_1$,
$I_4(\mathcal{K}'_{5,3})=I_1J_{3,2}+I_2J_2+I_{3,2}J_1$,
$I_5(\mathcal{K}'_{5,3})=I_1J_{4,2}+I_2J_{3,2}+I_{3,2}J_2+I_{4,2}J_1$,
$I_6(\mathcal{K}'_{5,3})=I_1J_{5,2}+I_2J_{4,2}+I_{3,2}J_{3,2}+I_{4,2}J_2+I_{5,2}J_1$,
$I_7(\mathcal{K}'_{5,3})=I_1J_{6,2}+I_2J_{5,2}+I_{3,2}J_{4,2}+I_{4,2}J_{3,2}+I_{5,2}J_2+I_{6,2}J_1$,
$I_8(\mathcal{K}'_{5,3})=I_2J_{6,2}+I_{3,2}J_{5,2}+I_{4,2}J_{4,2}+I_{5,2}J_{3,2}+I_{6,2}J_2+I_{7,2}J_1$,
$I_9(\mathcal{K}'_{5,3})=I_{3,2}J_{6,2}+I_{4,2}J_{5,2}+I_{5,2}J_{4,2}+I_{6,2}J_{3,2}+I_{7,2}J_2+I_{8,2}J_1$,
$I_{10}(\mathcal{K}'_{5,3})=I_{4,2}J_{6,2}+I_{5,2}J_{5,2}+I_{6,2}J_{4,2}+I_{7,2}J_{3,2}+I_{8,2}J_2+I_{9,2}J_1$,
$I_{11}(\mathcal{K}'_{5,3})=I_{5,2}J_{6,2}+I_{6,2}J_{5,2}+I_{7,2}J_{4,2}+I_{8,2}J_{3,2}+I_{9,2}J_2+I_{10,2}J_1$,
$I_{12}(\mathcal{K}'_{5,3})=I_{6,2}J_{6,2}+I_{7,2}J_{5,2}+I_{8,2}J_{4,2}+I_{9,2}J_{3,2}+I_{10,2}J_2$,
$I_{13}(\mathcal{K}'_{5,3})=I_{7,2}J_{6,2}+I_{8,2}J_{5,2}+I_{9,2}J_{4,2}+I_{10,2}J_{3,2}$,
$I_{14}(\mathcal{K}'_{5,3})=I_{8,2}J_{6,2}+I_{9,2}J_{5,2}+I_{10,2}J_{4,2}$,
$I_{15}(\mathcal{K}'_{5,3})=I_{9,2}J_{6,2}+I_{10,2}J_{5,2}$.

Let $(I_q)_c(\mathcal{K}'_{n,m})$ indicate the ideal of vertex covers of $I_q(\mathcal{K}'_{n,m})$, $\forall\, q$.

The following result establishes the structure of $(I_q)_c(\mathcal{K}'_{n,m})$.

Theorem 4.4 *Let $(I_q)_c(\mathcal{K}'_{n,m})$ be the ideals of vertex covers for the generalized graph ideals associated to the strong quasi-bipartite graph $\mathcal{K}'_{n,m}$, $n\geqslant m$.*
When $q=2(n+m)-2\ell+1$ or $q=2(n+m)-2(\ell-1)$, for any $\ell=1,\dots,n+m-1$, $(I_q)_c(\mathcal{K}'_{n,m})$ are structured as follows:

if $\ell=1,\dots,m-1$, there are
$\binom{n}{\ell}$ *generators of degree ℓ, $X_{i_1}\cdots X_{i_\ell}$,*
$\binom{m}{\ell}$ *generators of degree ℓ, $Y_{j_1}\cdots Y_{j_\ell}$,*
$\sum_{h=1}^{\ell-1}\binom{n}{\ell-h}\binom{m}{h}$ *generators of degree ℓ, $X_{i_1}\cdots X_{i_{\ell-h}}Y_{j_1}\cdots Y_{j_h}$,*
$\sum_{k=1}^{\ell-2}\binom{n}{\ell-k}\binom{m}{k+1}$ *generators of degree $\ell+1$, $X_{i_1}\cdots X_{i_{\ell-k}}Y_{j_1}\cdots Y_{j_{k+1}}$;*

if $\ell=m,\dots,n+m-1$, there are
$$\begin{cases}\binom{n}{\ell}\ \text{generators of degree}\ \ell,\ X_{i_1}\cdots X_{i_\ell}, & \text{for}\ \ell<n,\\ 1\ \text{generator of degree}\ n,\ X_1\cdots X_n, & \text{otherwise,}\end{cases}$$

