

## Sharp bounds for the first nonzero Steklov eigenvalues for $f$ -Laplacians

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**Abstract:** Let  $M$  be an  $n$ -dimensional compact Riemannian manifold with a boundary. In this paper, we consider the Steklov first eigenvalue with respect to the  $f$ -divergence form:

$$e^f \operatorname{div}(e^{-f} A \nabla u) = 0 \text{ in } M, \quad \langle A(\nabla u), \nu \rangle - \eta u = 0 \text{ on } \partial M,$$

where  $A$  is a smooth symmetric and positive definite endomorphism of  $TM$ , and the following three fourth order Steklov eigenvalue problems:

$$(\Delta_f)^2 u = 0 \text{ in } M, \quad u = \Delta_f u - q \frac{\partial u}{\partial \nu} = 0 \text{ on } \partial M;$$

$$(\Delta_f)^2 u = 0 \text{ in } M, \quad u = \frac{\partial^2 u}{\partial \nu^2} - \mu \frac{\partial u}{\partial \nu} = 0 \text{ on } \partial M;$$

$$(\Delta_f)^2 u = 0 \text{ in } M, \quad \frac{\partial u}{\partial \nu} = \frac{\partial(\Delta_f u)}{\partial \nu} + \xi u = 0 \text{ on } \partial M.$$

Under the assumption that the  $m$ -dimensional Bakry–Emery Ricci curvature and the weighted mean curvature are bounded from below, we obtain sharp bounds for Steklov first nonzero eigenvalues. Moreover, we also study the case in which the bounds are achieved.

**Key words:**  $m$ -dimensional Bakry–Emery Ricci curvature,  $f$ -Laplacian, Steklov eigenvalue

### 1. Introduction

Let  $(M, \langle \cdot, \cdot \rangle)$  be an  $n$ -dimensional compact Riemannian manifold with a boundary and  $n \geq 2$ . Denote  $\Delta$ ,  $\nu$  by the Laplace operator on  $M$  and the outward unit normal on  $\partial M$ , respectively. The Steklov problem is to find a solution of the equation

$$\Delta u = 0 \text{ in } M, \quad \frac{\partial u}{\partial \nu} = pu \text{ on } \partial M, \tag{1.1}$$

where  $p$  is a real number. This problem was introduced by Steklov in [19], in 1902, for bounded domains in the plane. The study of the Steklov eigenvalue comes from physics and has appeared in quite a few physical fields, such as fluid mechanics, electromagnetism, and elasticity. For related research and some improvements on the Steklov problem of (1.1), see [7, 8, 12, 13, 18] and the references therein.

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Let  $f \in C^2(M)$ . The  $f$ -Laplacian operator is defined by

$$\Delta_f = e^f \operatorname{div}(e^{-f} \nabla) = \Delta - \nabla f \nabla,$$

which is symmetric in  $L^2(M, e^{-f} dv)$ . Namely,

$$\int_M u \Delta_f v e^{-f} dv = - \int_M \nabla u \nabla v e^{-f} dv = \int_M v \Delta_f u e^{-f} dv, \quad \forall u, v \in C_0^\infty(M),$$

where  $dv$  is the volume form induced by the metric on  $M$ . In general, the triple  $(M, \langle, \rangle, e^{-f} dv)$  is customarily called a smooth metric measure space. Following [2, 3, 16], the  $m$ -dimensional Bakry–Emery Ricci curvature associated to the  $f$ -Laplacian is given by

$$\operatorname{Ric}_f^m = \operatorname{Ric} + \nabla^2 f - \frac{1}{m-n} df \otimes df,$$

where  $m \geq n$  is a constant, and  $m = n$  if and only if  $f$  is a constant. We define

$$\operatorname{Ric}_f = \operatorname{Ric} + \nabla^2 f.$$

Then  $\operatorname{Ric}_f$  can be seen as the  $\infty$ -dimensional Bakry–Emery Ricci curvature. The equation  $\operatorname{Ric}_f = k \langle, \rangle$  for some constant  $k$  is just the gradient Ricci soliton equation, which plays an important role in the study of Ricci flow (see [4]). The equation  $\operatorname{Ric}_f^m = k \langle, \rangle$  corresponds to the quasi-Einstein equation (cf. [5]), which has been studied by many authors.

In recent years, many interesting estimates for eigenvalues of the  $f$ -Laplacian operator have been obtained, for example, [6, 15, 17, 22]. In this paper, we study the following three fourth order Steklov eigenvalue problems:

$$(\Delta_f)^2 u = 0 \text{ in } M, \quad u = \Delta_f u - q \frac{\partial u}{\partial \nu} = 0 \text{ on } \partial M. \tag{1.2}$$

$$(\Delta_f)^2 u = 0 \text{ in } M, \quad u = \frac{\partial^2 u}{\partial \nu^2} - \mu \frac{\partial u}{\partial \nu} = 0 \text{ on } \partial M; \tag{1.3}$$

$$(\Delta_f)^2 u = 0 \text{ in } M, \quad \frac{\partial u}{\partial \nu} = \frac{\partial(\Delta_f u)}{\partial \nu} + \xi u = 0 \text{ on } \partial M. \tag{1.4}$$

