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A note on reduction numbers and Hilbert–Samuel functions of ideals over Cohen–Macaulay rings

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Abstract: Let $(R,\mathfrak{m})$ be a Cohen–Macaulay local ring of dimension $d \geq 2$ with infinite residue field and I an $\mathfrak{m}$-primary ideal of R. Let I be integrally closed and J be a minimal reduction of I. In this paper, we show that the following are equivalent: *(i)* $P_I(n) = H_I(n)$ for $n = 1, 2$; *(ii)* $P_I(n) = H_I(n)$ for all $n \geq 1$; *(iii)* $I^3 = JI^2$. Moreover, if $\dim R = 3$, $n(I) \leq 1$ and $\operatorname{grade} gr_I(R)_+ > 0$, then the reduction number $r(I)$ is independent.

Key words: Cohen–Macaulay rings, Hilbert–Samuel functions

1. Introduction

Throughout this note, we assume that $(R,\mathfrak{m})$ is a Cohen–Macaulay (abbreviated CM) local ring of dimension $d > 0$ with infinite residue field and I is an $\mathfrak{m}$-primary ideal of R. Let $\ell(R/I^n)$ denote the length of the R-module R/I^n. The Hilbert–Samuel function of I is the function $H_I(n) = \ell(R/I^n)$. It is well known that this function coincides with a polynomial $P_I(n)$ of degree d for all sufficiently large integers n and we set $n(I) = \max\{n \in \mathbb{Z} : P_I(n) \neq H_I(n)\}$; this number first introduced by Ooishi [11]. $P_I(n)$ is called the Hilbert–Samuel polynomial of I. Northcott and Rees [10] defined a minimal reduction of I (abbreviated MR(I)) to be a d-generated ideal $J \subseteq I$ of R such that $JI^n = I^{n+1}$ for some nonnegative integers n. We denote $r_J(I) = \min\{n \in \mathbb{Z} : JI^n = I^{n+1}\}$. The reduction number $r(I)$ is defined as $r(I) = \min\{r_J(I) : J \in MR(I)\}$. The reduction number $r(I)$ is said to be independent if $r(I) = r_J(I)$ for all $J \in MR(I)$. We denote the associated graded ring of I by $gr_I(R) = \oplus_{n\geq 0} I^n/I^{n+1}$ and $gr_I(R)_+$ denotes the ideal $\oplus_{n\geq 1} I^n/I^{n+1}$. Huckaba [3] and Trung [14] showed that if $\operatorname{grade} gr_I(R)_+ \geq d-1$, then $r(I)$ is independent (see also [8]).

Hoa [2] proved that if $d = 2$ and I agrees with its Ratliff–Rush closure $\widetilde{I}$, then the following are equivalent: *(i)* $H_I(n) = P_I(n)$ for $n = 1, 2$; *(ii)* $H_I(n) = P_I(n)$ for all $n \geq 1$; *(iii)* $r(I) \leq 2$ and $\operatorname{grade} gr_I(R)_+ \geq 1$ (see also [4]). For basic definitions, we refer the reader to [1, 9].

The main aim of this paper is to prove the following theorems.

Theorem 1.1 *Let $d \geq 2$ and J be a minimal reduction of I. If $\overline{I} = I$, then the following are equivalent:*

(i) $P_I(n) = H_I(n)$ *for* $n = 1, 2$.

(ii) $P_I(n) = H_I(n)$ *for all* $n \geq 1$.

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(iii) $r_J(I) \leq 2$.

Theorem 1.2 *(Compare with Corollary 5 of [16].) Let* $d = 3$ *and* $\overline{I} = I$. *If* $n(I) \leq 1$ *and* $\operatorname{grade} gr_I(R)_+ > 0$, *then* $r(I)$ *is independent.*