1 generator of degree m, $Y_1 \cdots Y_m$,
$\sum_{h=1}^{m-1} \binom{n}{\ell-h}\binom{m}{h}$ *generators of degree* ℓ, $X_{i_1} \cdots X_{i_{\ell-h}} Y_{j_1} \cdots Y_{j_h}$,
$\sum_{k=1}^{m-2} \binom{n}{\ell-k}\binom{m}{k+1}$ *generators of degree* $\ell+1$, $X_{i_1} \cdots X_{i_{\ell-k}} Y_{j_1} \cdots Y_{j_{k+1}}$;

by assuming that $\binom{a}{b}$ *is not zero only when* $0 \neq b \lneqq a$.

Proof The generators of $(I_q)_c(\mathcal{K}'_{n,m})$ are square-free monomials of R that derive from the associated minimal prime ideals of the generalized graph ideals $I_q(\mathcal{K}'_{n,m})$, when $q = 3, \ldots, 2(n+m)-1$. By symmetry on X_i and Y_j, we do not lack generality if we suppose $n \geqslant m$. It can be seen that q is linked to the degree of some monomials in $(I_q)_c(\mathcal{K}'_{n,m})$; in fact, the pairs of ideals of vertex covers in which $q = 2(n+m)-2\ell+1$ and $q = 2(n+m)-2(\ell-1)$, for any $\ell = 1, \ldots, n+m-1$, have the generators with both the sets of variables X_i, Y_j only of degrees ℓ and $\ell+1$. The structure of the ideals $I_q(\mathcal{K}'_{n,m})$ brings to the following composition of the ideals of vertex covers for them.

– $(I_{2(n+m)-1})_c(\mathcal{K}'_{n,m})$ and $(I_{2(n+m)})_c(\mathcal{K}'_{n,m})$

have $\binom{n}{1}$ generators $X_1, \ldots, X_n$, and $\binom{m}{1}$ generators $Y_1, \ldots, Y_m$.

– $(I_{2(n+m)-2\cdot2+1})_c(\mathcal{K}'_{n,m})$ and $I_{2(n+m)-2})_c(\mathcal{K}'_{n,m})$

have $\binom{n}{2}$ generators $X_1X_2, \ldots, X_{n-1}X_n$, $\binom{m}{2}$ generators $Y_1Y_2, \ldots, Y_{m-1}Y_m$, and
$\binom{n}{1}\binom{m}{1}$ generators $X_1Y_1, \ldots, X_nY_1, X_1Y_2, \ldots, X_nY_2, \ldots, X_1Y_m, \ldots, X_nY_m$.

– $(I_{2(n+m)-2\cdot3+1})_c(\mathcal{K}'_{n,m})$ and $(I_{2(n+m)-4})_c(\mathcal{K}'_{n,m})$

have $\binom{n}{3}$ generators $X_1X_2X_3, \ldots, X_{n-2}X_{n-1}X_n$,
$\binom{m}{3}$ generators $Y_1Y_2Y_3, \ldots, Y_{m-2}Y_{m-1}Y_m$,
$\binom{n}{2}\binom{m}{1}$ generators $X_1X_2Y_1, \ldots, X_{n-1}X_nY_1, X_1X_2Y_2, \ldots, X_1X_2Y_m, \ldots, X_{n-1}X_nY_m$,
$\binom{n}{1}\binom{m}{2}$ generators $X_1Y_1Y_2, \ldots, X_nY_1Y_2, X_1Y_1Y_3, \ldots, X_1Y_{m-1}Y_m, \ldots, X_nY_{m-1}Y_m$, and
$\binom{n}{2}\binom{m}{2}$ generators $X_1X_2Y_1Y_2, \ldots, X_{n-1}X_nY_1Y_2, \ldots, X_1X_2Y_{m-1}Y_m, \ldots, X_{n-1}X_nY_{m-1}Y_m$.