For  $f$  constant, the problem (1.2) was studied by Kuttler [13]; the problem (1.3) was studied by Kuttler [13] and Payne [18]; the problem (1.4) was first studied by Kuttler and Sigillito in [14], where some estimates for the first eigenvalue  $\xi_1$  were obtained. In the present paper, we also study the Steklov first eigenvalue with respect to the  $f$ -divergence form operator  $e^f \operatorname{div}(e^{-f} A \cdot)$ , where  $A$  is a smooth symmetric and positive definite endomorphism of  $TM$  with  $A \leq \delta I$  (in the matrix sense), namely the problem:

$$e^f \operatorname{div}(e^{-f} A \nabla u) = 0 \text{ in } M, \quad \langle A(\nabla u), \nu \rangle - \eta u = 0 \text{ on } \partial M. \tag{1.5}$$

When  $f$  is constant, the Steklov eigenvalue of the problem (1.5) on bounded domains in a Euclidean plane was studied by Alessandrini and Magnanini in [1].

The Steklov eigenvalue problem for elliptic equations in divergence form on bounded domains in a Euclidean plane has been studied in [1]. In the present paper, we will obtain the following upper bound of the first nonzero eigenvalue of the problem (1.5) on a compact manifold that generalizes a result in [21].

**Theorem 1.1** *Let  $(M, \langle \cdot, \cdot \rangle)$  be an  $n$ -dimensional compact Riemannian manifold with a boundary. Assume that the  $m$ -dimensional Bakry–Emery Ricci curvature is bounded below by  $-k_0$ , where  $k_0$  is nonnegative, and that the principal curvatures of  $\partial M$  are bounded below by a positive constant  $c$  and  $f_\nu \leq -(m - n)c$ . Denote by  $\lambda_1$  the first nonzero eigenvalue of the  $f$ -Laplacian on functions of  $\partial M$  and let  $\eta_1$  be the first nonzero eigenvalue of the eigenvalue problem (1.5) with  $A \leq \delta I$  (in the matrix sense). Then we have*

$$(2\lambda_1 + k_0)^2 \geq 4(m - 1)\lambda_1 c^2, \tag{1.6}$$

and

$$\eta_1 \leq \frac{2\lambda_1 + k_0 + \sqrt{(2\lambda_1 + k_0)^2 - 4(m - 1)\lambda_1 c^2}}{2(m - 1)c} \delta. \tag{1.7}$$

Furthermore, if (1.6) or (1.7) take an equality sign, then  $M$  is isometric to an  $n$ -dimensional Euclidean ball of radius  $\frac{1}{c}$  and  $f$  is constant.

On the other hand, for the first nonzero eigenvalues of three fourth order Steklov eigenvalue problems (1.2), (1.3), and (1.4), we prove the following:

**Theorem 1.2** *Let  $(M, \langle \cdot, \cdot \rangle)$  be an  $n$ -dimensional compact Riemannian manifold with a boundary and nonnegative  $m$ -dimensional Bakry–Emery Ricci curvature. Assume that  $H_f \geq \frac{m-1}{n-1}c$ ; then the first nonzero eigenvalue  $q_1$  of the eigenvalue problem (1.2) satisfies*

$$q_1 \geq mc, \tag{1.8}$$

with equality holding if and only if  $M$  is isometric to an  $n$ -dimensional Euclidean ball of radius  $\frac{1}{c}$  and  $f$  is constant.

**Theorem 1.3** *Let  $(M, \langle \cdot, \cdot \rangle)$  be an  $n$ -dimensional compact Riemannian manifold with a boundary. Then the first nonzero eigenvalue  $q_1$  of the eigenvalue problem (1.2) satisfies*

$$q_1 \leq \frac{A_f}{V_f}, \tag{1.9}$$

where  $A_f$  and  $V_f$  are  $f$ -area of  $\partial M$  and the volume of  $M$ , respectively. That is,

$$A_f = \int_{\partial M} e^{-f} d\mu, \quad V_f = \int_M e^{-f} dv.$$

Moreover, if in addition the  $m$ -dimensional Bakry–Emery Ricci curvature is nonnegative and there exists a point  $x_0$  such that the  $f$ -mean curvature  $H_f(x_0) \geq \frac{(m-1)A_f}{m(n-1)V_f}$ , then  $q_1 = \frac{A_f}{V_f}$  implies that  $M$  is isometric to an  $n$ -dimensional Euclidean ball and  $f$  is constant.

**Theorem 1.4** *Let  $(M, \langle \cdot, \cdot \rangle)$  be an  $n$ -dimensional compact Riemannian manifold with a boundary and nonnegative  $m$ -dimensional Bakry–Emery Ricci curvature. Assume that  $H_f \geq \frac{m-1}{n-1}c$ ; then the first nonzero eigenvalue  $\mu_1$  of the eigenvalue problem (1.3) satisfies*

$$\mu_1 \geq c \tag{1.10}$$

with equality holding if and only if  $M$  is isometric to an  $n$ -dimensional Euclidean ball of radius  $\frac{1}{c}$  and  $f$  is constant.