2. The results

Following Marley [8], let $S = \oplus_{n\geq 0} S_n$ be a Noetherian graded ring where S_0 is an Artinian local ring, S is generated by 1-forms over S_0, and $S_+ = \oplus_{n\geq 1} S_n$. If $M = \oplus_{n\in\mathbb{Z}} M_n$ is a finitely generated graded S-module, then for $i \geq 0$ we let $H^i_{S_+}(M) = \oplus_{n\in\mathbb{Z}} H^i_{S_+}(M)_n$ be the ith local cohomology modules of M with support in S_+. It was shown that these modules are Artinian and that each $H^i_{S_+}(M)_n$ is finitely generated. Thus, $H^i_{S_+}(M)_n = 0$ for n sufficiently large. If $H^i_{S_+}(M) \neq 0$, we let $a_i(M) = \max\{n \in \mathbb{Z} : H^i_{S_+}(M)_n \neq 0\}$. For convenience, we define $a_i(M) = \infty$ for $i < \operatorname{depth}_{S_+} M$ and $a_i(M) = \infty$ for $i > \dim M$. We will use $a_i(I)$ to denote $a_i(gr_I(R))$.

The following lemma extends Theorem 2.1 of [13].

Lemma 2.1 *Let* $\overline{I} = I$ *and* $r(I) \leq 2$. *Then* $gr_I(R)$ *is Cohen–Macaulay.*

Proof Suppose J is a minimal reduction of I. Then $\overline{I^2} = \overline{J^2}$ and by Theorem 1 of [6] we have $\overline{I^2} \cap J = \overline{I}J = IJ$. Therefore, $I^2 \cap J = IJ$, and since $JI^n = I^{n+1}$ for all $n \geq 2$, we have $I^n \cap J = I^{n-1}J$ for all $n \geq 1$. Hence, by Corollary 2.7 of [15], $gr_I(R)$ is Cohen–Macaulay. □

The following result extends Remark 3.2 of [8].

Lemma 2.2 *Let* $d = 2$ *and* $P_I(n) = H_I(n)$ *for all* $n \geq 3$. *If* $\overline{I} = I$, *then* $r(I)$ *is independent.*

Proof If $r(I) \leq 2$, then by Lemma 2.1 $gr_I(R)$ is CM and so $r(I)$ is independent. If $r(I) \geq 3$, then since $n(I) \leq 2$, we have $r(I) \geq n(I) + 1$. Hence, by Theorem 3.1 of [16] $r(I)$ is independent. □

Proposition 2.3 *Let* $d = 2$ *and* J *be a minimal reduction of* I. *If* $r_J(I) \leq n(I) + 1$, *then* $a_1(I) = n(I)$.

Proof By Theorem 2.1 of [8] and our hypothesis, we have $\operatorname{grade} gr_I(R)_+ = 0$. Therefore, $a_0(I) < a_1(I)$. If $a_1(I) < a_2(I)$, then by Lemma 2.1 and Remark 1.4 of [8] $n(I)+2 \leq r_J(I)$ and this is a contradiction with our hypothesis. Hence, $a_2(I) \leq a_1(I)$. If $a_2(I) = a_1(I)$, then $r_J(I) = a_2(I)+2$ and so $a_2(I)+2 \leq n(I)+1 \leq a_2(I)+1$ and this is also a contradiction. Thus, $a_2(I) < a_1(I)$ and so $a_1(I) = n(I)$, as required. □

Proposition 2.4 *Let* $d = 2$ *and* $n(I) + 2 \leq r(I)$. *Then* $a_2(I) = n(I)$ *or* $a_2(I) = r(I) - 2$.

Proof If $\operatorname{grade} gr_I(R)_+ \geq 1$, then $r(I) = n(I) + 2 = a_2(I) + 2$. Hence, $a_2(I) = n(I)$. Suppose $\operatorname{grade} gr_I(R)_+ = 0$ and then $a_0(I) < a_1(I)$. If $a_1(I) < a_2(I)$, then $a_2(I) = n(I)$. If $a_2(I) < a_1(I)$, then $n(I) = a_1(I)$ and so $a_1(I) + 2 = n(I) + 2 \leq r(I) \leq a_1(I) + 1$ and this is a contradiction. If $a_1(I) = a_2(I)$, then $a_2(I) + 2 \leq r(I) \leq a_2(I) + 2$. Therefore, $a_2(I) = r(I) - 2$. □

Ratliff and Rush [12] defined $\widetilde{I}$ as follows: $\widetilde{I} = \cup_{n\geq 1} I^{n+1} : I^n$. They proved, for all $n > 0$, $\widetilde{I^n} = \cup_{k>0} I^{n+k} : I^k$. It is well known that $\operatorname{grade} gr_I(R)_+ > 0$ if and only if $\widetilde{I^n} = I^n$ for all $n > 0$. In the following results we denote the leading form of an element $x \in I \setminus I^2$ with $x^* \in I/I^2 \subseteq gr_I(R)$.