– $(I_{2(n+m)-2\cdot4+1})_c(\mathcal{K}'_{n,m})$ and $(I_{2(n+m)-6})_c(\mathcal{K}'_{n,m})$

have $\binom{n}{4}$ generators $X_1X_2X_3X_4, \ldots, X_{n-3}X_{n-2}X_{n-1}X_n$,
$\binom{m}{4}$ generators $Y_1Y_2Y_3Y_4, \ldots, Y_{m-3}Y_{m-2}Y_{m-1}Y_m$,
$\binom{n}{3}\binom{m}{1}$ generators $X_1X_2X_3Y_1, \ldots, X_{n-2}X_{n-1}X_nY_1, \ldots, X_1X_2X_3Y_m, \ldots, X_{n-2}X_{n-1}X_nY_m$,
$\binom{n}{2}\binom{m}{2}$ generators $X_1X_2Y_1Y_2, \ldots, X_{n-1}X_nY_1Y_2, \ldots, X_1X_2Y_{m-1}Y_m, \ldots, X_{n-1}X_nY_{m-1}Y_m$,
$\binom{n}{1}\binom{m}{3}$ generators $X_1Y_1Y_2Y_3, \ldots, X_nY_1Y_2Y_3, \ldots, X_1Y_{m-2}Y_{m-1}Y_m, \ldots, X_nY_{m-2}Y_{m-1}Y_m$,
$\binom{n}{3}\binom{m}{2}$ generators $X_1X_2X_3Y_1Y_2, \ldots, X_1X_2X_3Y_{m-1}Y_m, \ldots, X_{n-2}X_{n-1}X_nY_{m-1}Y_m$,
$\binom{n}{2}\binom{m}{3}$ generators $X_1X_2Y_1Y_2Y_3, \ldots, X_1X_2Y_{m-2}Y_{m-1}Y_m, \ldots, X_{n-1}X_nY_{m-2}Y_{m-1}Y_m$.
. . . .

– $(I_{2(n+m)-2m+1})_c(\mathcal{K}'_{n,m})$ and $(I_{2(n+1)})_c(\mathcal{K}'_{n,m})$

have $\binom{n}{m}$ generators $X_1 \cdots X_m, \ldots, X_{n-m+1} \cdots X_n$, $\binom{m}{m}$ generators $Y_1 \cdots Y_m$,
$\binom{n}{m-1}\binom{m}{1}$ generators $X_1 \cdots X_{m-1} Y_1, \ldots, X_1 \cdots X_{m-1} Y_m, \ldots, X_{n-m+2} \cdots X_n Y_m$,
$\cdots\cdots\cdots$
$\binom{n}{1}\binom{m}{m-1}$ generators $X_1 Y_1 \cdots Y_{m-1}, \ldots, X_n Y_1 \cdots Y_{m-1}, \ldots, X_1 Y_2 \cdots Y_m, \ldots, X_n Y_2 \cdots Y_m$,
$\binom{n}{m-1}\binom{m}{2}$ generators $X_1 \cdots X_{m-1} Y_1 Y_2, \ldots, X_1 \cdots X_{m-1} Y_{m-1} Y_m, \ldots, X_{n-m+2} \cdots X_n Y_{m-1} Y_m$,
$\cdots\cdots\cdots$
$\binom{n}{2}\binom{m}{m-1}$ generators $X_1 X_2 Y_1 \cdots Y_{m-1}, \ldots, X_1 X_2 Y_2 \cdots Y_m, \ldots, X_{n-1} X_n Y_2 \cdots Y_m$.
$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$