**Theorem 1.5** *Let  $(M, \langle \cdot, \cdot \rangle)$  be an  $n$ -dimensional compact Riemannian manifold with a boundary and nonnegative  $m$ -dimensional Bakry–Emery Ricci curvature. Assume that the principal curvatures of  $\partial M$  are bounded below by a positive constant  $c$  and  $f_\nu \leq -(m - n)c$ ; then the first nonzero eigenvalue  $\xi_1$  of the eigenvalue problem (1.4) satisfies*

$$\xi_1 > \frac{mc\lambda_1}{m - 1}, \tag{1.11}$$

where  $\lambda_1$  denotes the first nonzero eigenvalue of the  $f$ -Laplacian on functions of  $\partial M$ .

**Remark.** When  $m = n$ , we have that  $f$  is constant and  $\text{Ric}_f^m = \text{Ric}$ . Hence, Theorem 1.1 generalizes Theorem 1.4 of Xia and Wang in [21], which concerns exactly the same problem (i.e. exactly the same divergence form operator) in case  $f$  constant. In particular, Theorem 1.2 and Theorem 1.3 generalize Theorem 1.2 and Theorem 1.3 of Wang and Xia in [20], respectively. Theorem 1.4 and Theorem 1.5 generalize Theorem 1.2 and Theorem 1.3 of Xia and Wang in [21], respectively.

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**2. Proof of results**

Let  $\langle \cdot, \cdot \rangle$  be the Riemannian metric on  $M$  and that induced on  $\partial M$ . We denote  $\nabla$  and  $\Delta$  as the connection and the Laplacian on  $M$ , respectively. For  $X, Y \in \Gamma(\partial M)$ , the second fundamental form of  $\partial M$  is defined by  $II(X, Y) = \langle \nabla_X \nu, Y \rangle$  and the shape operator of  $\partial M$  is related to  $II$  by  $II(X, Y) = \langle S(X), Y \rangle$ . The eigenvalues of  $S$  are called the principal curvatures and the mean curvature  $H$  of  $\partial M$  is given by  $H = \frac{1}{n-1} \text{tr}(S)$ . Recently, using the integration by parts for the Bochner formula for the  $f$ -Laplacian:

$$\frac{1}{2} \Delta_f |\nabla u|^2 = |\nabla^2 u|^2 + \langle \nabla u, \nabla \Delta_f u \rangle + \text{Ric}_f(\nabla u, \nabla u), \tag{2.1}$$

Ma and Du [17] studied the  $f$ -Laplacian and extended the classical Reilly’s formula to

$$\begin{aligned} & \int_M [(\Delta_f u)^2 - |\nabla^2 u|^2 - \text{Ric}_f(\nabla u, \nabla u)] e^{-f} dv \\ &= \int_{\partial M} [2u_\nu(\overline{\Delta}_f u) + (n - 1)H_f(u_\nu)^2 + II(\overline{\nabla} u, \overline{\nabla} u)] e^{-f} d\mu, \end{aligned} \tag{2.2}$$

where the  $f$ -mean curvature  $H_f$  given by  $H_f = \frac{1}{n-1}[\text{tr}(S) - f_\nu]$ ;  $\overline{\Delta}$  and  $\overline{\nabla}$  represent the Laplacian and the gradient on  $\partial M$  with respect to the induced metric on  $\partial M$ , respectively;  $d\mu$  is the volume form on  $\partial M$ . Using the inequality of  $|\nabla^2 u|^2 \geq \frac{1}{n}(\Delta u)^2$  and

$$(a + b)^2 \geq \frac{a^2}{1 + \alpha} - \frac{b^2}{\alpha}, \quad \forall \alpha > 0$$

we obtain (cf.[10, 15, 16])

$$\begin{aligned} |\nabla^2 u|^2 + \text{Ric}_f(\nabla u, \nabla u) &\geq \frac{1}{n}(\Delta u)^2 + \text{Ric}_f(\nabla u, \nabla u) \\ &\geq \frac{1}{m}(\Delta_f u)^2 + \text{Ric}_f^m(\nabla u, \nabla u), \end{aligned} \tag{2.3}$$

where the equality case holds in the second inequality if and only if  $\Delta u + \frac{n}{m-n}\langle \nabla f, \nabla u \rangle = 0$ . Inserting (2.3) into (2.2) yields

$$\begin{aligned} &\int_M \left[ \frac{m-1}{m}(\Delta_f u)^2 - \text{Ric}_f^m(\nabla u, \nabla u) \right] e^{-f} dv \\ &\geq \int_{\partial M} [2u_\nu(\overline{\Delta}_f u) + (n-1)H_f(u_\nu)^2 + II(\overline{\nabla} u, \overline{\nabla} u)] e^{-f} d\mu. \end{aligned} \tag{2.4}$$