Theorem 2.5 *Let $d \geq 2$ and J be a minimal reduction of I. If $\overline{I} = I$, then the following are equivalent:*

(i) $P_I(n) = H_I(n)$ *for* $n = 1, 2$.

(ii) $P_I(n) = H_I(n)$ *for all* $n \geq 1$.

(iii) $r_J(I) \leq 2$.

In particular, if one of the above cases holds, then $gr_I(R)$ is CM.

Proof $(iii) \Longrightarrow (ii)$. $r_J(I) \leq 2$ and by Lemma 2.1 $gr_I(R)$ is CM. Thus, $r(I) = n(I) + d$ and so $n(I) \leq 2 - d$. Therefore, $n(I) \leq 0$ and hence $P_I(n) = H_I(n)$ for all $n \geq 1$.
$(ii) \Longrightarrow (i)$. It is clear.
$(i) \Longrightarrow (iii)$. We use induction on d. If $d = 2$, then by Corollary 4 of [7] $P_I(n) = \ell(R/\widetilde{I^n})$ for all $n \geq 1$. Therefore, $\widetilde{I^{n+1}} = J\widetilde{I^n}$ for all $n \geq 2$ and so $I^3 = JI^2$. We may assume that $d \geq 3$, and that the assertion holds for CM local rings of dimension less than d. By Lemma 8.4.2 of [5] replacing R with $R(X)$, if necessary, and we may assume that there exists $x \in J$ such that x is a part of a system of minimal generators of J and, if we put $A = R/(x)$, by Lemma 11 of [7] $\mathfrak{a} = IA$ is integrally closed. By Lemma 2.3 of [16] $P_{\mathfrak{a}}(n) = H_{\mathfrak{a}}(n)$ for $n = 1, 2$. Therefore, by our induction hypothesis, $\mathfrak{a}^3 = J\mathfrak{a}^2$ and $gr_{\mathfrak{a}}(A)$ is CM; in particular, $\widetilde{\mathfrak{a}^n} = \mathfrak{a}^n$ for all n. Since $\widetilde{I^n}A \subseteq \widetilde{\mathfrak{a}^n} = \mathfrak{a}^n = I^n A$, we have $\widetilde{I^n} + (x) = I^n + (x)$. Therefore, $\widetilde{I^n} = I^n + (x)\widetilde{I^{n-1}}$ for all $n \geq 1$ and so by induction on n, we have $\widetilde{I^n} = I^n$ for all n. Hence, $gr_I(R)$ is CM and x^* is a regular element of $gr_I(R)$. By Lemma 2.2 of [16] $r_J(I) \leq 2$, as required. □

Theorem 2.6 *Let $d = 3$ and $n(I) \leq 1$. If* $\operatorname{grade} gr_I(R)_+ > 0$ *and $\overline{I} = I$, then $r(I)$ is independent.*

Proof Let $J_1 = (x_1, x_2, x_3)$ and $J_2 = (y_1, y_2, y_3)$ be minimal reductions of I. Since $\operatorname{grade} gr_I(R)_+ > 0$, by Lemma 2.7 of [16] we may assume that x_1^* and y_1^* are regular elements in $gr_I(R)$ and $J = (x_1, x_2, y_1)$ is a minimal reduction of I. By Lemma 2.1 of [16] $n(I/(x_1) = n(I) + 1 \leq 2$. Replacing R with $R(T)$, by Lemma 11 of [7] $I/(x_1)$ is integrally closed and by Lemma 2.1 $r(I/(x_1))$ is independent. Hence, $r(I/(x_1)) = r_{J_1/(x_1)}(I/(x_1) = r_{J/(x_1)}(I/(x_1))$. Since x_1^* is regular element in $gr_I(R)$, we have $r_{J_1}(I) = r_J(I)$. By the same argument $r_{J_2}(I) = r_J(I)$ and so $r_{J_1}(I) = r_{J_2}(I)$. Thus, $r(I)$ is independent, as required. □

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