– $(I_{2(n+m)-2n+1})_c(\mathcal{K}'_{n,m})$ and $(I_{2(m+1)})_c(\mathcal{K}'_{n,m})$

have $\binom{n}{n}$ generators $X_1 \cdots X_n$, $\binom{m}{m}$ generators $Y_1 \cdots Y_m$,
$\binom{n}{n-1}\binom{m}{1}$ generators $X_1 \cdots X_{n-1} Y_1, \ldots, X_2 \cdots X_n Y_1, \ldots, X_1 \cdots X_{n-1} Y_m, \ldots, X_2 \cdots X_n Y_m$,
$\cdots\cdots\cdots$
$\binom{n}{n-m+1}\binom{m}{m-1}$ generators $X_1 \cdots X_{n-m+1} Y_1 \cdots Y_{m-1}, \ldots, X_m \cdots X_n Y_1 \cdots Y_{m-1}, \ldots,$
$X_1 \cdots X_{n-m+1} Y_2 \cdots Y_m, \ldots, X_m \cdots X_n Y_2 \cdots Y_m$,
$\binom{n}{n-1}\binom{m}{2}$ generators $X_1 \cdots X_{n-1} Y_1 Y_2, \ldots, X_2 \cdots X_n Y_1 Y_2, \ldots, X_2 \cdots X_n Y_{m-1} Y_m$,
$\cdots\cdots\cdots$
$\binom{n}{n-m+2}\binom{m}{m-1}$ generators $X_1 \cdots X_{n-m+2} Y_1 \cdots Y_{m-1}, \ldots, X_{m-1} \cdots X_n Y_1 \cdots Y_{m-1}, \ldots,$
$X_1 \cdots X_{n-m+2} Y_2 \cdots Y_m, \ldots, X_{m-1} \cdots X_n Y_2 \cdots Y_m$.
$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$

– $(I_{2(n+m)-2(n+m-4)+1})_c(\mathcal{K}'_{n,m})$ and $(I_{10})_c(\mathcal{K}'_{n,m})$

have $\binom{n}{n}$ generators $X_1 \cdots X_n$, $\binom{m}{m}$ generators $Y_1 \cdots Y_m$,
$\binom{n}{n-1}\binom{m}{m-3}$ generators $X_1 \cdots X_{n-1} Y_1 \cdots Y_{m-3}, \ldots, X_2 \cdots X_n Y_1 \cdots Y_{m-3}, \ldots,$
$X_1 \cdots X_{n-1} Y_4 \cdots Y_m, \ldots, X_2 \cdots X_n Y_4 \cdots Y_m$,
$\binom{n}{n-2}\binom{m}{m-2}$ generators $X_1 \cdots X_{n-2} Y_1 \cdots Y_{m-2}, \ldots, X_3 \cdots X_n Y_1 \cdots Y_{m-2}, \ldots,$
$X_1 \cdots X_{n-2} Y_3 \cdots Y_m, \ldots, X_3 \cdots X_n Y_3 \cdots Y_m$,
$\binom{n}{n-3}\binom{m}{m-1}$ generators $X_1 \cdots X_{n-3} Y_1 \cdots Y_{m-1}, \ldots, X_4 \cdots X_n Y_1 \cdots Y_{m-1}, \ldots,$
$X_1 \cdots X_{n-3} Y_2 \cdots Y_m, \ldots, X_4 \cdots X_n Y_2 \cdots Y_m$,
$\binom{n}{n-1}\binom{m}{m-2}$ generators $X_1 \cdots X_{n-1} Y_1 \cdots Y_{m-2}, \ldots, X_2 \cdots X_n Y_1 \cdots Y_{m-2}, \ldots,$
$X_1 \cdots X_{n-1} Y_3 \cdots Y_m, \ldots, X_2 \cdots X_n Y_3 \cdots Y_m$,
$\binom{n}{n-2}\binom{m}{m-1}$ generators $X_1 \cdots X_{n-2} Y_1 \cdots Y_{m-1}, \ldots, X_3 \cdots X_n Y_1 \cdots Y_{m-1}, \ldots,$
$X_1 \cdots X_{n-2} Y_2 \cdots Y_m, \ldots, X_3 \cdots X_n Y_2 \cdots Y_m$.