*Proof of Theorem 1.1.* Let  $u$  be the solution of the following Laplace equation

$$\Delta_f u = 0 \text{ in } M, \quad u|_{\partial M} = z, \tag{2.5}$$

where  $z$  is a first eigenfunction of  $\partial M$  corresponding to  $\lambda_1$ , that is,  $\overline{\Delta}_f z = -\lambda_1 z$ . Substituting  $u$  into the formula (2.4) and noticing the assumption on  $\partial M$  and that  $\text{Ric}_f^m \geq -k_0$ , we have

$$\begin{aligned} &k_0 \int_M |\nabla u|^2 e^{-f} dv \\ &\geq \int_{\partial M} [2u_\nu(\overline{\Delta}_f u) + (n-1)H_f(u_\nu)^2 + II(\overline{\nabla} u, \overline{\nabla} u)] e^{-f} d\mu \\ &\geq -2\lambda_1 \int_{\partial M} z u_\nu e^{-f} d\mu + (m-1)c \int_{\partial M} (u_\nu)^2 e^{-f} d\mu + c \int_{\partial M} |\overline{\nabla} u|^2 e^{-f} d\mu \\ &= -2\lambda_1 \int_{\partial M} z u_\nu e^{-f} d\mu + (m-1)c \int_{\partial M} (u_\nu)^2 e^{-f} d\mu + c\lambda_1 \int_{\partial M} z^2 e^{-f} d\mu. \end{aligned} \tag{2.6}$$

By the divergence theorem, we have

$$\int_M |\nabla u|^2 e^{-f} dv = - \int_M u \Delta_f u e^{-f} dv + \int_{\partial M} z u_\nu e^{-f} d\mu = \int_{\partial M} z u_\nu e^{-f} d\mu. \tag{2.7}$$

Therefore, (2.6) can be written as

$$-(2\lambda_1 + k_0) \int_{\partial M} z u_\nu e^{-f} d\mu + (m-1)c \int_{\partial M} (u_\nu)^2 e^{-f} d\mu + c\lambda_1 \int_{\partial M} z^2 e^{-f} d\mu \leq 0, \tag{2.8}$$

which gives

$$\begin{aligned}
 0 &\geq -(2\lambda_1 + k_0) \int_{\partial M} z u_\nu e^{-f} d\mu + (m-1)c \int_{\partial M} (u_\nu)^2 e^{-f} d\mu + c\lambda_1 \int_{\partial M} z^2 e^{-f} d\mu \\
 &= (m-1)c \int_{\partial M} \left[ \left( u_\nu - \frac{2\lambda_1 + k_0}{2(m-1)c} z \right)^2 + \left( \frac{\lambda_1}{m-1} - \frac{(2\lambda_1 + k_0)^2}{4(m-1)^2 c^2} \right) z^2 \right] e^{-f} d\mu \\
 &\geq \left( c\lambda_1 - \frac{(2\lambda_1 + k_0)^2}{4(m-1)c} \right) \int_{\partial M} z^2 e^{-f} d\mu.
 \end{aligned} \tag{2.9}$$

Thus, from (2.9), we have

$$c\lambda_1 - \frac{(2\lambda_1 + k_0)^2}{4(m-1)c} \leq 0, \tag{2.10}$$

and the inequality (1.6) is obtained.

Note that  $z$  is the eigenfunction of the  $f$ -Laplacian on  $\partial M$  and  $\int_{\partial M} z e^{-f} d\mu = 0$ . It follows from the variational characterization of  $\eta_1$  (cf. [1]) that

$$\begin{aligned}
 \eta_1 &\leq \frac{\int_{\partial M} A(\nabla u, \nabla u) e^{-f} dv}{\int_{\partial M} z^2 e^{-f} d\mu} \leq \delta \frac{\int_{\partial M} |\nabla u|^2 e^{-f} dv}{\int_{\partial M} z^2 e^{-f} d\mu} \\
 &= \delta \frac{\int_{\partial M} z u_\nu e^{-f} d\mu}{\int_{\partial M} z^2 e^{-f} d\mu} \leq \delta \left( \frac{\int_{\partial M} (u_\nu)^2 e^{-f} d\mu}{\int_{\partial M} z^2 e^{-f} d\mu} \right)^{\frac{1}{2}}.
 \end{aligned} \tag{2.11}$$

By virtue of the Cauchy inequality, we have from (2.8) that

$$\begin{aligned}
 0 &\geq -(2\lambda_1 + k_0) \left( \int_{\partial M} (u_\nu)^2 e^{-f} d\mu \right)^{\frac{1}{2}} \left( \int_{\partial M} z^2 e^{-f} d\mu \right)^{\frac{1}{2}} \\
 &\quad + (m-1)c \int_{\partial M} (u_\nu)^2 e^{-f} d\mu + c\lambda_1 \int_{\partial M} z^2 e^{-f} d\mu
 \end{aligned} \tag{2.12}$$

and hence

$$\left( \int_{\partial M} (u_\nu)^2 e^{-f} d\mu \right)^{\frac{1}{2}} \leq C_{\lambda_1, k_0, m} \left( \int_{\partial M} z^2 e^{-f} d\mu \right)^{\frac{1}{2}}, \tag{2.13}$$

where

$$C_{\lambda_1, k_0, m} = \frac{2\lambda_1 + k_0 + \sqrt{(2\lambda_1 + k_0)^2 - 4(m-1)\lambda_1 c^2}}{2(m-1)c}.$$

Inserting (2.13) into (2.11) yields (1.7).

If the equality holds in (1.6), then we have

$$\nabla^2 u = \frac{\Delta u}{n} \langle, \rangle; \tag{2.14}$$

$$\Delta_f u + \frac{m}{m-n} \langle \nabla f, \nabla u \rangle = \Delta u + \frac{n}{m-n} \langle \nabla f, \nabla u \rangle = 0; \tag{2.15}$$

$$\text{Ric}_f^m = -k_0 \langle, \rangle, \quad f_\nu = -(m-n)c, \quad II = cI; \tag{2.16}$$

$$u_\nu = \frac{2\lambda_1 + k_0}{2(m-1)c} z. \tag{2.17}$$

The proof is trivial for  $m = n$  since  $f$  is constant. Hence, we assume  $m > n$  and obtain  $\Delta u = 0$  and  $\langle \nabla f, \nabla u \rangle = 0$  from  $\Delta_f u = 0$  and (2.15). Taking an orthonormal frame  $\{e_1, \dots, e_{n-1}\}$  tangent to  $\partial M$ , then we have from (2.14)

$$\begin{aligned} 0 &= \sum_{i=1}^{n-1} \nabla^2 u(e_i, e_i) = \bar{\Delta} u + (n-1)H u_\nu \\ &= \bar{\Delta}_f u + (n-1)H_f u_\nu \\ &= -\lambda_1 z + (m-1)c \frac{2\lambda_1 + k_0}{2(m-1)c} z \\ &= \frac{k_0}{2} z, \end{aligned} \tag{2.18}$$

which shows that  $k_0 = 0$  and  $\lambda_1 = (m-1)c^2$ . Then  $M$  is isometric to an  $n$ -dimensional Euclidean ball of radius  $\frac{1}{c}$  and  $f$  is constant from Huang and Ruan’s result (cf. [11, Theorem 1.6]). Similarly, if the equality holds in (1.7), then the equality case in (2.12) should be

$$u_\nu = C_{\lambda_1, k_0, m} z, \tag{2.19}$$

which also gives that  $k_0 = 0$  and  $\lambda_1 = (m-1)c^2$ . Consequently,  $M$  is isometric to an  $n$ -dimensional Euclidean ball of radius  $\frac{1}{c}$  and  $f$  is constant (cf. [11, Theorem 1.6]). This completes the proof of Theorem 1.1.

*Proof of Theorem 1.2.* Let  $u$  be an eigenfunction corresponding to the first eigenvalue  $q_1$  of the following equation

$$(\Delta_f)^2 u = 0 \text{ in } M, \quad u = \Delta_f u - q \frac{\partial u}{\partial \nu} = 0 \text{ on } \partial M. \tag{2.20}$$

That is,

$$q_1 = \frac{\int_M (\Delta_f u)^2 e^{-f} dv}{\int_{\partial M} (u_\nu)^2 e^{-f} d\mu}. \tag{2.21}$$

By virtue of (2.4) and the assumptions in Theorem 1.2, we have

$$\int_M \left[ \frac{m-1}{m} (\Delta_f u)^2 \right] e^{-f} dv \geq \int_{\partial M} (m-1)c (u_\nu)^2 e^{-f} d\mu. \tag{2.22}$$

Combining with (2.21) gives

$$q_1 \geq mc.$$

Now we assume that  $q_1 = mc$ . In this case, the inequality (2.22) must take an equality sign. In particular, we have

$$\nabla^2 u = \frac{\Delta u}{n} \langle \cdot, \cdot \rangle; \tag{2.23}$$

$$\Delta_f u + \frac{m}{m-n} \langle \nabla f, \nabla u \rangle = \Delta u + \frac{n}{m-n} \langle \nabla f, \nabla u \rangle = 0; \tag{2.24}$$

$$\text{Ric}_f^m(\nabla u, \nabla u) = 0, \quad H_f = \frac{m-1}{n-1} c. \tag{2.25}$$

Taking an orthonormal frame  $\{e_1, \dots, e_{n-1}, e_n\}$  on  $M$  such that when restricted to  $\partial M$ , we have  $e_n = \nu$ . By (2.23), we obtain for  $i = 1, \dots, n-1$ ,

$$0 = \nabla^2 u(e_i, \nu) = e_i(u_\nu), \tag{2.26}$$

which shows that  $u_\nu|_{\partial M}$  is constant and hence  $(\Delta_f u)|_{\partial M} = q_1 u_\nu = mc u_\nu := \rho$  is also constant from (2.20). By the divergence theorem and the fact that the function  $\Delta_f u$  is harmonic on  $M$ , we have