– $(I_{2(n+m)-2(n+m-3)+1})_c(\mathcal{K}'_{n,m})$ and $(I_8)_c(\mathcal{K}'_{n,m})$

have $\binom{n}{n}$ generators $X_1 \cdots X_n$, $\binom{m}{m}$ generators $Y_1 \cdots Y_m$,
$\binom{n}{n-1}\binom{m}{m-2}$ generators $X_1 \cdots X_{n-1} Y_1 \cdots Y_{m-2}, \ldots, X_2 \cdots X_n Y_1 \cdots Y_{m-2}, \ldots,$
$X_1 \cdots X_{n-1} Y_3 \cdots Y_m, \ldots, X_2 \cdots X_n Y_3 \cdots Y_m$,
$\binom{n}{n-2}\binom{m}{m-1}$ generators $X_1 \cdots X_{n-2} Y_1 \cdots Y_{m-1}, \ldots, X_3 \cdots X_n Y_1 \cdots Y_{m-1}, \ldots,$

$X_1 \cdots X_{n-2}Y_2 \cdots Y_m, \ldots, X_3 \cdots X_nY_2 \cdots Y_m$, and

$\binom{n}{n-1}\binom{m}{m-1}$ generators $X_1 \cdots X_{n-1}Y_1 \cdots Y_{m-1}, \ldots, X_2 \cdots X_nY_1 \cdots Y_{m-1}, \ldots,$
$X_1 \cdots X_{n-1}Y_2 \cdots Y_m, \ldots, X_2 \cdots X_nY_2 \cdots Y_m$.

– $(I_{2(n+m)-2(n+m-2)+1})_c(\mathcal{K}'_{n,m})$ and $(I_6)_c(\mathcal{K}'_{n,m})$

have $\binom{n}{n}$ generators $X_1 \cdots X_n$, $\binom{m}{m}$ generators $Y_1 \cdots Y_m$, and

$\binom{n}{n-1}\binom{m}{m-1}$ generators $X_1 \cdots X_{n-1}Y_1 \cdots Y_{m-1}, \ldots, X_2 \cdots X_nY_1 \cdots Y_{m-1}, \ldots,$
$X_1 \cdots X_{n-1}Y_2 \cdots Y_m, \ldots, X_2 \cdots X_nY_2 \cdots Y_m$.

– $(I_{2(n+m)-2(n+m-1)+1})_c(\mathcal{K}'_{n,m})$ and $(I_4)_c(\mathcal{K}'_{n,m})$

have $\binom{n}{n}$ generators $X_1 \cdots X_n$, and $\binom{m}{m}$ generators $Y_1 \cdots Y_m$. □

Example 4.5 Let $R = K[X_1, X_2, X_3, X_4, X_5; Y_1, Y_2, Y_3]$ and $\mathcal{K}'_{5,3}$ be the strong quasi-bipartite graph on vertex set $\{x_1, x_2, x_3, x_4, x_5; y_1, y_2, y_3\}$. The ideals of vertex covers $(I_q)_c$ of $I_q(\mathcal{K}'_{5,3})$, for $q = 3, \ldots, 15$, are the following ones:

$(I_3)_c = (I_4)_c = (X_1X_2X_3X_4X_5;\ Y_1Y_2Y_3)$,

$(I_5)_c = (I_6)_c = (X_1X_2X_3X_4X_5;\ Y_1Y_2Y_3;$
$X_1X_2X_3X_4Y_1Y_2, X_1X_2X_3X_5Y_1Y_2, X_1X_2X_4X_5Y_1Y_2, X_1X_3X_4X_5Y_1Y_2, X_2X_3X_4X_5Y_1Y_2,$
$X_1X_2X_3X_4Y_1Y_3, \ldots, X_2X_3X_4X_5Y_1Y_3, X_1X_2X_3X_4Y_2Y_3, \ldots, X_2X_3X_4X_5Y_2Y_3)$,