$$\begin{aligned} 0 &= \int_M [(\Delta_f u) - \rho] \Delta_f [(\Delta_f u) - \rho] e^{-f} dv \\ &= - \int_M |\nabla [(\Delta_f u) - \rho]|^2 e^{-f} dv \\ &\quad + \int_{\partial M} [(\Delta_f u) - \rho] \frac{\partial [(\Delta_f u) - \rho]}{\partial \nu} e^{-f} d\mu \\ &= - \int_M |\nabla [(\Delta_f u) - \rho]|^2 e^{-f} dv, \end{aligned} \tag{2.27}$$

which means that  $\Delta_f u$  is constant on  $M$ . Without loss of generality, we assume that  $\Delta_f u = 1$  and so we have from (2.23), (2.24), and (2.25) and the Bochner formula for the  $f$ -Laplacian,

$$\begin{aligned} \Delta_f \left( \frac{1}{2} |\nabla u|^2 - \frac{1}{m} u \right) &= |\nabla^2 u|^2 + \langle \nabla u, \nabla \Delta_f u \rangle + \text{Ric}_f(\nabla u, \nabla u) - \frac{1}{m} \Delta_f u \\ &= \frac{1}{m} (\Delta_f u)^2 + \langle \nabla u, \nabla \Delta_f u \rangle + \text{Ric}_f^m(\nabla u, \nabla u) - \frac{1}{m} \Delta_f u \\ &= 0. \end{aligned} \tag{2.28}$$



Integrating both sides of (2.28) yields

$$\begin{aligned} 0 &= \int_{\partial M} \left( u_\nu u_{,\nu\nu} - \frac{1}{m} u_\nu \right) e^{-f} d\mu \\ &= \int_{\partial M} \left( \frac{m-1}{m} u_\nu - (n-1)H_f u_\nu^2 \right) e^{-f} d\mu \\ &= \frac{m-1}{m} V_f - (n-1)H_f \frac{V_f^2}{A_f}, \end{aligned} \tag{2.29}$$

where we used the fact that  $u_{,\nu\nu} + (n-1)H_f u_\nu = 1$  from  $\Delta_f u = 1$  and  $u|_{\partial M} = 0$ ;

$$A_f u_\nu = \int_{\partial M} u_\nu e^{-f} d\mu = \int_M \Delta_f u e^{-f} dv = V_f.$$

Therefore, we derive from (2.29) that

$$H_f = \frac{(m-1)A_f}{m(n-1)V_f}. \tag{2.30}$$

Using the Corollary 1.2 in [11] completes the proof of Theorem 1.2.

*Proof of Theorem 1.3.* Let  $u$  be a special solution of the following Laplace equation

$$\Delta_f u = 1 \text{ in } M, \quad u = 0 \text{ on } \partial M. \tag{2.31}$$

It follows from the Rayleigh–Ritz inequality of  $q_1$  that

$$q_1 \leq \frac{\int_M (\Delta_f u)^2 e^{-f} dv}{\int_{\partial M} (u_\nu)^2 e^{-f} d\mu} = \frac{V_f}{\int_{\partial M} (u_\nu)^2 e^{-f} d\mu}. \tag{2.32}$$

Integrating both sides of  $\Delta_f u = 1$  and using the divergence theorem, we get

$$V_f = \int_M \Delta_f u e^{-f} dv = \int_{\partial M} u_\nu e^{-f} d\mu \leq (A_f)^{\frac{1}{2}} \left( \int_{\partial M} (u_\nu)^2 e^{-f} d\mu \right)^{\frac{1}{2}}. \tag{2.33}$$

Applying (2.33) into (2.32) gives

$$q_1 \leq \frac{A_f}{V_f}. \tag{2.34}$$

When  $q_1 = \frac{A_f}{V_f}$ , (2.33) shows that

$$u_\nu = \frac{V_f}{A_f} \text{ on } \partial M. \tag{2.35}$$

We define

$$\varphi = \frac{1}{2} |\nabla u|^2 - \frac{1}{m} u. \tag{2.36}$$

Under the assumption that the  $m$ -dimensional Bakry–Emery Ricci curvature is nonnegative, we have

$$\begin{aligned} \Delta_f \varphi &= |\nabla^2 u|^2 + \langle \nabla u, \nabla \Delta_f u \rangle + \text{Ric}_f(\nabla u, \nabla u) - \frac{1}{m} \Delta_f u \\ &\geq \frac{1}{m} (\Delta_f u)^2 + \langle \nabla u, \nabla \Delta_f u \rangle + \text{Ric}_f^m(\nabla u, \nabla u) - \frac{1}{m} \Delta_f u \\ &= \text{Ric}_f^m(\nabla u, \nabla u) \\ &\geq 0 \end{aligned} \tag{2.37}$$

and

$$\begin{aligned} \varphi_\nu|_{x_0} &= u_\nu u_{,\nu\nu} - \frac{1}{m} u_\nu \\ &= u_\nu \left( \frac{m-1}{m} - (n-1)H_f u_\nu \right) \\ &\leq 0, \end{aligned} \tag{2.38}$$

where we used the fact that  $u_{,\nu\nu} + (n-1)H_f u_\nu = 1$  from  $\Delta_f u = 1$  and  $u|_{\partial M} = 0$ . The strong maximum principal and Hopf lemma (cf. [9, pp. 34-35]) imply that  $\varphi_\nu|_{\partial M} > 0$  unless  $\varphi$  is constant on  $M$ . Hence we obtain  $\varphi$  is constant on  $M$  and hence

$$\Delta_f \varphi = 0. \tag{2.39}$$

That is, the equalities in (2.37) hold and

$$\Delta_f u + \frac{m}{m-n} \langle \nabla f, \nabla u \rangle = \Delta u + \frac{n}{m-n} \langle \nabla f, \nabla u \rangle = 0. \tag{2.40}$$