$(I_7)_c = (I_8)_c = (X_1X_2X_3X_4X_5;\ Y_1Y_2Y_3;$
$X_1X_2X_3X_4Y_1, X_1X_2X_3X_5Y_1, X_1X_2X_4X_5Y_1, X_1X_3X_4X_5Y_1, X_2X_3X_4X_5Y_1,$
$X_1X_2X_3X_4Y_2, \ldots, X_2X_3X_4X_5Y_2, X_1X_2X_3X_4Y_3, \ldots, X_2X_3X_4X_5Y_3;$
$X_1X_2X_3Y_1Y_2, X_1X_2X_4Y_1Y_2, X_1X_2X_5Y_1Y_2, X_1X_3X_4Y_1Y_2, X_1X_3X_5Y_1Y_2,$
$X_1X_4X_5Y_1Y_2, X_2X_3X_4Y_1Y_2, X_2X_3X_5Y_1Y_2, X_2X_4X_5Y_1Y_2, X_3X_4X_5Y_1Y_2,$
$X_1X_2X_3Y_1Y_3, \ldots, X_3X_4X_5Y_1Y_3, X_1X_2X_3Y_2Y_3, \ldots, X_3X_4X_5Y_2Y_3;$
$X_1X_2X_3X_4Y_1Y_2, X_1X_2X_3X_5Y_1Y_2, X_1X_2X_4X_5Y_1Y_2, X_1X_3X_4X_5Y_1Y_2, X_2X_3X_4X_5Y_1Y_2,$
$X_1X_2X_3X_4Y_1Y_3, \ldots, X_2X_3X_4X_5Y_1Y_3, X_1X_2X_3X_4Y_2Y_3, \ldots, X_2X_3X_4X_5Y_2Y_3)$,

$(I_9)_c = (I_{10})_c = (X_1X_2X_3X_4, X_1X_2X_3X_5, X_1X_2X_4X_5, X_1X_3X_4X_5, X_2X_3X_4X_5;\ Y_1Y_2Y_3;$
$X_1X_2X_3Y_1, X_1X_2X_4Y_1, X_1X_2X_5Y_1, X_1X_3X_4Y_1, X_1X_3X_5Y_1, X_1X_4X_5Y_1, X_2X_3X_4Y_1,$
$X_2X_3X_5Y_1, X_2X_4X_5Y_1, X_3X_4X_5Y_1, X_1X_2X_3Y_2, \ldots, X_3X_4X_5Y_2, X_1X_2X_3Y_3, \ldots, X_3X_4X_5Y_3;$
$X_1X_2Y_1Y_2, X_1X_3Y_1Y_2, X_1X_4Y_1Y_2, X_1X_5Y_1Y_2, X_2X_3Y_1Y_2, X_2X_4Y_1Y_2, X_2X_5Y_1Y_2,$
$X_3X_4Y_1Y_2, X_3X_5Y_1Y_2, X_4X_5Y_1Y_2, X_1X_2Y_1Y_3, \ldots, X_4X_5Y_1Y_3, X_1X_2Y_2Y_3, \ldots, X_4X_5Y_2Y_3;$
$X_1X_2X_3Y_1Y_2, X_1X_2X_4Y_1Y_2, X_1X_2X_5Y_1Y_2, X_1X_3X_4Y_1Y_2, X_1X_3X_5Y_1Y_2,$
$X_1X_4X_5Y_1Y_2, X_2X_3X_4Y_1Y_2, X_2X_3X_5Y_1Y_2, X_2X_4X_5Y_1Y_2, X_3X_4X_5Y_1Y_2,$
$X_1X_2X_3Y_1Y_3, \ldots, X_3X_4X_5Y_1Y_3, X_1X_2X_3Y_2Y_3, \ldots, X_3X_4X_5Y_2Y_3)$,

$(I_{11})_c = (I_{12})_c = (X_1X_2X_3, X_1X_2X_4, X_1X_2X_5, X_1X_3X_4, X_1X_3X_5, X_1X_4X_5,$
$X_2X_3X_4, X_2X_3X_5, X_2X_4X_5, X_3X_4X_5;\ Y_1Y_2Y_3;$
$X_1X_2Y_1, X_1X_3Y_1, X_1X_4Y_1, X_1X_5Y_1, X_2X_3Y_1, X_2X_4Y_1, X_2X_5Y_1, X_3X_4Y_1, X_3X_5Y_1,$
$X_4X_5Y_1, X_1X_2Y_2, \ldots, X_4X_5Y_2, X_1X_2Y_3, \ldots, X_4X_5Y_3;$
$X_1Y_1Y_2, X_2Y_1Y_2, X_3Y_1Y_2, X_4Y_1Y_2, X_5Y_1Y_2, X_1Y_1Y_3, \ldots, X_5Y_1Y_3, X_1Y_2Y_3, \ldots, X_5Y_2Y_3;$

$X_1X_2Y_1Y_2, X_1X_3Y_1Y_2, X_1X_4Y_1Y_2, X_1X_5Y_1Y_2, X_2X_3Y_1Y_2, X_2X_4Y_1Y_2, X_2X_5Y_1Y_2,$
$X_3X_4Y_1Y_2, X_3X_5Y_1Y_2, X_4X_5Y_1Y_2, X_1X_2Y_1Y_3, \dots, X_4X_5Y_1Y_3, X_1X_2Y_2Y_3, \dots, X_4X_5Y_2Y_3)\,,$

$(I_{13})_c = (I_{14})_c = (X_1X_2, X_1X_3, X_1X_4, X_1X_5, X_2X_3, X_2X_4, X_2X_5, X_3X_4, X_3X_5, X_4X_5;$
$Y_1Y_2, Y_1Y_3, Y_2Y_3;\ X_1Y_1, X_2Y_1, X_3Y_1, X_4Y_1, X_5Y_1, X_1Y_2, \dots, X_5Y_2, X_1Y_3, \dots, X_5Y_3)\,,$

$(I_{15})_c = (X_1, X_2, X_3, X_4, X_5;\ Y_1, Y_2, Y_3)\,.$

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References

[1] Bandari S, Herzog J. Monomial localization and polymatroidal ideals. Eur J Comb 2013; 34: 752-763.

[2] Conca A, Herzog J. Castelnuovo-Mumford regularity of product of ideals. Collect Math 2003; 54: 137-152.

[3] Herzog J, Hibi T. Discrete polymatroids. J Alg Comb 2002; 16: 239-268.

[4] Herzog J, Rauf A, Vladoiu M. The stable set of associated prime ideals of a polymatroidal ideal. J Alg Comb 2013; 37: 289-312.

[5] Imbesi M, La Barbiera M. Algebraic properties of non-squarefree graph ideals. Math Rep 2013; 15: 107-113.

[6] Imbesi M, La Barbiera M. Monomial ideals of graphs with loops. Anal Ştiinţ Univ Al I Cuza Iaşi (S.N) Mat 2014; 60, 2: 321-336.

[7] Imbesi M, La Barbiera M. On a class of vertex cover ideal. Anal Ştiinţ Univ Al I Cuza Iaşi (S.N) Mat 2015; 61: 319-326.

[8] Imbesi M, La Barbiera M. Vertex covers in graphs with loops. Preprint arXiv: 1210.8198.

[9] Imbesi M, La Barbiera M, Staglianò PL. On generalized graph ideals of complete bipartite graphs Preprint.

[10] La Barbiera M. Integral closure and normality of a class of Veronese-type ideals. Riv Mat Univ Parma 2008; 9: 31-47.

[11] La Barbiera M. On a class of monomial ideals generated by s-sequences. Math Rep 2010; 12, 62: 201-216.

[12] La Barbiera M. On standard invariants of bi-polymatroidal ideals. Alg Colloq 2014; 21: 267-274.

[13] La Barbiera M. A note on unmixed ideals of Veronese bi-type. Turk J Math 2013; 37: 1-7.

[14] La Barbiera M, Paratore M, Restuccia G. The edge ideal associated to graphs with loops. Rend Circ Mat Palermo 2006; Serie II, Suppl. 77: 363-372.

[15] La Barbiera M, Staglianò PL. Monomial ideals of linear type. Turk J Math 2014; 38: 203-211.

[16] Restuccia G, Villarreal RH. On the normality of monomial ideals of mixed products. Commun Alg 2001; 29: 3571-3580.

[17] Sturmfels B. Gröbner bases and convex polytopes. Providence, RI, USA: American Mathematical Society, 1995.

[18] Villarreal RH. Monomial Algebras. Monographs and Textbooks in Pure and Applied Mathematics 238. New York, NY, USA: Marcel Dekker, Inc., 2001.