If  $m > n$ , multiplying (2.40) with  $u$  and integrating on  $M$  with respect to  $e^{\frac{n}{m-n}f} dv$  give that

$$\begin{aligned} 0 &= \int_M u \left( \Delta u + \frac{n}{m-n} \langle \nabla f, \nabla u \rangle \right) e^{\frac{n}{m-n}f} dv \\ &= \int_M u \operatorname{div} \left( e^{\frac{n}{m-n}f} \nabla u \right) dv \\ &= - \int_M |\nabla u|^2 e^{\frac{n}{m-n}f} dv + \int_{\partial M} u u_\nu e^{\frac{n}{m-n}f} d\mu \\ &= - \int_M |\nabla u|^2 e^{\frac{n}{m-n}f} dv. \end{aligned} \tag{2.41}$$

Therefore, we have that  $u$  is a constant function on  $M$ , which is a contradiction since  $\Delta_f u = 1$ .

Thus, we conclude that the equalities in (2.37) hold only when  $m = n$ ,  $f$  is constant, and  $\text{Ric}_f^m = \text{Ric}$ . Then by Wang and Xia’s arguments in [20], we complete the proof of Theorem 1.3.

*Proof of Theorem 1.4.* Let  $u$  be an eigenfunction corresponding to the first eigenvalue  $q_1$  of the equation (1.3). That is,

$$(\Delta_f)^2 u = 0 \text{ in } M, \quad u = \frac{\partial^2 u}{\partial \nu^2} - \mu_1 \frac{\partial u}{\partial \nu} = 0 \text{ on } \partial M. \tag{2.42}$$

Noticing that  $u_\nu = \frac{\partial u}{\partial \nu} \neq 0$ . Otherwise,  $0 = u_\nu = \langle \nabla u, \nu \rangle$  which combining with  $u|_{\partial M} = 0$  implies that  $u$  is constant on  $M$ . This is a contradiction.

By the divergence theorem and the boundary conditions in (2.42), we have

$$\begin{aligned} 0 &= \int_M u(\Delta_f)^2 u e^{-f} dv \\ &= \int_{\partial M} u \frac{\partial(\Delta_f u)}{\partial \nu} e^{-f} d\mu - \int_M \langle \nabla u, \nabla \Delta_f u \rangle e^{-f} dv \\ &= \int_{\partial M} u \frac{\partial(\Delta_f u)}{\partial \nu} e^{-f} d\mu - \int_{\partial M} u_\nu \Delta_f u e^{-f} d\mu + \int_M (\Delta_f u)^2 e^{-f} dv, \end{aligned} \tag{2.43}$$

which shows that

$$\int_M (\Delta_f u)^2 e^{-f} dv = \int_{\partial M} u_\nu \Delta_f u e^{-f} d\mu. \tag{2.44}$$

Inserting

$$\Delta_f u|_{\partial M} = \frac{\partial^2 u}{\partial \nu^2} + (n-1)H_f u_\nu = \mu_1 u_\nu + (n-1)H_f u_\nu \tag{2.45}$$

into (2.44) gives

$$\mu_1 \int_{\partial M} u_\nu^2 e^{-f} d\mu = \int_M (\Delta_f u)^2 e^{-f} dv - (n-1) \int_{\partial M} H_f u_\nu^2 e^{-f} d\mu. \tag{2.46}$$

Applying  $u$  into the formula (2.4) and noticing the assumption that the  $m$ -dimensional Bakry–Emery Ricci curvature is nonnegative, we have

$$\begin{aligned} &\frac{m-1}{m} \int_M (\Delta_f u)^2 e^{-f} dv \\ &\geq \int_{\partial M} [2u_\nu(\overline{\Delta}_f u) + (n-1)H_f(u_\nu)^2 + II(\overline{\nabla} u, \overline{\nabla} u)] e^{-f} d\mu \\ &= (n-1) \int_{\partial M} H_f(u_\nu)^2 e^{-f} d\mu. \end{aligned} \tag{2.47}$$

Hence, we obtain from (2.46), (2.47), and  $(n-1)H_f \geq (m-1)c$  that

$$\mu_1 \geq c. \tag{2.48}$$

Assume that  $\mu_1 = c$ . In this case, the inequalities in (2.4) and (2.47) must take an equality sign. In particular, we have

$$\Delta_f u + \frac{m}{m-n} \langle \nabla f, \nabla u \rangle = \Delta u + \frac{n}{m-n} \langle \nabla f, \nabla u \rangle = 0. \tag{2.49}$$

If  $m > n$ , multiplying (2.49) with  $u$  and integrating on  $M$  with respect to  $e^{\frac{n}{m-n}f} dv$  give that

$$\begin{aligned} 0 &= \int_M u \left( \Delta u + \frac{n}{m-n} \langle \nabla f, \nabla u \rangle \right) e^{\frac{n}{m-n}f} dv \\ &= - \int_M |\nabla u|^2 e^{\frac{n}{m-n}f} dv, \end{aligned} \tag{2.50}$$

which shows that  $u$  is a constant function on  $M$ . This is a contradiction since  $\Delta_f u$  is a nonzero constant.

Thus, we conclude that the equalities hold only when  $m = n$ ,  $f$  is constant, and  $\text{Ric}_f^m = \text{Ric}$ . Then by Xia and Wang’s arguments in [21], we complete the proof of Theorem 1.4.

*Proof of Theorem 1.5.* Let  $u$  be an eigenfunction corresponding to the first eigenvalue  $\xi_1$  of the equation (1.4). That is,

$$(\Delta_f)^2 u = 0 \text{ in } M, \quad \frac{\partial u}{\partial \nu} = \frac{\partial(\Delta_f u)}{\partial \nu} + \xi_1 u = 0 \text{ on } \partial M. \tag{2.51}$$

Let  $u|_{\partial M} = z$ . By virtue of (2.43), we have

$$\xi_1 \int_{\partial M} z^2 e^{-f} d\mu = \int_M (\Delta_f u)^2 e^{-f} dv. \tag{2.52}$$

Substituting  $u$  into the formula (2.4) and noticing the assumption that the  $m$ -dimensional Bakry0-Emery Ricci curvature is nonnegative, we have

$$\begin{aligned} &\frac{m-1}{m} \int_M (\Delta_f u)^2 e^{-f} dv \\ &\geq \int_{\partial M} [2u_\nu(\bar{\Delta}_f u) + (n-1)H_f(u_\nu)^2 + II(\bar{\nabla} u, \bar{\nabla} u)] e^{-f} d\mu \\ &\geq c \int_{\partial M} |\bar{\nabla} u|^2 e^{-f} d\mu. \end{aligned} \tag{2.53}$$

Integrating the boundary condition (2.51) gives

$$\xi_1 \int_{\partial M} z e^{-f} d\mu = - \int_{\partial M} \frac{\partial(\Delta_f u)}{\partial \nu} e^{-f} d\mu = - \int_M (\Delta_f)^2 u e^{-f} dv = 0, \tag{2.54}$$

which shows that  $\int_{\partial M} z e^{-f} d\mu = 0$ . Let  $\lambda_1$  denote the first nonzero eigenvalue of the  $\Delta_f$  on  $\partial M$ , i.e.

$$\lambda_1 = \inf \left\{ \frac{\int_{\partial M} |\nabla \varphi|^2 e^{-f} d\mu}{\int_{\partial M} \varphi^2 e^{-f} d\mu}, \varphi \text{ is not identically zero and } \int_{\partial M} \varphi e^{-f} d\mu = 0 \right\}.$$

Then we have

$$\int_{\partial M} |\bar{\nabla} u|^2 e^{-f} d\mu \geq \lambda_1 \int_{\partial M} z^2 e^{-f} d\mu, \tag{2.55}$$

and (2.53) yields

$$\frac{m-1}{m} \int_M (\Delta_f u)^2 e^{-f} dv \geq c\lambda_1 \int_{\partial M} z^2 e^{-f} d\mu. \tag{2.56}$$

Hence, we obtain (1.11) from (2.52) and (2.56).

Assume that  $\xi_1 = \frac{mc\lambda_1}{m-1}$  occurs. In this case, the inequalities in (2.53) and (2.55) must take an equality sign. In particular, we have

$$\nabla^2 u = \frac{\Delta u}{n} \langle \cdot, \cdot \rangle; \tag{2.57}$$

$$\Delta_f u + \frac{m}{m-n} \langle \nabla f, \nabla u \rangle = \Delta u + \frac{n}{m-n} \langle \nabla f, \nabla u \rangle = 0; \tag{2.58}$$

$$\text{Ric}_f^m(\nabla u, \nabla u) = 0, \quad f_\nu = -(m-n)c, \quad II = cI. \tag{2.59}$$

Taking an orthonormal frame  $\{e_1, \dots, e_{n-1}, e_n\}$  on  $M$  such that when restricted to  $\partial M$ , we have  $e_n = \nu$ . By (2.57), we obtain for  $i = 1, \dots, n-1$ ,

$$0 = \nabla^2 u(e_i, \nu) = e_i(u_\nu) - II_{ij}u_j = -II_{ij}u_j, \tag{2.60}$$

which shows that  $II(\bar{\nabla}z, \bar{\nabla}z) = 0$ . This is impossible since  $II \geq cI$  and  $z$  is not constant. We complete the proof of Theorem 1.5.

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